A relation involving Rankin-Selberg L-functions of cusp forms and Maass forms

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Abstract

In previous articles, an identity relating the canonical metric to the hyperbolic metric associated to any compact Riemann surface of genus at least two has been derived and studied. In this article this identity is extended to any hyperbolic Riemann surface of finite volume. The method of proof is to study the identity given in the compact case through degeneration and to understand the limiting behavior of all quantities involved. In the second part of the paper, the Rankin-Selberg transform of the non-compact identity is studied, meaning that both sides of the relation after multiplication by a non-holomorphic, parabolic Eisenstein series are being integrated over the Riemann surface in question. The resulting formula yields an asymptotic relation involving the Rankin-Selberg L-functions of weight two holomorphic cusp forms, of weight zero Maass forms, and of non-holomorphic weight zero parabolic Eisenstein series.

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1 Introduction

1.1. Beginning with the article [13], we derived and studied a basic identity, stated in (1) below, coming from the spectral theory of the Laplacian associated to any compact hyperbolic Riemann surface. In the subsequent papers, this identity was employed to address a number of problems, including the following: Establishing precise relations between analytic invariants arising in the Arakelov theory of algebraic curves and hyperbolic geometry (see [13]), proving the non-completeness of a newly-defined metric on the moduli space of algebraic curves of a fixed genus (see [14]), deriving bounds for canonical and hyperbolic Green's functions (see [15]), and obtaining bounds for Faltings's delta function with applications associated to Arakelov theory (see [16]). In the present article, we expand our application of the results from [13] to analytic number theory. In brief, we first generalize the identity (1) to general non-compact, finite volume hyperbolic Riemann surfaces without elliptic fixed points; this relation is stated in equation (2) below. We then compute the Rankin-Selberg convolution with respect to (2), and show that the result yields a new relation involving Rankin-Selberg L-functions of cusp forms of weight two and Maass forms, as well as the scattering matrix of the non-holomorphic Eisenstein series of weight zero.

1.2. The basic identity. Let X denote a compact hyperbolic Riemann surface, necessarily of genus $g \geq 2$. Let $\{f_j\}$ be a basis of the g-dimensional space of cusp forms of weight two, which we assume to be orthonormal with respect to the Petersson inner product. Then, we set

$$\mu_{\mathrm{can}}(z) = \frac{1}{g} \cdot \frac{i}{2} \sum_{j=1}^{g} |f_j(z)|^2 \mathrm{d}z \wedge \mathrm{d}\overline{z}$$

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for any point $z \in X$. Let Δ_{hyp} denote the hyperbolic Laplacian acting on the space of smooth functions on X, and $K(t;z,w)$ the corresponding heat kernel; set $K(t;z) = K(t;z,z)$. We use μ_{shyp} to denote the $(1,1)$-form of the constant negative curvature metric on X such that X has volume one, and μ_{hyp} to denote the $(1,1)$-form of the metric on X with constant negative curvature equal to -1. With this notation, the key identity of [13] states

$$\mu_{\mathrm{can}}(z) = \mu_{\mathrm{shyp}}(z) + \frac{1}{2g}\int\limits_{0}^{\infty} \Delta_{\mathrm{hyp}} K(t;z)\mathrm{d}t\mu_{\mathrm{hyp}}(z) \qquad (z \in X). \tag{1}$$

The first result in the present paper is to generalize (1) to general non-compact, finite volume hyperbolic Riemann surfaces without elliptic fixed points. Specifically, if X is such a non-compact, finite volume hyperbolic Riemann surface of genus g with p cusps and no elliptic fixed points, then

$$\mu_{\mathrm{can}}(z) = \left(1 + \frac{p}{2g}\right)\mu_{\mathrm{shyp}}(z) + \frac{1}{2g}\int\limits_{0}^{\infty} \Delta_{\mathrm{hyp}} K(t;z)\mathrm{d}t\,\mu_{\mathrm{hyp}}(z) \qquad (z \in X). \tag{2}$$

The proof of (2) we present here is to study (1) for a degenerating family of hyperbolic Riemann surfaces and to use known results for the asymptotic behavior of the canonical metric form μ_{can} (see [12]), the hyperbolic heat kernel (see [18]), and small eigenvalues and eigenfunctions of the Laplacian (see [21]).

In [2], the author extends the identity (2) to general finite volume quotients of the hyperbolic upper half-plane, allowing for the presence of elliptic elements. The proof does not employ degeneration techniques, as in the present paper, but rather follows the original method of proof given in [13] and [15]. The article [2] is part of the Ph.D. dissertation completed under the direction of the second named author of the present article.

1.3. The Rankin-Selberg convolution. For the remainder of the present article, we assume $p > 0$. Let P denote a cusp of X and $E_{P,s}(z)$ the associated non-holomorphic Eisenstein series of weight zero. In essence, the purpose of the present article is to evaluate the Rankin-Selberg convolution with respect to (2), by which we mean to multiply both sides of (2) by $E_{P,s}(z)$ and to integrate over all $z \in X$.

By means of the uniformization theorem, there is a Fuchsian group of the first kind $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ such that X is isometric to $\Gamma\backslash\mathbb{H}$. Furthermore, we can choose Γ so that the point $i\infty$ in the boundary of $\mathbb{H}$ projects to the cusp P, which we assume to have width b. Writing $z = x + iy$, well-known elementary considerations then show that the expression

$$\int\limits_{X} E_{P,s}(z)\mu_{\mathrm{can}}(z) = \int\limits_{X} E_{P,s}(z)\left(\left(1 + \frac{p}{2g}\right)\mu_{\mathrm{shyp}}(z) + \frac{1}{2g}\int\limits_{0}^{\infty} \Delta_{\mathrm{hyp}} K(t;z)\mathrm{d}t\,\mu_{\mathrm{hyp}}(z)\right)$$

is equivalent to

$$\int\limits_{y=0}^{\infty}\int\limits_{x=0}^{b} y^s \mu_{\mathrm{can}}(z) = \int\limits_{y=0}^{\infty}\int\limits_{x=0}^{b} y^s \left(\left(1 + \frac{p}{2g}\right)\mu_{\mathrm{shyp}}(z) + \frac{1}{2g}\int\limits_{0}^{\infty} \Delta_{\mathrm{hyp}} K(t;z)\mathrm{d}t\,\mu_{\mathrm{hyp}}(z)\right). \tag{3}$$

The majority of the computations carried out in the present article are related to the evaluation of (3). To be precise, for technical reasons we consider the integrals in (3) multiplied by the factor $2gb^{-1}\pi^{-s}\Gamma(s)\zeta(2s)$, where $\Gamma(s)$ is the Γ-function and $\zeta(s)$ is the Riemann ζ-function.

1.4. The main result. Having posed the problem under consideration, we can now state the main result of this article after establishing some additional notation.

The cusp forms f_j, being invariant under the map $z \mapsto z + b$, allow a Fourier expansion of the form

$$f_j(z) = \sum_{n=1}^{\infty} a_{j,n} e^{2\pi i n z/b}.$$

Following notations and conventions in [4], we let

$$\widetilde{L}(s, f_j \otimes \overline{f}_j) = G_\infty(s) \cdot L(s, f_j \otimes \overline{f}_j), \tag{4}$$

where

$$G_\infty(s) = (2\pi)^{-2s-1}\Gamma(s)\Gamma(s+1)\zeta(2s),$$
$$L(s, f_j \otimes \overline{f}_j) = \sum_{n=1}^{\infty} \frac{|a_{j,n}|^2}{(n/b)^{s+1}}.$$

As shown in [4], the Rankin-Selberg L-function $\widetilde{L}(s, f_j \otimes \overline{f}_j)$ is holomorphic for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, admits a meromorphic continuation to all $s \in \mathbb{C}$, and is symmetric under $s \mapsto 1-s$.

Let φ_j be a non-holomorphic weight zero form which is an eigenfunction of Δ_{hyp} with eigenvalue $\lambda_j = s_j(1-s_j)$, hence $s_j = 1/2 + ir_j$. From [11], we recall the expansion

$$\varphi_j(z) = \alpha_{j,0}(y) + \sum_{n \neq 0} \alpha_{j,n} W_{s_j}(nz/b),$$

where

$$\alpha_{j,0}(y) = \alpha_{j,0} y^{1-s_j},$$
$$W_{s_j}(w) = 2\sqrt{\cosh(\pi r_j)}\sqrt{|\mathrm{Im}(w)|}K_{ir_j}(2\pi|\mathrm{Im}(w)|)e^{2\pi i \mathrm{Re}(w)} \quad (w \in \mathbb{C}),$$

and $K_\cdot(\cdot)$ denotes the classical K-Bessel function. Again, following notations and conventions in [4], we let

$$\widetilde{L}(s, \varphi_j \otimes \overline{\varphi}_j) = G_{r_j}(s) \cdot L(s, \varphi_j \otimes \overline{\varphi}_j),$$

where

$$G_{r_j}(s) = s(1-s)\pi^{-2s}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2} + ir_j\right)\Gamma\left(\frac{s}{2} - ir_j\right)\zeta(2s),$$
$$L(s, \varphi_j \otimes \overline{\varphi}_j) = \sum_{n \neq 0} \frac{|\alpha_{j,n}|^2}{(n/b)^{s-1}}.$$

As shown in [4], the Rankin-Selberg L-function $\widetilde{L}(s, \varphi_j \otimes \overline{\varphi}_j)$ is holomorphic for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, admits a meromorphic continuation to all $s \in \mathbb{C}$, and is symmetric under $s \mapsto 1-s$. Observe that our completed L-function $\widetilde{L}(s, \varphi_j \otimes \overline{\varphi}_j)$ differs from the L-function defined in [4] because of the appearance of the multiplicative factor $s(1-s)$ in the definition of $G_{r_j}(s)$.

Similarly, one can define completed Rankin-Selberg L-functions associated to the non-holomorphic Eisenstein series $E_{P,s}(z)$ for any cusp P on X having a Fourier expansion of the form

$$E_{P,s}(z) = \delta_{P,\infty} y^s + \phi_{P,\infty}(s) y^{1-s} + \sum_{n \neq 0} \alpha_{P,s,n} W_s(nz/b)$$

with $\phi_{P,\infty}(s)$ denoting the (P,∞)-th entry of the scattering matrix.

