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Bounds on canonical Green's functions

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Abstract

A fundamental object in the theory of arithmetic surfaces is the Green's function associated to the canonical metric. Previous expressions for the canonical Green's function have relied on general functional analysis or, when using specific properties of the canonical metric, the classical Riemann theta function. In this article, we derive a new identity for the canonical Green's function involving the hyperbolic heat kernel. As an application of our results, we obtain bounds for the canonical Green's function through covers and for families of modular curves.

1. Introduction

1.1

In [Ara74], Arakelov defined an intersection theory for divisors on arithmetic surfaces by including a contribution at infinity, which is computed using certain Green's functions defined on the corresponding Riemann surfaces. Arakelov's theory has been extended to higher dimensions, primarily through the work of H. Gillet, C. Soulé, and G. Faltings. Motivated by the recent work of B. Edixhoven, which will be explained below, we derive here several analytic relations and estimates for the Green's functions used by Arakelov.

More specifically, let X be a compact Riemann surface of genus $g_X > 1$. The canonical volume form μ_{can} on X is the positive $(1,1)$-form obtained by the pull-back of the standard Euclidean volume form on the Jacobian variety $\mathrm{Jac}(X)$ associated to X via the classical Abel–Jacobi map. The canonical Green's function $g_{\mathrm{can}}(z,w)$, also written as $g_{\mathrm{can},X}(z,w)$, is the function of $z,w \in X$, which is uniquely characterized by the differential equation

$$d_z d_z^c g_{\mathrm{can}}(z,w) + \delta_w(z) = \mu_{\mathrm{can}}(z) \quad (z,w \in X),$$

where $\delta_w(z)$ is the usual Dirac delta distribution, and the normalization condition

$$\int_X g_{\mathrm{can}}(z,w)\,\mu_{\mathrm{can}}(z) = 0 \quad (w \in X).$$

The fundamental properties of the canonical Green's function, such as existence and symmetry, follow from general functional analysis. By identifying the points $z,w \in X$ with their pre-images in the universal cover, which we take to be the hyperbolic upper half-plane $\mathbb{H}$, we have that the function

$$g_{\mathrm{can}}(z,w) + \log|z-w|^2$$

is bounded and continuous as z approaches w.

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The results we present here involve a development of bounds for the canonical Green's function after removing its logarithmic singularity. In effect, we obtain three types of bound. First, we study the setting of a fixed compact hyperbolic Riemann surface X, ultimately deriving a sup-norm bound involving quantities associated to the hyperbolic spectral theory and hyperbolic geometry on X. Second, we investigate these bounds in the relative situation, when X is a finite-degree cover of a fixed compact hyperbolic Riemann surface. Third, we consider these bounds for families of hyperbolic modular curves, meaning the sequences of modular curves $\{X_0(N)\}$, $\{X_1(N)\}$, or $\{X(N)\}$ of genus bigger than one.

To prove our results, we develop the bounds by first deriving bounds for the difference between the canonical Green's function and the hyperbolic Green's function, whose definition parallels that of the canonical Green's function when replacing the canonical $(1,1)$-form by the appropriately scaled hyperbolic $(1,1)$-form. We then express the difference between the canonical and the hyperbolic Green's functions using various expressions involving the hyperbolic heat kernel (including special values of Selberg's zeta function). The remainder of the article is devoted to proving bounds for hyperbolic heat kernels, from which our main results follow.

1.2 Arithmetic applications

Analytic problems related to Arakelov theory can be interesting both for their own sake and for potential applications to arithmetic algebraic geometry. Concerning the specific work we undertake in the present article, we were informed of some analytic problems with immediate arithmetic implications in current work by Edixhoven, which we now briefly discuss.

Edixhoven has a strategy to compute Galois representations modulo ℓ associated to a fixed modular form of arbitrary weight, with the goal of devising an algorithm, which has complexity that is polynomial in ℓ. A typical modular form to consider is Δ, the (up to scale) unique cusp form of weight 12 associated to the modular group $\mathrm{PSL}_2(\mathbb{Z})$. In this case, Edixhoven's strategy amounts to computing the field of definition of a suitable torsion point of order ℓ on the Jacobian variety $\mathrm{Jac}(X_1(\ell))$ of the modular curve $X_1(\ell)$. Naturally, such torsion points can be described in terms of a divisor on $X_1(\ell)$. Since the dimension of $\mathrm{Jac}(X_1(\ell))$ grows quadratically with ℓ, it seems as if existing methods to compute torsion points, such as with computer algebra systems, will be unfeasible. Edixhoven's idea is to numerically approximate the divisor in question with sufficiently high precision so that the approximation can turn into an exact result. More precisely, in order to get a polynomial time algorithm, one needs that the precision in the above approximation (that is, the number of digits with which the numerical computations need to be carried out) is to be at most polynomial in ℓ.