With all this, the main result of the present article is the following theorem. For any $\varepsilon > 0$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, define the Θ-function

$$\Theta_\varepsilon(s) = \sum_{\lambda_j > 0}^{\infty} \frac{\cosh(\pi r_j) e^{-\lambda_j \varepsilon}}{2\lambda_j} \widetilde{L}(s, \varphi_j \otimes \overline{\varphi}_j)$$
$$+ \frac{1}{8\pi} \sum_{P \text{ cusp}} \int_{-\infty}^{\infty} \frac{\cosh(\pi r) e^{-(r^2+1/4)\varepsilon}}{r^2 + 1/4} \widetilde{L}(s, E_{P,1/2+ir} \otimes \overline{E}_{P,1/2+ir}) \mathrm{d}r$$

and the universal function

$$F_\varepsilon(s) = \frac{\zeta(s)b^{s-1}}{2\pi^2} \int\limits_0^\infty \frac{r\sinh(\pi r)e^{-(r^2+1/4)\varepsilon}}{r^2+1/4} G_r(s)\mathrm{d}r.$$

Then, the L-function relation involving Rankin-Selberg L-functions of cusp forms and Maass forms

$$\lim_{\varepsilon\to 0}\big(\Theta_\varepsilon(s) - F_\varepsilon(s)\big) = \\ \sum_{j=1}^{g} \widetilde{L}(s, f_j \otimes \overline{f}_j) - 4\pi\zeta(s)b^{s-1}G_\infty(s) - \pi^{-s}\frac{2s}{s+1}\Gamma(s)\zeta(2s)\phi_{\infty,\infty}\left(\frac{s+1}{2}\right) \tag{5}$$

holds true. By taking $\varepsilon > 0$ in (5), one has an error term which is $o(1)$ as ε approaches zero. This error term is explicit and given in terms of integrals involving the hyperbolic heat kernel.

A natural question to ask is to what extent the relation of L-functions (5) implies relations between the Fourier coefficients of the holomorphic weight two forms and the Fourier coefficients of the Maass forms under consideration. In general, extracting such information from a limiting relationship such as (5) could be very difficult. However, as stated, our analysis yields an explicit expression for the error term by rewriting (5) for a fixed $\varepsilon > 0$, which allows for additional considerations. The problem of using (5) to study possible relations amongst the Fourier coefficients is currently under investigation.

1.5. General comments. If X is the Riemann surface associated to a congruence subgroup, then the series $\phi_{\infty,\infty}(s)$ can be expressed in terms of Dirichlet L-functions associated to even characters with conductors dividing the level (see [8] or [10]). With these computations, one can rewrite (5) further so that one obtains an expression involving Rankin-Selberg L-functions associated to cusp forms of weight two, Maass forms, non-holomorphic Eisenstein series, and classical zeta functions. However, the relation stated in (5) holds for any finite volume hyperbolic Riemann surface without elliptic fixed points. In order to eliminate the restriction that X has no elliptic fixed points, one needs to revisit the proof of (2), and possibly (1), in order to allow for elliptic fixed points. As stated above, this project currently is under investigation in [2]; however, we choose to focus in this paper on deriving (5) with the simplifying assumption that X has no elliptic fixed points in order to draw attention to the presence of an L-function relation coming from the basic identity (2). We will leave for future work the generalization of (2) to arbitrary finite volume hyperbolic Riemann surfaces, which may have elliptic fixed points, and derive the relation analogous to (5).

From Riemannian geometry, theta functions naturally appear as the trace of a heat kernel, and the small time expansion of the heat kernel has a first-order term which is somewhat universal and a second-order term which involves integrals of a curvature of the Riemannian metric. In this regard, (5) suggests that the sum of Rankin-Selberg L-functions

$$\sum_{j=1}^{g} \widetilde{L}(s, f_j \otimes \overline{f}_j)$$

represents some type of curvature integral relative to the theta function $\Theta_\varepsilon(s)$. Further investigation of this heuristic observation is warranted.

1.6. Outline of the paper. In section 2, we recall necessary background material and establish additional notation. In section 3, we prove (2) and further develop the identity (2) using the spectral expansion of the heat kernel $K(t; z, w)$. In section 4, we evaluate the integrals in (3) using the revised analytic expressions of (2), and in section 5, we gather the computations from section 4 and prove (5).

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2 Notations and preliminaries

2.1. Hyperbolic and canonical metrics. Let Γ be a Fuchsian subgroup of the first kind of $\mathrm{PSL}_2(\mathbb{R})$ acting by fractional linear transformations on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, z = x + iy,\, y > 0\}$. We let X be the quotient space $\Gamma\backslash\mathbb{H}$ and denote by g the genus of X. We assume that Γ has no elliptic elements and that X has $p \geq 1$ cusps. We identify X locally with its universal cover $\mathbb{H}$.

In the sequel μ denotes a (smooth) metric on X, i.e., μ is a positive $(1,1)$-form on X. In particular, we let $\mu = \mu_{\mathrm{hyp}}$ denote the hyperbolic metric on X, which is compatible with the complex structure of X, and has constant negative curvature equal to minus one. Locally, we have

$$\mu_{\mathrm{hyp}}(z) = \frac{i}{2} \cdot \frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{y^2}\,.$$

We write $\mathrm{vol}_{\mathrm{hyp}}(X)$ for the hyperbolic volume of X; recall that $\mathrm{vol}_{\mathrm{hyp}}(X)$ is given by $2\pi(2g-2+p)$. The scaled hyperbolic metric $\mu = \mu_{\mathrm{shyp}}$ is simply the rescaled hyperbolic metric $\mu_{\mathrm{hyp}}/\mathrm{vol}_{\mathrm{hyp}}(X)$, which measures the volume of X to be one.

Let $S_k(\Gamma)$ denote the $\mathbb{C}$-vector space of cusp forms of weight k with respect to Γ equipped with the Petersson inner product

$$\langle f, g\rangle = \frac{i}{2}\int\limits_X f(z)\,\overline{g(z)}\,y^k\,\frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{y^2} \qquad \big(f, g \in S_k(\Gamma)\big)\,.$$

By choosing an orthonormal basis $\{f_1, ..., f_g\}$ of $S_2(\Gamma)$ with respect to the Petersson inner product, the canonical metric $\mu = \mu_{\mathrm{can}}$ of X is given by

$$\mu_{\mathrm{can}}(z) = \frac{1}{g} \cdot \frac{i}{2} \sum_{j=1}^{g} |f_j(z)|^2\,\mathrm{d}z \wedge \mathrm{d}\overline{z}.$$

We denote the hyperbolic Laplacian on X by Δ_{hyp}; locally, we have

$$\Delta_{\mathrm{hyp}} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{6}$$

The discrete spectrum of Δ_{hyp} is given by the increasing sequence of eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$$

2.2. Modular forms, Maass forms, and Eisenstein series. Throughout we assume, as before, that the cusp width of the cusp $i\infty$ equals b. In subsection 1.4, we established the notation for holomorphic cusp forms of weight two and Maass forms with respect to Γ, as well as the corresponding Rankin-Selberg L-functions, so we do not repeat the discussion here.

The eigenfunctions for the continuous spectrum of Δ_{hyp} are provided by the Eisenstein series $E_{P,s'}$ (associated to each cusp P of X) with eigenvalue $\lambda = s'(1-s')$, hence $s' = 1/2 + ir$ $(r \in \mathbb{R})$. They have Fourier expansions of the form

$$E_{P,s'}(z) = \alpha_{P,s',0}(y) + \sum_{n \neq 0} \alpha_{P,s',n} W_{s'}(nz/b),$$

where

$$\begin{aligned}
&\alpha_{P,s',0}(y) = \delta_{P,\infty} y^{s'} + \phi_{P,\infty}(s') y^{1-s'},\\
&W_{s'}(w) = 2\sqrt{\cosh(\pi r)}\sqrt{|\mathrm{Im}(w)|}K_{ir}(2\pi|\mathrm{Im}(w)|)e^{2\pi i\mathrm{Re}(w)} \quad (w \in \mathbb{C});
\end{aligned}$$

here $\delta_{P,\infty}$ is the Kronecker delta and $\phi_{P,\infty}(s')$ is the (P,∞)-th entry of the scattering matrix (see [11]). For example, the function $\phi_{\infty,\infty}(s')$ is given by a Dirichlet series of the form

$$\phi_{\infty,\infty}(s') = \sqrt{\pi}\,\frac{\Gamma(s'-1/2)}{\Gamma(s')} \sum_{n=1}^{\infty} \frac{a_n}{\mu_n^{2s'}}, \tag{7}$$

where the quantities a_n and μ_n are explicitly given in [11]), p. 60.
For $s\in\mathbb{C}$, $\mathrm{Re}(s)>1$, we define the completed Rankin-Selberg L-function attached to $E_{P,s'}$ by

$$\widetilde{L}(s,E_{P,s'}\otimes\overline{E}_{P,s'})=G_r(s)\cdot L(s,E_{P,s'}\otimes\overline{E}_{P,s'}), \tag{8}$$

where

$$G_r(s)=s(1-s)\pi^{-2s}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+ir\right)\Gamma\left(\frac{s}{2}-ir\right)\zeta(2s),$$
$$L(s,E_{P,s'}\otimes\overline{E}_{P,s'})=\sum_{n\neq0}\frac{|\alpha_{P,s',n}|^2}{(n/b)^{s-1}}.$$

2.3. Hyperbolic heat kernel and variants. The hyperbolic heat kernel $K_{\mathbb{H}}(t;z,w)$ ($t\in\mathbb{R}_{>0}$; $z,w\in\mathbb{H}$) on $\mathbb{H}$ is given by the formula

$$K_{\mathbb{H}}(t;z,w)=K_{\mathbb{H}}(t;\rho)=\frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}}\int_{\rho}^{\infty}\frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r)-\cosh(\rho)}}\,\mathrm{d}r\,,$$

where $\rho=d_{\mathrm{hyp}}(z,w)$ denotes the hyperbolic distance from z to w. The hyperbolic heat kernel $K(t;z,w)$ ($t\in\mathbb{R}_{>0}$; $z,w\in X$) on X is obtained by averaging over the elements of Γ, namely

$$K(t;z,w)=\sum_{\gamma\in\Gamma}K_{\mathbb{H}}\big(t;z,\gamma(w)\big).$$

The heat kernel on X satisfies the equations

$$\left(\frac{\partial}{\partial t}+\Delta_{\mathrm{hyp},z}\right)K(t;z,w)=0\qquad(w\in X),$$
$$\lim_{t\to0}\int_X K(t;z,w)\,f(w)\,\mu_{\mathrm{hyp}}(w)=f(z)\quad(z\in X)$$

for all C^∞-functions f on X. As a shorthand, we write $K(t;z)=K(t;z,z)$.
With the notations from subsection 2.2, we introduce the modified heat kernel function

$$K^{\mathrm{cusp}}(t;z)=K(t;z)-$$
$$\sum_{0\leq\lambda_j<1/4}|\alpha_{j,0}|^2y^{2-2s_j}e^{-\lambda_j t}-\frac{1}{4\pi}\sum_{P\ \mathrm{cusp}}\int_{-\infty}^{\infty}|\delta_{P,\infty}y^{1/2+ir}+\phi_{P,\infty}(s)y^{1/2-ir}|^2e^{-(r^2+1/4)t}\mathrm{d}r. \tag{9}$$

Denoting by Γ_∞ the stabilizer of the cusp ∞, we can define the following partial heat kernel functions

$$K_0(t;z)=\sum_{\gamma\in\Gamma\setminus\Gamma_\infty}K_{\mathbb{H}}\big(t;z,\gamma(z)\big), \tag{10}$$
$$K_\infty(t;z)=\sum_{\gamma\in\Gamma_\infty}K_{\mathbb{H}}\big(t;z,\gamma(z)\big) \tag{11}$$

giving rise to the decomposition

$$K(t;z)=K_0(t;z)+K_\infty(t;z).$$

3 The fundamental identity

In this section we derive the identity (2) by studying the relation (1) for a degenerating family of compact hyperbolic Riemann surfaces. The corresponding statement is proven in Lemma 3.1. In the remainder of the section, we manipulate the terms in (2) assuming $p > 0$ in order to obtain an equivalent formulation of the relation which then will be suited for our computations in the subsequent sections. Specifically, we first express the heat kernel on the underlying Riemann surface in terms of its spectral expansion, which involves Maass forms and non-holomorphic Eisenstein series, and we remove the terms associated to the constant terms in the Fourier expansions of the Maass forms and the non-holomorphic Eisenstein series (see Proposition 3.3). We then express the heat kernel as a periodization over the uniformizing group and remove the contribution from the parabolic subgroup associated to a single cusp (see Lemma 3.8 as well as the preliminary computations and remarks). The main result of this section is Theorem 3.9.

3.1. Lemma. With the above notations, we have

$$\mu_{\mathrm{can}}(z) = \left(1 + \frac{p}{2g}\right)\mu_{\mathrm{shyp}}(z) + \frac{1}{2g}\int\limits_0^\infty \Delta_{\mathrm{hyp}} K(t;z)\mathrm{d}t\,\mu_{\mathrm{hyp}}(z). \tag{12}$$

Proof. The proof of identity (12) in case X is compact, i.e. $p = 0$, for any $g \geq 2$ is given in [13] as well as the appendix to [16]. We will now prove (12) by induction on p by considering degenerating sequences of finite volume hyperbolic Riemann surfaces. More specifically, we assume that (12) holds for any hyperbolic Riemann surface of genus g with p cusps, and then prove the relation for hyperbolic Riemann surfaces of any genus with $p+1$ cusps. Whereas the method of proof can be viewed as standard perturbation theory, we choose to include all details in order to determine all constants, specifically the multiplicative factor of μ_{hyp} in (2).

If X has genus g and $p+1$ cusps, then, following the methodology of [12] and [18], one can construct a degenerating family $\{X_\ell\}$ with the following properties:

- For $\ell > 0$, each surface X_ℓ has genus $g+1$ and p cusps,
- the degenerating family has precisely one pinching geodesic of length ℓ approaching zero,
- the limiting surface X_0, which necessarily has two components, is such that X is isometric to one of the two components.

Let X and X' be the two components of X_0 with hyperbolic volumes $v = \mathrm{vol}_{\mathrm{hyp}}(X)$ and $v' = \mathrm{vol}_{\mathrm{hyp}}(X')$, respectively; by construction, X' has genus one and one cusp. the hyperbolic volume of X_ℓ equals $v + v'$, and the induction hypothesis for X_ℓ reads (using an obvious change in notation)

$$2(g+1)\mu_{\mathrm{can},X_\ell}(z) = \big(2(g+1)+p\big)\mu_{\mathrm{shyp},X_\ell}(z) + \int\limits_0^\infty \Delta_{\mathrm{hyp},X_\ell} K_{X_\ell}(t;z)\mathrm{d}t\,\mu_{\mathrm{hyp},X_\ell}(z). \tag{13}$$

We now determine the limiting value of (13) through degeneration. Throughout, we will let $z \in X_\ell$ be any point which limits to a point $z \in X$.

From [12], we have that

$$\lim_{\ell\to 0}\big(2(g+1)\mu_{\mathrm{can},X_\ell}(z)\big) = 2g\mu_{\mathrm{can},X}(z). \tag{14}$$

From [1], we recall that

$$\lim_{\ell\to 0}\big(\mu_{\mathrm{hyp},X_\ell}(z)\big) = \mu_{\mathrm{hyp},X}(z),$$

which leads to

$$\lim_{\ell\to 0}\big((2(g+1)+p)\mu_{\mathrm{shyp},X_\ell}(z)\big) = \frac{2(g+1)+p}{v+v'}\mu_{\mathrm{hyp},X}(z). \tag{15}$$

Let now λ_{1,X_ℓ} denote the smallest non-zero eigenvalue of the hyperbolic Laplacian $\Delta_{\mathrm{hyp},X_\ell}$ on X_ℓ, with corresponding eigenfunction φ_{1,X_ℓ}. From [18], we have that

$$\lim_{\ell\to 0}\left(K_{X_\ell}(t;z)-\frac{1}{v+v'}-\varphi^2_{1,X_\ell}(z)e^{-\lambda_{1,X_\ell}t}\right)=K_X(t;z)-\frac{1}{v}$$

with uniformity of the convergence for all $t>0$ (see [18], Lemma 3.2). The proof given in [18] extends (see Remark 3.2) to show that

$$\lim_{\ell\to 0}\Delta_{\mathrm{hyp},X_\ell}\left(K_{X_\ell}(t;z)-\frac{1}{v+v'}-\varphi^2_{1,X_\ell}(z)e^{-\lambda_{1,X_\ell}t}\right)=\Delta_{\mathrm{hyp},X}\left(K_X(t;z)-\frac{1}{v}\right), \tag{16}$$

with a corresponding uniformity result, which allows us to arrive at the conclusion that

$$\lim_{\ell\to 0}\left(\int_0^\infty \Delta_{\mathrm{hyp},X_\ell}K_{X_\ell}(t;z)\mathrm{d}t-\frac{\Delta_{\mathrm{hyp},X_\ell}\varphi^2_{1,X_\ell}(z)}{\lambda_{1,X_\ell}}\right)=\int_0^\infty \Delta_{\mathrm{hyp},X}K_X(t;z)\mathrm{d}t. \tag{17}$$

By substituting the limit computations (14), (15), and (17) into (13), we are led to

$$\begin{aligned}&2g\mu_{\mathrm{can},X}(z)=\\&\int_0^\infty \Delta_{\mathrm{hyp},X}K_X(t;z)\mathrm{d}t\,\mu_{\mathrm{hyp},X}(z)+\left(\frac{2(g+1)+p}{v+v'}+\lim_{\ell\to 0}\left(\frac{\Delta_{\mathrm{hyp},X_\ell}\varphi^2_{1,X_\ell}(z)}{\lambda_{1,X_\ell}}\right)\right)\mu_{\mathrm{hyp},X}(z),\end{aligned}$$

so we are left to prove that

$$\frac{2(g+1)+p}{v+v'}+\lim_{\ell\to 0}\left(\frac{\Delta_{\mathrm{hyp},X_\ell}\varphi^2_{1,X_\ell}(z)}{\lambda_{1,X_\ell}}\right)=\frac{2g+(p+1)}{v}. \tag{18}$$

The construction of the degenerating family $\{X_\ell\}$ from [18] begins by constructing a degenerating family of compact Riemann surfaces with distinguished points, after which one obtains a degenerating family of finite volume hyperbolic Riemann surfaces by employing the uniformization theorem. As a result, there is an underlying real parameter u which describes the degenerating family $\{X_\ell\}$. An asymptotic relation between u and ℓ is established in [21]; for our purposes, it suffices to use that $\ell\to 0$ as $u\to 0$, and conversely. With all this, it is proven in [21] that one has the asymptotic expansion

$$\lambda_{1,X_\ell}=\alpha_1 u+O(u^2)\quad\text{as}\quad u\to 0 \tag{19}$$

for some constant α_1. In addition, one has from [21] the asymptotic expansions

$$\varphi_{1,X_\ell}(z)=c_{0,X}(z)+c_{1,X}(z)u+O(u^2)\quad\text{as}\quad u\to 0\qquad (z\in X), \tag{20}$$

and

$$\varphi_{1,X_\ell}(z)=c_{0,X'}(z)+c_{1,X'}(z)u+O(u^2)\quad\text{as}\quad u\to 0\qquad (z\in X'). \tag{21}$$

In [18], it is proven that small eigenvalues and small eigenfunctions converge through degeneration; hence, the functions $c_{0,X}$ and $c_{0,X'}$ are constants. More precisely, since φ_{1,X_ℓ} is orthogonal to the constant functions on X_ℓ and has L^2-norm one, we have the relations

$$c_{0,X}v+c_{0,X'}v'=0\quad\text{and}\quad c^2_{0,X}v+c^2_{0,X'}v'=1,$$

from which we immediately derive

$$c_{0,X}=\pm\left(\frac{v'}{v(v+v')}\right)^{1/2}\quad\text{and}\quad c_{0,X'}=\mp\left(\frac{v}{v'(v+v')}\right)^{1/2}. \tag{22}$$

The uniformity of the convergence of heat kernels through degeneration from [18] and the convergence of hyperbolic metrics through degeneration from [1], allow one to conclude that, since φ_{1,X_ℓ} is an eigenfunction of $\Delta_{\mathrm{hyp},X_\ell}$ with eigenvalue λ_{1,X_ℓ}, the asymptotic expansions (19) and (20) yield the relation (keeping in mind that the function $c_{0,X}$ is constant)

$$\Delta_{\mathrm{hyp},X} c_{1,X}(z) = \alpha_1 c_{0,X}. \tag{23}$$

In the same way, we derive from (20) the asymptotic expansion

$$\begin{aligned}\Delta_{\mathrm{hyp},X_\ell}\varphi^2_{1,X_\ell}(z) &= \Delta_{\mathrm{hyp},X} c^2_{0,X}(z) + \Delta_{\mathrm{hyp},X}\big(2c_{0,X}(z)c_{1,X}(z)\big)u + O(u^2) \\ &= 2c_{0,X}\Delta_{\mathrm{hyp},X} c_{1,X}(z)u + O(u^2) \quad \text{as} \quad u \to 0. \end{aligned}\tag{24}$$