In Edixhoven's work, the required precision is roughly equal to the height of the divisor, which is estimated using Arakelov theory. The arithmetic Riemann–Roch theorem, Noether's formula, and estimates for the Faltings height of $X_1(\ell)$ and for norms of theta functions are applied. To complete this analysis, Edixhoven needs various estimates involving an upper bound for Green's functions on $X_1(\ell)$, as a function of ℓ. As an application of our general results, we derive an upper bound for the Green's functions on $X_1(\ell)$, after removing its logarithmic singularity. Indeed, our upper bound is uniform in ℓ, thus showing that the analytic contribution from the Green's functions in Edixhoven's algorithm is an order smaller than required by the algorithm.

In communicating his ideas, Edixhoven informed us that F. Merkl has studied methods, which yield upper bounds for Green's functions, that are polynomial in ℓ. Our method of proof, which builds on previous investigations, notably [JK01, JK04, JK05], provides a sharper upper bound, which we hope will lead to a better estimate of the complexity of Edixhoven's algorithm.

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1.3 Summary of the main results

The hyperbolic Green's function $g_{\mathrm{hyp}}(z,w)$ on X is the function of $z,w\in X$, which satisfies the differential equation

$$d_z d_z^c g_{\mathrm{hyp}}(z,w)+\delta_w(z)=\frac{\mu_{\mathrm{hyp}}(z)}{\mathrm{vol}_{\mathrm{hyp}}(X)}\quad (z,w\in X),$$

and the normalization condition

$$\int_X g_{\mathrm{hyp}}(z,w)\,\mu_{\mathrm{hyp}}(z)=0\quad (w\in X),$$

where μ_{hyp} is the $(1,1)$-form associated to the metric with constant negative curvature equal to minus one giving X the volume $\mathrm{vol}_{\mathrm{hyp}}(X)$. In particular, if $z,w\in\mathbb{H}$, the hyperbolic Green's function on $\mathbb{H}$ is given by

$$g_{\mathbb{H}}(z,w)=-\log\left(\left|\frac{z-w}{z-\bar{w}}\right|^2\right).$$

Our first main result, Theorem 3.8, expresses the difference $g_{\mathrm{can}}(z,w)-g_{\mathrm{hyp}}(z,w)$ in terms of a function associated to hyperbolic geometry, namely the hyperbolic heat kernel on X. This construction of $g_{\mathrm{can}}(z,w)$ allows for the study of the canonical Green's function through techniques of hyperbolic geometry. We then study the identity from Theorem 3.8 and prove bounds for the hyperbolic Green's function and the canonical Green's function on X in terms of small eigenvalues and corresponding eigenfunctions of the hyperbolic Laplacian on X, as well as other data coming from hyperbolic geometry, such as the length of the shortest closed geodesic and the injectivity radius of X. These results are summarized in Theorems 4.5, 4.8, and 4.9.

We then study these bounds for families of compact hyperbolic Riemann surfaces. In general, let X_1 be a finite degree cover of X_0, a fixed compact hyperbolic Riemann surface. Let g_{X_1} denote the genus of X_1 and $\lambda_{X_1,1}$ be the smallest non-zero eigenvalue of the hyperbolic Laplacian on X_1. Given a uniformization $X_1=\Gamma_{X_1}\backslash\mathbb{H}$ (with Γ_{X_1} a cocompact torsion-free Fuchsian subgroup of the first kind of $\mathrm{PSL}_2(\mathbb{R})$), we shall, by abuse of notation, identify X_1 with a choice of a fundamental domain for X_1 in $\mathbb{H}$, and identify points on X_1 with their pre-images in $\mathbb{H}$. Given $\delta>0$, and points $z,w\in X_1$, define the set

$$S_{\Gamma_{X_1}}(\delta;z,w)=\{\gamma\in\Gamma_{X_1}\mid d_{\mathbb{H}}(z,\gamma w)<\delta\}\,;$$

here $d_{\mathbb{H}}(\cdot,\cdot)$ denotes the hyperbolic distance on $\mathbb{H}$. Let $\{\lambda_{X_1,n}\}$ denote the set of eigenvalues of the hyperbolic Laplacian, which acts on the space of smooth functions on X_1, with associated orthonormal eigenfunctions $\{\varphi_{X_1,n}\}$. We prove that for any $\varepsilon>0$, $\delta>0$, and for all $z,w\in X_1$, we have the bounds