Using (19), (22), (23), and (24), we arrive at

$$\lim_{\ell\to 0}\left(\frac{\Delta_{\mathrm{hyp},X_\ell}\varphi^2_{1,X_\ell}(z)}{\lambda_{1,X_\ell}}\right) = \lim_{u\to 0}\left(\frac{2c_{0,X}\Delta_{\mathrm{hyp},X}c_{1,X}(z)u + O(u^2)}{\alpha_1 u + O(u^2)}\right) = 2c^2_{X,0} = \frac{2v'}{v(v+v')}.$$

Recalling the formulae

$$v = 2\pi\big(2g - 2 + (p+1)\big) \quad \text{and} \quad v' = 2\pi,$$

we finally compute

$$\begin{aligned}&\frac{2(g+1)+p}{v+v'} + \frac{2v'}{v(v+v')} = \frac{v(v/(2\pi)+3)}{v(v+v')} + \frac{2v'}{v(v+v')} = \\ &\frac{1}{2\pi}\frac{v^2+3vv'+2v'^2}{v(v+v')} = \frac{1}{2\pi}\frac{v+2v'}{v} = \frac{2g+(p+1)}{v}\,,\end{aligned}$$

which completes the proof of claim (18) and hence the proof of the lemma. □

3.2. Remark. We describe here how one can extend the arguments from [18] and references therein to prove formula (16); we continue to use the notation from the proof of Lemma 3.1. The pointwise convergence

$$\lim_{\ell\to 0}\Delta_{\mathrm{hyp},X_\ell}K_{X_\ell}(t;z) = \Delta_{\mathrm{hyp},X}K_X(t;z) \tag{25}$$

follows immediately from [17], Theorem 1.3 (iii). Using the inverse Laplace transform, one concludes from (25) the convergence of small eigenvalues and small eigenfunctions (see, for example, [9] for complete details) to conclude that (16) holds pointwise for all $t > 0$. Theorem 1.3 in [17] states further conditions under which the convergence in (25) is uniform, which immediately implies that the convergence in (16) holds for fixed z and t lying in any bounded, compact subset of $t > 0$, so it remains to prove uniform convergence for t near zero and near infinity. The uniformity of the convergence near zero is established as part of the proof of Theorem 1.3 in [17] since the identity term does not contribute to the realization of the heat kernel through group periodization. What remains is to prove uniformity of the convergence in (16) as t approaches infinity. For this, the method of proof of Lemma 3.2 in [18] applies. More specifically, one writes the function

$$\Delta_{\mathrm{hyp},X_\ell}\left(K_{X_\ell}(t;z) - \frac{1}{v+v'} - \varphi^2_{1,X_\ell}(z)e^{-\lambda_{1,X_\ell}t}\right)$$

as the Laplace transform of a measure as in [18], p. 649. In this case, the measure is not bounded, but standard bounds for the sup-norm of L^2-eigenfunctions of the Laplacian imply that the measure is bounded by a positive measure, which suffices to apply the method of proof of Lemma 3.2 in [18]. With all this, one concludes the pointwise convergence asserted in (16) and integrable, uniform bounds for all $t > 0$, from which (17) follows.

3.3. Proposition. With the above notations, in particular using the form (7) for the function $\phi_{\infty,\infty}(s')$, we have

$$\mu_{\mathrm{can}}(z) = \frac{1}{4\pi g}\mu_{\mathrm{hyp}}(z) + \frac{1}{2g}\int\limits_0^\infty \Delta_{\mathrm{hyp}}K^{\mathrm{cusp}}(t;z)\mathrm{d}t\,\mu_{\mathrm{hyp}}(z) + \frac{1}{g}\sum_{\mu_n<1/y}\frac{2a_n\mu_n y^3}{\sqrt{1-(\mu_n y)^2}}\mu_{\mathrm{hyp}}(z). \quad (26)$$

We point out that the sum in (26) vanishes if $y \gg 0$.

Proof. The proof is based on formula (12) from Lemma 3.1 and consists in substituting the integrand $K(t;z)$ by $K^{\mathrm{cusp}}(t;z)$. We compute

$$\Delta_{\mathrm{hyp}}K(t;z) = \Delta_{\mathrm{hyp}}K^{\mathrm{cusp}}(t;z) - \sum_{0\leq\lambda_j<1/4}|\alpha_{j,0}|^2(2-2s_j)(1-2s_j)y^{2-2s_j}e^{-\lambda_j t} -$$

$$\frac{1}{4\pi}\sum_{P\ \mathrm{cusp}}\int\limits_{-\infty}^{\infty} y^2\frac{\partial^2}{\partial y^2}(\delta_{P,\infty}y + |\phi_{P,\infty}(1/2+ir)|^2 y + \delta_{P,\infty}\phi_{P,\infty}(1/2+ir)y^{1-2ir} +$$

$$\delta_{P,\infty}\overline{\phi}_{P,\infty}(1/2+ir)y^{1+2ir})e^{-(r^2+1/4)t}\mathrm{d}r =$$

$$\Delta_{\mathrm{hyp}}K^{\mathrm{cusp}}(t;z) - \sum_{0\leq\lambda_j<1/4}|\alpha_{j,0}|^2(2-2s_j)(1-2s_j)y^{2-2s_j}e^{-\lambda_j t} -$$

$$\frac{1}{4\pi i}\int\limits_{\mathrm{Re}(s)=1/2}(\phi_{\infty,\infty}(s)(2-2s)(1-2s)y^{2-2s} + \phi_{\infty,\infty}(1-s)2s(2s-1)y^{2s})e^{-s(1-s)t}\mathrm{d}s.$$

Next, we integrate against t to get

$$\int\limits_0^\infty \Delta_{\mathrm{hyp}}K(t;z)\mathrm{d}t = \int\limits_0^\infty \Delta_{\mathrm{hyp}}K^{\mathrm{cusp}}(t;z)\mathrm{d}t - \sum_{0\leq\lambda_j<1/4}|\alpha_{j,0}|^2\frac{(2-2s_j)(1-2s_j)}{\lambda_j}y^{2-2s_j} -$$

$$\frac{1}{4\pi i}\int\limits_{\mathrm{Re}(s)=1/2}(\phi_{\infty,\infty}(s)(2-2s)(1-2s)y^{2-2s} + \phi_{\infty,\infty}(1-s)2s(2s-1)y^{2s})\frac{\mathrm{d}s}{s(1-s)} =$$

$$\int\limits_0^\infty \Delta_{\mathrm{hyp}}K^{\mathrm{cusp}}(t;z)\mathrm{d}t - \sum_{0\leq\lambda_j<1/4}|\alpha_{j,0}|^2\frac{(2-2s_j)(1-2s_j)}{\lambda_j}y^{2-2s_j} -$$

$$\frac{4}{4\pi i}\int\limits_{\mathrm{Re}(s)=1/2}\phi_{\infty,\infty}(s)\frac{1-2s}{s}y^{2-2s}\mathrm{d}s.$$

Now we use the residue theorem to evaluate the last integral (be aware of the orientation).

$$-\frac{4}{4\pi i}\int\limits_{\mathrm{Re}(s)=1/2}\phi_{\infty,\infty}(s)\frac{1-2s}{s}y^{2-2s}\mathrm{d}s =$$

$$-\sum_{\mathrm{residues}\ s_j}(-2)\mathrm{Res}_{s=s_j}(\phi_{\infty,\infty}(s))\frac{1-2s_j}{s_j}y^{2-2s_j} + 2\left(-\frac{1}{2\pi i}\right)\int\limits_{\mathrm{Re}(s)=a}\phi_{\infty,\infty}(s)\frac{1-2s}{s}y^{2-2s}\mathrm{d}s;$$

here $a>1$. It is known that the residues of $\phi_{\infty,\infty}$ occur at $s=1$ with residue $1/\mathrm{vol}_{\mathrm{hyp}}(X)$ and at $s=s_j$ such that $0<\lambda_j=s_j(1-s_j)<1/4$ with residue $|\alpha_{j,0}|^2$ (see [20], p. 652). Therefore, we get

$$-\frac{4}{4\pi i}\int\limits_{\mathrm{Re}(s)=1/2}\phi_{\infty,\infty}(s)\frac{1-2s}{s}y^{2-2s}\mathrm{d}s =$$

$$-\frac{2}{\mathrm{vol}_{\mathrm{hyp}}(X)} + \sum_{0<\lambda_j<1/4}|\alpha_{j,0}|^2\frac{(2-2s_j)(1-2s_j)}{\lambda_j}y^{2-2s_j} + \frac{2}{2\pi i}\int\limits_{\mathrm{Re}(s)=a}\phi_{\infty,\infty}(s)\frac{2s-1}{s}y^{2-2s}\mathrm{d}s.$$

We are left to determine the latter integral. By substituting formula (7) for $\phi_{\infty,\infty}$ and using the functional equation for the Γ-function, we first compute

$$\frac{1}{2\pi i}\int\limits_{\mathrm{Re}(s)=a}\phi_{\infty,\infty}(s)\frac{2s-1}{s}y^{2-2s}\mathrm{d}s=\sum_{n=1}^{\infty}2\sqrt{\pi}a_n y^2\cdot\frac{1}{2\pi i}\int\limits_{\mathrm{Re}(s)=a}\frac{\Gamma(s+1/2)}{\Gamma(s+1)}\left(\frac{1}{(\mu_n y)^2}\right)^s\mathrm{d}s$$
$$=\sum_{n=1}^{\infty}2a_n y^2\cdot\frac{1}{2\pi i}\int\limits_{\mathrm{Re}(s)=a}\sqrt{\pi}\,\frac{\Gamma(s+1/2)}{\Gamma(s+1)}e^{st_n}\mathrm{d}s,$$

where $t_n=-\log\big((\mu_n y)^2\big)$. Recalling formula (10.5) of [19], p. 307, namely

$$\frac{1}{2\pi i}\int\limits_{\mathrm{Re}(s)=a}\sqrt{\pi}\,\frac{\Gamma(s+1/2)}{\Gamma(s+1)}e^{st}\mathrm{d}s=\begin{cases}\dfrac{1}{\sqrt{e^t-1}}, & t>0,\\ 0, & t<0,\end{cases}$$

we obtain

$$\frac{1}{2\pi i}\int\limits_{\mathrm{Re}(s)=a}\phi_{\infty,\infty}(s)\frac{2s-1}{s}y^{2-2s}\mathrm{d}s=\sum_{t_n>0}\frac{2a_n y^2}{\sqrt{e^{t_n}-1}}=\sum_{\mu_n<1/y}\frac{2a_n\mu_n y^3}{\sqrt{1-(\mu_n y)^2}}.$$

Summing up, we get

$$\int\limits_0^{\infty}\Delta_{\mathrm{hyp}}K(t;z)\mathrm{d}t=\int\limits_0^{\infty}\Delta_{\mathrm{hyp}}K^{\mathrm{cusp}}(t;z)\mathrm{d}t-\frac{2}{\mathrm{vol}_{\mathrm{hyp}}(X)}+\sum_{\mu_n<1/y}\frac{4a_n\mu_n y^3}{\sqrt{1-(\mu_n y)^2}}.$$

The claim now follows by observing that

$$\left(1+\frac{p}{2g}\right)\mu_{\mathrm{shyp}}(z)-\frac{1}{2g}\cdot\frac{2}{\mathrm{vol}_{\mathrm{hyp}}(X)}\mu_{\mathrm{hyp}}(z)=\frac{1}{4\pi g}\mu_{\mathrm{hyp}}(z).$$

This completes the proof of the proposition. $\square$

3.4. Remark. By our definition, the partial heat kernel $K_\infty(t;z)$ is given by

$$K_\infty(t;z)=\sum_{n=-\infty}^{\infty}K_{\mathbb{H}}(t;z,z+nb).$$

Recalling the formula for the hyperbolic distance $d_{\mathrm{hyp}}(z,w)$, namely (see [3], p. 130)

$$\cosh\big(d_{\mathrm{hyp}}(z,w)\big)=1+\frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)},$$

which specializes to

$$\cosh\big(d_{\mathrm{hyp}}(z,z+nb)\big)=1+\frac{(nb)^2}{2y^2}\,,$$

shows that the function $K_{\mathbb{H}}(t;z,z+nb)$ is independent of x, and hence can be represented in the form

$$K_{\mathbb{H}}(t;z,z+nb)=f_t\left(\frac{b}{\sqrt{2}y}n\right) \tag{27}$$

with $f_t(w)=K_{\mathbb{H}}(t;\cosh^{-1}(1+w^2))$. Therefore, we can write

$$K_\infty(t;z)=\sum_{n=-\infty}^{\infty}f_t\left(\frac{b}{\sqrt{2}y}n\right). \tag{28}$$

By the general Poisson formula we then have

$$\sum_{n=-\infty}^{\infty} f_t\left(\frac{b}{\sqrt{2}y}n\right) = \frac{\sqrt{2}y}{b}\sum_{n=-\infty}^{\infty} \widehat{f}_t\left(\frac{2\pi\sqrt{2}y}{b}n\right),$$

where $\widehat{f}_t(v)$ denotes the Fourier transform of $f_t(w)$ given by

$$\widehat{f}_t(v) = \int_{-\infty}^{\infty} f_t(w)e^{-iwv}\mathrm{d}w.$$

Summarizing we arrive at

$$K_\infty(t;z) = \frac{\sqrt{2}y}{b}\widehat{f}_t(0) + \frac{2\sqrt{2}y}{b}\sum_{n=1}^{\infty} \widehat{f}_t\left(\frac{2\pi\sqrt{2}y}{b}n\right). \tag{29}$$

3.5. Definition. With the above notations, we set

$$\begin{aligned} K_\infty^{\mathrm{cusp}}(t;z) &= K_\infty(t;z) - \frac{\sqrt{2}y}{b}\widehat{f}_t(0), \\ K_0^{\mathrm{cusp}}(t;z) &= K^{\mathrm{cusp}}(t;z) - K_\infty^{\mathrm{cusp}}(t;z). \end{aligned}$$

3.6. Lemma. For the Fourier transform $\widehat{f}_t$ of f_t, we have the formula

$$\widehat{f}_t(v) = \frac{\sqrt{2}}{\pi^2}\int_0^\infty r\sinh(\pi r)e^{-(r^2+1/4)t}K_{ir}^2(v/\sqrt{2})\mathrm{d}r.$$

Proof. Using the explicit formula for the heat kernel on the upper half-plane (see [5], p. 246), we have

$$K_{\mathbb{H}}(t;z,w) = \frac{1}{2\pi}\int_0^\infty r\tanh(\pi r)e^{-(r^2+1/4)t}P_{-1/2+ir}\big(\cosh(d_{\mathrm{hyp}}(z,w))\big)\mathrm{d}r,$$

from which we get

$$f_t(w) = \frac{1}{2\pi}\int_0^\infty r\tanh(\pi r)e^{-(r^2+1/4)t}P_{-1/2+ir}(1+w^2)\mathrm{d}r. \tag{30}$$

Taking into account that $f_t(w)$ is an even function, the Fourier transform $\widehat{f}_t$ of f_t can be written in the form

$$\widehat{f}_t(v) = \int_{-\infty}^{\infty} f_t(w)e^{-iwv}\mathrm{d}w = 2\int_0^\infty f_t(w)\cos(wv)\mathrm{d}w.$$

By means of formula 7.162 (5) of [7], p. 807, the proof of the lemma can now be easily completed. $\square$

3.7. Lemma. The function $K_\infty^{\mathrm{cusp}}(t;z)$ decays exponentially as y tends to infinity.

Proof. From Lemma 3.6, we note that the function $\widehat{f}_t(v)$ decays exponentially as v tends to infinity. From this we immediately conclude that $K_\infty^{\mathrm{cusp}}(t;z)$ decays exponentially as y tends to infinity. $\square$

3.8. Lemma. With the above notations, we have

$$\int_0^\infty \Delta_{\mathrm{hyp}} K_\infty(t;z)\mathrm{d}t = \frac{1}{2\pi}\left(\frac{2\pi y/b}{\sinh(2\pi y/b)}\right)^2 - \frac{1}{2\pi}. \tag{31}$$

Proof. First, we recall for $z, w \in \mathbb{H}$, $z \neq w$, the relation

$$\int_0^\infty K_{\mathbb{H}}(t;z,w)\mathrm{d}t = -\frac{1}{4\pi}\log\left(\left|\frac{z-w}{z-\overline{w}}\right|^2\right).$$

Substituting $w = \gamma(z)$, summing over $\gamma \in \Gamma_\infty$, $\gamma \neq \mathrm{id}$, and applying Δ_{hyp}, then yields the formula

$$\int_0^\infty \Delta_{\mathrm{hyp}} K_\infty(t;z)\mathrm{d}t = -\frac{1}{4\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \Delta_{\mathrm{hyp}}\log\left(\left|\frac{z-(z+nb)}{z-(\overline{z}+nb)}\right|^2\right) =$$

$$-\frac{2y^2}{\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(nb)^2-4y^2}{((nb)^2+4y^2)^2} = -\frac{2y^2}{\pi b^2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{n^2-(2y/b)^2}{(n^2+(2y/b)^2)^2}.$$

Applying now formula 1.421 (5) of [7], p. 36, namely

$$\sum_{n=-\infty}^{\infty}\frac{n^2-w^2}{(n^2+w^2)^2} = -\left(\frac{\pi}{\sinh(\pi w)}\right)^2,$$

with $w = 2y/b$, immediately completes the proof of the lemma. □

3.9. Theorem. We set

$$\Phi(y) = \left(\frac{2\pi y/b}{\sinh(2\pi y/b)}\right)^2.$$

With the above notations, we then have the fundamental identity

$$\mu_{\mathrm{can}}(z) =$$
$$\frac{1}{2g}\int_0^\infty \Delta_{\mathrm{hyp}} K_0^{\mathrm{cusp}}(t;z)\mathrm{d}t\,\mu_{\mathrm{hyp}}(z) + \frac{1}{4\pi g}\Phi(y)\mu_{\mathrm{hyp}}(z) + \frac{1}{g}\sum_{\mu_n<1/y}\frac{2a_n\mu_n y^3}{\sqrt{1-(\mu_n y)^2}}\mu_{\mathrm{hyp}}(z). \tag{32}$$

Proof. The proof consists in combining Proposition 3.3 with Lemma 3.8 together with the observation that

$$\begin{aligned}\Delta_{\mathrm{hyp}} K^{\mathrm{cusp}}(t;z) &= \Delta_{\mathrm{hyp}}\big(K_0^{\mathrm{cusp}}(t;z) + K_\infty^{\mathrm{cusp}}(t;z)\big)\\ &= \Delta_{\mathrm{hyp}}\big(K_0^{\mathrm{cusp}}(t;z) + K_\infty(t;z)\big),\end{aligned}$$

since $\Delta_{\mathrm{hyp}}\big(y\widehat{f}_t(0)\big) = 0$. □

4 Preliminary computations

We will multiply the fundamental identity (32) of Theorem 3.9 with the function

$$h(s,y) = \frac{2g}{b}\pi^{-s}\Gamma(s)\zeta(2s)y^s \tag{33}$$

and integrate the resulting form along x and y. In this section we first calculate the integrals involving the form μ_{can}, the function Φ, and the sum over the μ_n's, respectively. In the second part of the section we treat the term involving K_0^{cusp} partly; this computation will be completed in the next section.

4.1. Lemma. With the above notations, we have

$$\int_0^\infty \int_0^b h(s,y)\mu_{\mathrm{can}}(z) = \sum_{j=1}^{g} \widetilde{L}(s, f_j \otimes \overline{f}_j). \tag{34}$$

Proof. The proof is elementary, so we omit further details. □

4.2. Lemma. With the above notations, we have

$$\frac{1}{4\pi g}\int_0^\infty \int_0^b h(s,y)\Phi(y)\mu_{\mathrm{hyp}}(z) = 4\pi\zeta(s)b^{s-1}G_\infty(s). \tag{35}$$

Proof. We start with the following observation. By differentiating the relation

$$\frac{1}{1-e^{-2w}} = \sum_{n=0}^{\infty} e^{-2nw}$$

we get

$$\frac{e^{-2w}}{(1-e^{-2w})^2} = \sum_{n=1}^{\infty} ne^{-2nw},$$

which gives

$$\frac{1}{\sinh^2(w)} = \frac{4}{(e^w - e^{-w})^2} = \frac{4e^{-2w}}{(1-e^{-2w})^2} = 4\sum_{n=1}^{\infty} ne^{-2nw}.$$