$$g_{\mathrm{hyp},X_1}(z,w)-\sum_{\gamma\in S_{\Gamma_{X_1}}(\delta;z,w)}g_{\mathbb{H}}(z,\gamma w)-\sum_{0<\lambda_{X_1,n}\leqslant\varepsilon}\frac{4\pi}{\lambda_{X_1,n}}\varphi_{X_1,n}(z)\varphi_{X_1,n}(w)=O_{X_0,\varepsilon,\delta}(1),$$

and

$$g_{\mathrm{can},X_1}(z,w)-g_{\mathrm{hyp},X_1}(z,w)=O_{X_0}\left(\frac{1}{g_{X_1}}\left(1+\frac{1}{\lambda_{X_1,1}}\right)\right);$$

therefore, by the triangle inequality, we show that

$$g_{\mathrm{can},X_1}(z,w)-\sum_{\gamma\in S_{\Gamma_{X_1}}(\delta;z,w)}g_{\mathbb{H}}(z,\gamma w)=O_{X_0,\delta}\left(1+\frac{1}{\lambda_{X_1,1}}\right).$$

As the notation indicates, all bounds are uniform on X_1, and depend solely on the choices of ε, δ, and the base surface X_0. The proofs of these bounds are given in § 5.

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As in [JK05], we extend our analysis to the study of the families of hyperbolic modular curves $\{X_0(N)\}$, $\{X_1(N)\}$, and $\{X(N)\}$. In this setting, it was shown in [Bro99] that the smallest non-zero eigenvalues are uniformly bounded away from zero. Therefore, our results imply, among others, the estimates

$$g_{\mathrm{can},X_1(N)}(z,w) - g_{\mathrm{hyp},X_1(N)}(z,w) = O(g_{X_1(N)}^{-1}),$$

and

$$g_{\mathrm{can},X_1(N)}(z,w) - \sum_{\gamma \in S_{\Gamma_{X_1(N)}}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w) = O_\delta(1),$$

with similar bounds for the other families of modular curves $\{X_0(N)\}$ and $\{X(N)\}$. Again, as the notation indicates, the bounds are uniform in N.

1.4 Outline of the paper

The article is organized as follows. In § 2, we establish our notation and discuss background material and results. In § 3, we derive an explicit, analytic expression relating the canonical Green's function to the hyperbolic Green's function and various other data coming from hyperbolic geometry. For the most part, the data from hyperbolic geometry that we use come directly from integral expressions involving the hyperbolic heat kernel, including the special value of the Selberg zeta function, which was studied in [JK02]. The main formula we derive is stated in Theorem 3.8. In § 4, we bound all quantities appearing in Theorem 3.8 in terms of fundamental invariants from hyperbolic geometry, such as the smallest non-zero eigenvalue, the length of the shortest closed geodesic, etc.; a list summarizing the invariants, which we use, is given in § 2.6. In § 5, we study the behavior of these invariants in two different settings, namely, a compact Riemann surface X_1, which is a finite degree cover of some fixed compact hyperbolic Riemann surface X_0, or a compact Riemann surface X_1, which lies in one of the families of hyperbolic modular surfaces $\{X_0(N)\}$, $\{X_1(N)\}$, or $\{X(N)\}$. The analysis of many of the hyperbolic invariants that appear in the present article have also been studied in detail in [JK05]. The corresponding results of [JK05] are then applied to the bounds obtained in § 4, thus completing the proofs of the results stated above.

2. Background material

2.1 Hyperbolic and canonical metrics

Let Γ be a Fuchsian subgroup of the first kind of $\mathrm{PSL}_2(\mathbb{R})$ acting by fractional linear transformations on the hyperbolic upper half-plane, which we denote by $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. We let X be the quotient space $\Gamma\backslash\mathbb{H}$ and denote by g_X the genus of X. In a slight abuse of notation, throughout this article we identify X with a fundamental domain (say, a Ford domain, bounded by geodesic paths) and identify points on X with their pre-images in $\mathbb{H}$. We assume that $g_X > 1$ and that Γ has no elliptic and, apart from the identity, no parabolic elements, that is, X is smooth and compact.

In the following, μ denotes a (smooth) metric on X, that is, μ is a positive $(1,1)$-form on X. We write $\mathrm{vol}_\mu(X)$ for the volume of X with respect to μ. In particular, we let $\mu = \mu_{\mathrm{hyp}}$ denote the hyperbolic metric on X, which is compatible with the complex structure of X, and has constant negative curvature equal to minus one. Locally, we have

$$\mu_{\mathrm{hyp}}(z) = \frac{i}{2} \cdot \frac{dz \wedge d\bar{z}}{\mathrm{Im}(z)^2}.$$

As a shorthand, we write v_X for the hyperbolic volume $\mathrm{vol}_{\mu_{\mathrm{hyp}}}(X)$; we recall that v_X is given by $4\pi(g_X - 1)$. The scaled hyperbolic metric $\mu = \mu_{\mathrm{shyp}}$ is simply the rescaled hyperbolic metric μ_{hyp}/v_X, which measures the volume of X to be one.