We now turn to the proof of the lemma. We compute

$$\begin{aligned}
&\frac{1}{4\pi g}\int_0^\infty \int_0^b h(s,y)\Phi(y)\mu_{\mathrm{hyp}}(z) = \frac{\pi^{-s}\Gamma(s)\zeta(2s)}{2\pi}\int_0^\infty y^s\Phi(y)\frac{\mathrm{d}y}{y^2} = \\
&\frac{\pi^{-s}\Gamma(s)\zeta(2s)}{2\pi}\cdot\frac{(2\pi)^2}{b^2}\int_0^\infty \frac{y^s}{\sinh^2(2\pi y/b)}\mathrm{d}y = \\
&\frac{\pi^{-s}\Gamma(s)\zeta(2s)}{2\pi}\cdot\frac{(2\pi)^2}{b^2}\int_0^\infty 4y^{s+1}\sum_{n=1}^{\infty} ne^{-4\pi ny/b}\frac{\mathrm{d}y}{y} = \\
&2^3\pi^{-s+1}\Gamma(s)\zeta(2s)b^{-2}\sum_{n=1}^{\infty} n\int_0^\infty y^{s+1}e^{-4\pi ny/b}\frac{\mathrm{d}y}{y} = \\
&2^3\pi^{-s+1}\Gamma(s)\zeta(2s)b^{-2}\Gamma(s+1)\sum_{n=1}^{\infty}\frac{n}{(4\pi n/b)^{(s+1)}} = \\
&2^{-2s+1}\pi^{-2s}\Gamma(s)\Gamma(s+1)\zeta(s)\zeta(2s)b^{s-1}.
\end{aligned}$$

The claim now follows using the definition of the function $G_\infty(s)$. □

4.3. Lemma. With the above notations, we have

$$\frac{1}{g}\int_0^\infty\int_0^b h(s,y)\sum_{\mu_n<1/y}\frac{a_n\mu_n y}{\sqrt{1-(\mu_n y)^2}}\mathrm{d}x\mathrm{d}y=\pi^{-s}\frac{s}{s+1}\Gamma(s)\zeta(2s)\phi_{\infty,\infty}\left(\frac{s+1}{2}\right). \tag{36}$$

Proof. Using the B-function, we compute

$$\frac{1}{g}\int_0^\infty\int_0^b h(s,y)\sum_{\mu_n<1/y}\frac{a_n\mu_n y}{\sqrt{1-(\mu_n y)^2}}\mathrm{d}x\mathrm{d}y=2\pi^{-s}\Gamma(s)\zeta(2s)\int_0^\infty\sum_{y<1/\mu_n}\frac{a_n\mu_n y^{s+1}}{\sqrt{1-(\mu_n y)^2}}\mathrm{d}y=$$

$$2\pi^{-s}\Gamma(s)\zeta(2s)\sum_{n=1}^\infty\frac{a_n}{\mu_n^{s+1}}\int_0^1\frac{w^{s+1}}{\sqrt{1-w^2}}\mathrm{d}w=\pi^{-s}\Gamma(s)\zeta(2s)B\left(\frac{s}{2}+1,\frac{1}{2}\right)\sum_{n=1}^\infty\frac{a_n}{\mu_n^{s+1}}=$$

$$\pi^{-s}\Gamma(s)\zeta(2s)\frac{\Gamma(s/2+1)\Gamma(1/2)}{\Gamma\big((s+3)/2\big)}\sum_{n=1}^\infty\frac{a_n}{\mu_n^{2\frac{s+1}{2}}}=$$

$$\pi^{-s}\Gamma(s)\zeta(2s)\frac{s/2\,\Gamma\big((s+1)/2-1/2\big)\sqrt{\pi}}{(s+1)/2\,\Gamma\big((s+1)/2\big)}\sum_{n=1}^\infty\frac{a_n}{\mu_n^{2\frac{s+1}{2}}}=$$

$$\pi^{-s}\frac{s}{s+1}\Gamma(s)\zeta(2s)\phi_{\infty,\infty}\left(\frac{s+1}{2}\right).$$

□

4.4. Remark. For $\varepsilon>0$, we can write

$$\frac{1}{2g}\int_0^\infty\int_0^b\int_0^\infty h(s,y)\Delta_{\mathrm{hyp}}K_0^{\mathrm{cusp}}(t;z)\mathrm{d}t\mathrm{d}x\frac{\mathrm{d}y}{y^2}=\frac{1}{2g}\int_\varepsilon^\infty\int_0^\infty\int_0^b h(s,y)\Delta_{\mathrm{hyp}}K_0^{\mathrm{cusp}}(t;z)\mathrm{d}x\frac{\mathrm{d}y}{y^2}\mathrm{d}t+o(1)$$

as $\varepsilon\to0$. Using now the specific form of the hyperbolic Laplacian, we integrate by parts in each real variable x and y. Since the integrand is invariant under $x\mapsto x+b$, the terms involving derivatives with respect to x will vanish. What remains to be done is the integration by parts with respect to y. Substituting

$$K_0^{\mathrm{cusp}}(t;z)=K^{\mathrm{cusp}}(t;z)-K_\infty^{\mathrm{cusp}}(t;z),$$

we arrive in this way at the formula

$$\frac{1}{2g}\int_0^\infty\int_0^b\int_0^\infty h(s,y)\Delta_{\mathrm{hyp}}K_0^{\mathrm{cusp}}(t;z)\mathrm{d}t\mathrm{d}x\frac{\mathrm{d}y}{y^2}=$$

$$\frac{s(1-s)}{2g}\lim_{\varepsilon\to0}\left[\int_\varepsilon^\infty\int_0^\infty\int_0^b h(s,y)K^{\mathrm{cusp}}(t;z)\mathrm{d}x\frac{\mathrm{d}y}{y^2}\mathrm{d}t-\int_\varepsilon^\infty\int_0^\infty\int_0^b h(s,y)K_\infty^{\mathrm{cusp}}(t;z)\mathrm{d}x\frac{\mathrm{d}y}{y^2}\mathrm{d}t\right]. \tag{37}$$

We point out that for the right-hand side of formula (37) the individual triple integrals over $h(s,y)K^{\mathrm{cusp}}(t;z)$ and $h(s,y)K_\infty^{\mathrm{cusp}}(t;z)$ do not exist for $\varepsilon=0$, which justifies the need to introduce the parameter ε. For further discussion of this point, see also Proposition 5.5 below.

4.5. Lemma. With the above notations, we have

$$\frac{1}{2g}\int_0^\infty\int_0^b h(s,y)\left(|\varphi_j(z)|^2-|\alpha_{j,0}(y)|^2\right)\mathrm{d}x\frac{\mathrm{d}y}{y^2}=\frac{\cosh(\pi r_j)}{2s(1-s)}\widetilde{L}(s,\varphi_j\otimes\overline{\varphi}_j). \tag{38}$$

Proof. We compute

$$\frac{1}{2g}\int_0^{\infty}\int_0^{b} h(s,y)\left(|\varphi_j(z)|^2-|\alpha_{j,0}(y)|^2\right)\mathrm{d}x\frac{\mathrm{d}y}{y^2}=$$

$$\frac{\pi^{-s}}{b}\Gamma(s)\zeta(2s)\int_0^{\infty}\int_0^{b} y^{s-2}\left(|\varphi_j(z)|^2-|\alpha_{j,0}(y)|^2\right)\mathrm{d}x\mathrm{d}y=$$

$$\frac{\pi^{-s}}{b}\Gamma(s)\zeta(2s)\int_0^{\infty}\int_0^{b} y^{s-2}\left[\sum_{n,m\neq 0}\alpha_{j,n}\overline{\alpha}_{j,m}W_{s_j}\left(\frac{nz}{b}\right)\overline{W}_{s_j}\left(\frac{mz}{b}\right)+\right.$$

$$\left.\overline{\alpha}_{j,0}(y)\sum_{n\neq 0}\alpha_{j,n}W_{s_j}\left(\frac{nz}{b}\right)+\alpha_{j,0}(y)\sum_{m\neq 0}\overline{\alpha}_{j,m}\overline{W}_{s_j}\left(\frac{mz}{b}\right)\right]\mathrm{d}x\mathrm{d}y=$$

$$\pi^{-s}\Gamma(s)\zeta(2s)\int_0^{\infty} y^{s-2}\sum_{n\neq 0}|\alpha_{j,n}|^2\left|W_{s_j}\left(\frac{nz}{b}\right)\right|^2\mathrm{d}y=$$

$$\pi^{-s}\Gamma(s)\zeta(2s)\cosh(\pi r_j)\int_0^{\infty} y^{s-2}\sum_{n\neq 0}|\alpha_{j,n}|^2\left(\frac{4|n|y}{b}\right)K_{ir_j}^2\left(\frac{2\pi|n|y}{b}\right)\mathrm{d}y=$$

$$4\pi^{-s}\Gamma(s)\zeta(2s)\cosh(\pi r_j)\sum_{n\neq 0}|\alpha_{j,n}|^2\left(\frac{|n|}{b}\right)\int_0^{\infty} y^{s}K_{ir_j}\left(\frac{2\pi|n|y}{b}\right)K_{ir_j}\left(\frac{2\pi|n|y}{b}\right)\frac{\mathrm{d}y}{y}.$$

With the change of variables

$$u=\frac{2\pi|n|y}{b},$$

we then obtain (see [11], p. 205)

$$\frac{1}{2g}\int_0^{\infty}\int_0^{b} h(s,y)\left(|\varphi_j(z)|^2-|\alpha_{j,0}(y)|^2\right)\mathrm{d}x\frac{\mathrm{d}y}{y^2}=$$

$$4\pi^{-s}\Gamma(s)\zeta(2s)\cosh(\pi r_j)\sum_{n\neq 0}|\alpha_{j,n}|^2\left(\frac{|n|}{b}\right)\int_0^{\infty} u^{s}K_{ir_j}(u)K_{ir_j}(u)\left(\frac{2\pi|n|}{b}\right)^{-s}\frac{\mathrm{d}u}{u}=$$

$$4\pi^{-s}(2\pi)^{-s}\Gamma(s)\zeta(2s)\cosh(\pi r_j)\sum_{n\neq 0}|\alpha_{j,n}|^2\left(\frac{|n|}{b}\right)\left(\frac{|n|}{b}\right)^{-s}\int_0^{\infty} u^{s}K_{ir_j}(u)K_{ir_j}(u)\frac{\mathrm{d}u}{u}=$$

$$4\pi^{-s}(2\pi)^{-s}\Gamma(s)\zeta(2s)\cosh(\pi r_j)\sum_{n\neq 0}|\alpha_{j,n}|^2\left(\frac{|n|}{b}\right)\left(\frac{|n|}{b}\right)^{-s}\times$$

$$\times\frac{2^{s-3}}{\Gamma(s)}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+ir_j\right)\Gamma\left(\frac{s}{2}-ir_j\right)=$$

$$\frac{2^{2-s+s-3}G_{r_j}(s)\cosh(\pi r_j)}{s(1-s)}\sum_{n\neq 0}\frac{|\alpha_{j,n}|^2}{(|n|/b)^{s-1}}=\frac{\cosh(\pi r_j)}{2s(1-s)}\widetilde{L}(s,\varphi_j\otimes\overline{\varphi}_j).$$

$\square$

4.6. Lemma. With the above notations, we have

$$\frac{1}{2g}\int_0^{\infty}\int_0^{b} h(s,y)\left(|E_{P,1/2+ir}(z)|^2-|\alpha_{P,1/2+ir,0}(y)|^2\right)\mathrm{d}x\frac{\mathrm{d}y}{y^2}=\frac{\cosh(\pi r)}{2s(1-s)}\widetilde{L}(s,E_{P,1/2+ir}\otimes\overline{E}_{P,1/2-ir}).$$

Proof. The proof runs along the same lines as the proof of Lemma 4.5. □

4.7. Proposition. With the above notations, we have for any $\varepsilon > 0$

$$\frac{s(1-s)}{2g}\int\limits_{\varepsilon}^{\infty}\int\limits_{0}^{\infty}\int\limits_{0}^{b} h(s,y)K^{\mathrm{cusp}}(t;z)\mathrm{d}x\frac{\mathrm{d}y}{y^2}\mathrm{d}t = \sum_{\lambda_j>0}\frac{\cosh(\pi r_j)e^{-\lambda_j\varepsilon}}{2\lambda_j}\widetilde{L}(s,\varphi_j\otimes\overline{\varphi}_j)$$
$$+\frac{1}{8\pi}\sum_{P\text{ cusp}}\int\limits_{-\infty}^{\infty}\frac{\cosh(\pi r)e^{-(r^2+1/4)\varepsilon}}{r^2+1/4}\widetilde{L}(s,E_{P,1/2+ir}\otimes\overline{E}_{P,1/2-ir})\mathrm{d}r. \tag{39}$$

Proof. Recall that

$K^{\mathrm{cusp}}(t;z) =$

$$K(t;z) - \sum_{0\le\lambda_j<1/4}|\alpha_{j,0}(y)|^2e^{-\lambda_j t} - \frac{1}{4\pi}\sum_{P\text{ cusp}}\int\limits_{-\infty}^{\infty}|\alpha_{P,1/2+ir,0}(y)|^2e^{-(r^2+1/4)t}\mathrm{d}r =$$

$$\sum_{\lambda_j>0}\left(|\varphi_j(z)|^2-|\alpha_{j,0}(y)|^2\right)e^{-\lambda_j t}+\frac{1}{4\pi}\sum_{P\text{ cusp}}\int\limits_{-\infty}^{\infty}\left(|E_{P,1/2+ir}(z)|^2-|\alpha_{P,1/2+ir,0}(y)|^2\right)e^{-(r^2+1/4)t}\mathrm{d}r. \tag{40}$$

By multiplying (38) by $e^{-\lambda_j t}$, adding over all positive eigenvalues λ_j, and integrating along t from ε to ∞, we get

$$\frac{1}{2g}\sum_{\lambda_j>0}\int\limits_{\varepsilon}^{\infty}\int\limits_{0}^{\infty}\int\limits_{0}^{b}h(s,y)\left(|\varphi_j(z)|^2-|\alpha_{j,0}(y)|^2\right)e^{-\lambda_j t}\mathrm{d}x\frac{\mathrm{d}y}{y^2}\mathrm{d}t =$$
$$\sum_{\lambda_j>0}\int\limits_{\varepsilon}^{\infty}\frac{\cosh(\pi r_j)}{2s(1-s)}\widetilde{L}(s,\varphi_j\otimes\overline{\varphi}_j)e^{-\lambda_j t}\mathrm{d}t = \frac{1}{s(1-s)}\sum_{\lambda_j>0}\frac{\cosh(\pi r_j)e^{-\lambda_j\varepsilon}}{2\lambda_j}\widetilde{L}(s,\varphi_j\otimes\overline{\varphi}_j). \tag{41}$$

Using Lemma 4.6, we analogously find

$$\frac{1}{4\pi}\frac{1}{2g}\sum_{P\text{ cusp}}\int\limits_{-\infty}^{\infty}\int\limits_{\varepsilon}^{\infty}\int\limits_{0}^{\infty}\int\limits_{0}^{b}h(s,y)\left(|E_{P,1/2+ir}(z)|^2-|\alpha_{P,1/2+ir,0}(y)|^2\right)e^{-(r^2+1/4)t}\mathrm{d}x\frac{\mathrm{d}y}{y^2}\mathrm{d}t\mathrm{d}r =$$
$$\frac{1}{4\pi}\sum_{P\text{ cusp}}\int\limits_{-\infty}^{\infty}\int\limits_{\varepsilon}^{\infty}\frac{\cosh(\pi r)}{2s(1-s)}\widetilde{L}(s,E_{P,1/2+ir}\otimes\overline{E}_{P,1/2-ir})e^{-(r^2+1/4)t}\mathrm{d}t\mathrm{d}r =$$
$$\frac{1}{8\pi}\frac{1}{s(1-s)}\sum_{P\text{ cusp}}\int\limits_{-\infty}^{\infty}\frac{\cosh(\pi r)e^{-(r^2+1/4)\varepsilon}}{r^2+1/4}\widetilde{L}(s,E_{P,1/2+ir}\otimes\overline{E}_{P,1/2-ir})\mathrm{d}r. \tag{42}$$

By combining (41) and (42) with (40), and multiplying by $s(1-s)$, we complete the proof of the proposition. □

5 The L-function relation

As stated before, our computations amount to computing the integral of the identity in Theorem 3.9 when multiplied by $h(s,y)$. As stated in Remark 4.4, we write

$$K_0^{\mathrm{cusp}}(t;z) = K^{\mathrm{cusp}}(t;z) - K_\infty^{\mathrm{cusp}}(t;z).$$

The computations in the previous section allow us to compute the integral involving the term $K^{\mathrm{cusp}}(t;z)$. In this section, we begin by computing the integral involving $K_{\infty}^{\mathrm{cusp}}(t;z)$, after which we complete the proof of our main theorem, which we state in Theorem 5.4. To conclude this section, we show the necessity of introducing the parameter $\varepsilon > 0$, as stated in Remark 4.4, by computing the asymptotic behavior of the integral arising from the $K_{\infty}^{\mathrm{cusp}}(t;z)$-term from Theorem 3.9. This computation is given in Proposition 5.5.

5.1. Lemma. With the above notations, we have

$$\frac{1}{2g}\int_0^{\infty}\int_0^{b} h(s,y)K_{\infty}^{\mathrm{cusp}}(t;z)\mathrm{d}x\frac{\mathrm{d}y}{y^2} = 2^{3/2(-s+1)}\pi^{-2s}\Gamma(s)\zeta(s)\zeta(2s)b^{s-1}\mathcal{M}(\widehat{f}_t)(s), \tag{43}$$

where $\mathcal{M}(\widehat{f}_t)$ is the Mellin transform of the function $\widehat{f}_t$ defined in Remark 3.4 given by

$$\mathcal{M}(\widehat{f}_t)(s) = \int_0^{\infty} v^s\widehat{f}_t(v)\frac{\mathrm{d}v}{v}. \tag{44}$$

Proof. By Remark 3.4 and Definition 3.5, we have

$$\begin{aligned}
\frac{1}{2g}\int_0^{\infty}\int_0^{b} h(s,y)K_{\infty}^{\mathrm{cusp}}(t;z)\mathrm{d}x\frac{\mathrm{d}y}{y^2} &= \frac{1}{2g}\int_0^{\infty}\int_0^{b} h(s,y)\frac{2\sqrt{2}y}{b}\sum_{n=1}^{\infty}\widehat{f}_t\left(\frac{2\pi\sqrt{2}y}{b}n\right)\mathrm{d}x\frac{\mathrm{d}y}{y^2} \\
&= 2\sqrt{2}\pi^{-s}\Gamma(s)\zeta(2s)b^{-1}\int_0^{\infty} y^s\sum_{n=1}^{\infty}\widehat{f}_t\left(\frac{2\pi\sqrt{2}y}{b}n\right)\frac{\mathrm{d}y}{y} \\
&= 2\sqrt{2}\pi^{-s}\Gamma(s)\zeta(2s)b^{-1}\sum_{n=1}^{\infty}\int_0^{\infty} y^s\widehat{f}_t\left(\frac{2\pi\sqrt{2}y}{b}n\right)\frac{\mathrm{d}y}{y}.
\end{aligned}$$

By the change of variables

$$v = \frac{2\pi\sqrt{2}y}{b}n,$$

we find

$$\begin{aligned}
\frac{1}{2g}\int_0^{\infty}\int_0^{b} h(s,y)K_{\infty}^{\mathrm{cusp}}(t;z)\mathrm{d}x\frac{\mathrm{d}y}{y^2} &= 2\sqrt{2}\pi^{-s}\Gamma(s)\zeta(2s)b^{-1}\sum_{n=1}^{\infty}\frac{b^s}{(2\pi\sqrt{2}n)^s}\int_0^{\infty} v^s\widehat{f}_t(v)\frac{\mathrm{d}v}{v} \\
&= 2^{3/2(-s+1)}\pi^{-2s}\Gamma(s)\zeta(s)\zeta(2s)b^{s-1}\mathcal{M}(\widehat{f}_t)(s),
\end{aligned}$$

which completes the proof. $\square$

5.2. Lemma. The Mellin transform $\mathcal{M}(\widehat{f}_t)$ of the function $\widehat{f}_t$ is given by

$$\mathcal{M}(\widehat{f}_t)(s) = \frac{2^{3s/2-5/2}}{\pi^2}\frac{\Gamma^2(s/2)}{\Gamma(s)}\int_0^{\infty} r\sinh(\pi r)e^{-(r^2+1/4)t}\Gamma(s/2+ir)\Gamma(s/2-ir)\mathrm{d}r.$$

Proof. By Lemma 3.6, we have

$$\widehat{f}_t(v) = \frac{\sqrt{2}}{\pi^2}\int_0^{\infty} r\sinh(\pi r)e^{-(r^2+1/4)t}K_{ir}^2(v/\sqrt{2})\mathrm{d}r.$$

This gives

$$\begin{aligned}\mathcal{M}(\widehat{f}_t)(s) &= \int\limits_0^\infty v^s \widehat{f}_t(v)\frac{\mathrm{d}v}{v} \\ &= \int\limits_0^\infty v^s \left(\frac{\sqrt{2}}{\pi^2}\int\limits_0^\infty r\sinh(\pi r)e^{-(r^2+1/4)t}K_{ir}^2(v/\sqrt{2})\mathrm{d}r\right)\frac{\mathrm{d}v}{v} \\ &= \frac{\sqrt{2}}{\pi^2}\int\limits_0^\infty r\sinh(\pi r)e^{-(r^2+1/4)t}\left(\int\limits_0^\infty v^s K_{ir}^2(v/\sqrt{2})\frac{\mathrm{d}v}{v}\right)\mathrm{d}r.\end{aligned}$$

From [11], p. 205, we find

$$\int\limits_0^\infty v^s K_{ir}^2(v/\sqrt{2})\frac{\mathrm{d}v}{v} = 2^{3s/2-3}\Gamma^2(s/2)\frac{\Gamma(s/2+ir)\Gamma(s/2-ir)}{\Gamma(s)}\,.$$

Summing up, we get

$$\mathcal{M}(\widehat{f}_t)(s) = \frac{2^{3s/2-5/2}}{\pi^2}\frac{\Gamma^2(s/2)}{\Gamma(s)}\int\limits_0^\infty r\sinh(\pi r)e^{-(r^2+1/4)t}\Gamma(s/2+ir)\Gamma(s/2-ir)\mathrm{d}r,$$

which is the claimed formula. $\square$

5.3. Proposition. With the above notations, we have for any $\varepsilon > 0$

$$\frac{s(1-s)}{2g}\int\limits_\varepsilon^\infty\int\limits_0^\infty\int\limits_0^b h(s,y)K_\infty^{\mathrm{cusp}}(t;z)\mathrm{d}x\frac{\mathrm{d}y}{y^2}\mathrm{d}t = \frac{\zeta(s)b^{s-1}}{2\pi^2}\int\limits_0^\infty\frac{r\sinh(\pi r)e^{-(r^2+1/4)\varepsilon}}{r^2+1/4}G_r(s)\mathrm{d}r.$$

Proof. Using Lemma 5.1, we compute for the inner double integral

$$\frac{1}{2g}\int\limits_0^\infty\int\limits_0^b h(s,y)K_\infty^{\mathrm{cusp}}(t;z)\mathrm{d}x\frac{\mathrm{d}y}{y^2} = 2^{3/2(-s+1)}\pi^{-2s}\Gamma(s)\zeta(s)\zeta(2s)b^{s-1}\mathcal{M}(\widehat{f}_t)(s) =$$

$$2^{-1}\pi^{-2}\pi^{-2s}\Gamma^2(s/2)\zeta(s)\zeta(2s)b^{s-1}\int\limits_0^\infty r\sinh(\pi r)e^{-(r^2+1/4)t}\Gamma(s/2+ir)\Gamma(s/2-ir)\mathrm{d}r.$$

The claim now follows using the definition of the function $G_r(s)$ and integrating along t from ε to ∞. $\square$

5.4. Theorem. With the above notations, we define for any $\varepsilon > 0$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the Θ-function

$$\begin{aligned}\Theta_\varepsilon(s) = &\sum_{\lambda_j>0}^\infty\frac{\cosh(\pi r_j)e^{-\lambda_j\varepsilon}}{2\lambda_j}\widetilde{L}(s,\varphi_j\otimes\overline{\varphi}_j) \\ &+\frac{1}{8\pi}\sum_{P\ \mathrm{cusp}}\int\limits_{-\infty}^\infty\frac{\cosh(\pi r)e^{-(r^2+1/4)\varepsilon}}{r^2+1/4}\widetilde{L}(s,E_{P,1/2+ir}\otimes\overline{E}_{P,1/2+ir})\mathrm{d}r\end{aligned}$$

and the universal function

$$F_\varepsilon(s) = \frac{\zeta(s)b^{s-1}}{2\pi^2}\int\limits_0^\infty\frac{r\sinh(\pi r)e^{-(r^2+1/4)\varepsilon}}{r^2+1/4}G_r(s)\mathrm{d}r. \tag{45}$$

Then, we have the relation

$$\lim_{\varepsilon\to 0}\big(\Theta_\varepsilon(s)-F_\varepsilon(s)\big) = \\ \sum_{j=1}^{g}\widetilde{L}(s,f_j\otimes\overline{f}_j)-4\pi\zeta(s)b^{s-1}G_\infty(s)-\pi^{-s}\frac{2s}{s+1}\Gamma(s)\zeta(2s)\phi_{\infty,\infty}\left(\frac{s+1}{2}\right).$$

Proof. The proof follows immediately from Lemma 4.1, Lemma 4.2, and Lemma 4.3, as well as Proposition 4.7 and Proposition 5.3 in conjunction with Remark 4.4. □

5.5. Proposition. With the above notations, we have the following asymptotics for the universal function (45) for $s\in\mathbb{R}$, $s>1$,

$$\lim_{\varepsilon\to 0}\left(\varepsilon^{\frac{s-1}{2}}F_\varepsilon(s)\right)=\frac{\zeta(s)b^{s-1}}{4\pi}\frac{G_{i/2}(s)}{\Gamma(s/2+1/2)}\,.$$

Proof. Substituting $v=\sqrt{\varepsilon}r$, we get

$$F_\varepsilon(s)=\frac{\zeta(s)b^{s-1}}{2\pi^2}e^{-\varepsilon/4}\int_0^\infty\frac{v\sinh(\pi v/\sqrt{\varepsilon})e^{-v^2}}{v^2+\varepsilon/4}G_{v/\sqrt{\varepsilon}}(s)\mathrm{d}v.$$

Now, recall the formula

$$\lim_{\varepsilon\to 0}\left(e^{-\frac{\pi v}{\sqrt{\varepsilon}}}\sinh\left(\frac{\pi v}{\sqrt{\varepsilon}}\right)\right)=\frac{1}{2}\,, \tag{46}$$

and, using Stirling's formula, the asymptotics

$$\lim_{|y|\to\infty}\left(|\Gamma(x+iy)|e^{\frac{\pi|y|}{2}}|y|^{\frac{1}{2}-x}\right)=\sqrt{2\pi} \tag{47}$$

for fixed $x\in\mathbb{R}$ (see formula (6) of [6], p. 47). Writing

$$\sinh\left(\frac{\pi v}{\sqrt{\varepsilon}}\right)\left|\Gamma\left(\frac{s}{2}+i\frac{v}{\sqrt{\varepsilon}}\right)\right|\left|\Gamma\left(\frac{s}{2}-i\frac{v}{\sqrt{\varepsilon}}\right)\right|\left(\frac{v}{\sqrt{\varepsilon}}\right)^{1-s}= \\ e^{-\frac{\pi v}{\sqrt{\varepsilon}}}\sinh\left(\frac{\pi v}{\sqrt{\varepsilon}}\right)\left|\Gamma\left(\frac{s}{2}+i\frac{v}{\sqrt{\varepsilon}}\right)\right|e^{\frac{\pi v}{2\sqrt{\varepsilon}}}\left(\frac{v}{\sqrt{\varepsilon}}\right)^{\frac{1-s}{2}}\left|\Gamma\left(\frac{s}{2}-i\frac{v}{\sqrt{\varepsilon}}\right)\right|e^{\frac{\pi v}{2\sqrt{\varepsilon}}}\left(\frac{v}{\sqrt{\varepsilon}}\right)^{\frac{1-s}{2}},$$

we obtain, using (46) and (47),

$$\lim_{\varepsilon\to 0}\left(\varepsilon^{\frac{s-1}{2}}\sinh\left(\frac{\pi v}{\sqrt{\varepsilon}}\right)\left|\Gamma\left(\frac{s}{2}+i\frac{v}{\sqrt{\varepsilon}}\right)\right|\left|\Gamma\left(\frac{s}{2}-i\frac{v}{\sqrt{\varepsilon}}\right)\right|\right)=\pi v^{s-1}. \tag{48}$$

We have

$$G_{v/\sqrt{\varepsilon}}(s)=H(s)\Gamma\left(\frac{s}{2}+i\frac{v}{\sqrt{\varepsilon}}\right)\Gamma\left(\frac{s}{2}-i\frac{v}{\sqrt{\varepsilon}}\right)=H(s)\left|\Gamma\left(\frac{s}{2}+i\frac{v}{\sqrt{\varepsilon}}\right)\right|\left|\Gamma\left(\frac{s}{2}-i\frac{v}{\sqrt{\varepsilon}}\right)\right|$$

with

$$H(s)=s(1-s)\pi^{-2s}\Gamma^2\left(\frac{s}{2}\right)\zeta(2s).$$

From (48), we then find

$$\lim_{\varepsilon\to 0}\left(\varepsilon^{\frac{s-1}{2}}\sinh\left(\frac{\pi v}{\sqrt{\varepsilon}}\right)G_{v/\sqrt{\varepsilon}}(s)\right)=\pi v^{s-1}H(s),$$

from which we derive

$$\lim_{\varepsilon\to 0}\left(\varepsilon^{\frac{s-1}{2}}F_\varepsilon(s)\right)=\frac{\zeta(s)b^{s-1}}{2\pi^2}\lim_{\varepsilon\to 0}\left(\varepsilon^{\frac{s-1}{2}}e^{-\varepsilon/4}\int\limits_0^\infty\frac{v\sinh(\pi v/\sqrt{\varepsilon})e^{-v^2}}{v^2+\varepsilon/4}G_{v/\sqrt{\varepsilon}}(s)\mathrm{d}v\right)=$$

$$\frac{\zeta(s)b^{s-1}}{2\pi^2}\int\limits_0^\infty\lim_{\varepsilon\to 0}\left(\varepsilon^{\frac{s-1}{2}}\sinh\left(\frac{\pi v}{\sqrt{\varepsilon}}\right)G_{v/\sqrt{\varepsilon}}(s)\right)e^{-v^2}\frac{\mathrm{d}v}{v}=\frac{H(s)\zeta(s)b^{s-1}}{2\pi}\int\limits_0^\infty e^{-v^2}v^{s-1}\frac{\mathrm{d}v}{v}.$$

Using the substitution $w=v^2$, the remaining integral simplifies to

$$\int\limits_0^\infty e^{-v^2}v^{s-1}\frac{\mathrm{d}v}{v}=\frac{1}{2}\int\limits_0^\infty e^{-w}v^{\frac{s-1}{2}}\frac{\mathrm{d}w}{w}=\frac{1}{2}\Gamma\left(\frac{s}{2}-\frac{1}{2}\right).$$

Summing up, we get

$$\lim_{\varepsilon\to 0}\left(\varepsilon^{\frac{s-1}{2}}F_\varepsilon(s)\right)=\frac{\zeta(s)b^{s-1}}{4\pi}\frac{G_{i/2}(s)}{\Gamma(s/2+1/2)},$$

which is the claimed formula. □

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