Let $S_k(\Gamma)$ denote the $\mathbb{C}$-vector space of cusp forms of weight k with respect to Γ equipped with the Petersson inner product

$$\langle f, g\rangle = \frac{i}{2}\int_X f(z)\,\overline{g(z)}\,\mathrm{Im}(z)^k \cdot \frac{dz \wedge d\bar{z}}{\mathrm{Im}(z)^2} \quad (f, g \in S_k(\Gamma)).$$

By choosing an orthonormal basis $\{f_1, \ldots, f_{g_X}\}$ of $S_2(\Gamma)$ with respect to the Petersson inner product, the canonical metric $\mu = \mu_{\mathrm{can}}$ of X is given by

$$\mu_{\mathrm{can}}(z) = \frac{1}{g_X} \cdot \frac{i}{2}\sum_{j=1}^{g_X} |f_j(z)|^2\, dz \wedge d\bar{z}.$$

We note that the canonical metric measures the volume of X to be one. For the purpose of comparing the hyperbolic and the canonical metrics, we define

$$d_X = \sup_{z\in X}\left|\frac{\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{shyp}}(z)}\right|.$$

In [JK04], optimal bounds for d_X through covers were obtained for arbitrary towers of compact and non-compact Riemann surfaces; see also [Don96], where the author considered the problem of towers of compact Riemann surfaces.

2.2 Green's functions and residual metrics

We denote the Green's function associated to the metric μ by g_μ. It is a function on $X \times X$ characterized by the two properties

$$d_z d_z^c g_\mu(z, w) + \delta_w(z) = \frac{\mu(z)}{\mathrm{vol}_\mu(X)},$$

$$\int_X g_\mu(z, w)\mu(z) = 0 \quad (w \in X).$$

Assuming that z, w are points on X, which are sufficiently close, our convention for the Green's function is such that the sum $g_\mu(z, w) + \log|z - w|^2$ is bounded as w approaches z.

The Green's function is an integral kernel that inverts the Laplacian associated to μ and is orthogonal to the constant functions. More precisely, for any smooth, bounded function f on X, we have the identity

$$\int_X g_\mu(z, \zeta)(-d_\zeta d_\zeta^c f(\zeta))\mu(\zeta) = f(z), \quad \text{provided that } \int_X f(\zeta)\mu(\zeta) = 0.$$

If $\mu = \mu_{\mathrm{hyp}}$, $\mu = \mu_{\mathrm{shyp}}$, or $\mu = \mu_{\mathrm{can}}$, we set

$$g_\mu = g_{\mathrm{hyp}}, \quad g_\mu = g_{\mathrm{shyp}}, \quad g_\mu = g_{\mathrm{can}},$$

respectively. By means of the function $G_\mu = \exp(g_\mu)$, we can now define a metric $\|\cdot\|_{\mu,\mathrm{res}}$ on the canonical line bundle Ω_X^1 of X in the following way. For $z \in X$, we set

$$\|dz\|^2_{\mu,\mathrm{res}} = \lim_{w\to z}(G_\mu(z, w) \cdot |z - w|^2).$$

We call the metric

$$\mu_{\mathrm{res}}(z) = \frac{i}{2} \cdot \frac{dz \wedge d\bar{z}}{\|dz\|^2_{\mu,\mathrm{res}}}$$

the residual metric associated to μ. If $\mu = \mu_{\mathrm{hyp}}$, $\mu = \mu_{\mathrm{shyp}}$, or $\mu = \mu_{\mathrm{can}}$, we set

$$\|\cdot\|_{\mu,\mathrm{res}} = \|\cdot\|_{\mathrm{hyp,res}}, \quad \|\cdot\|_{\mu,\mathrm{res}} = \|\cdot\|_{\mathrm{shyp,res}}, \quad \|\cdot\|_{\mu,\mathrm{res}} = \|\cdot\|_{\mathrm{can,res}},$$

$$\mu_{\mathrm{res}} = \mu_{\mathrm{hyp,res}}, \quad \mu_{\mathrm{res}} = \mu_{\mathrm{shyp,res}}, \quad \mu_{\mathrm{res}} = \mu_{\mathrm{can,res}},$$

respectively. We recall that the Arakelov metric μ_{Ar} is defined as the residual metric associated to the canonical metric μ_{can}; the corresponding metric on Ω^1_X is denoted by $\|\cdot\|_{\mathrm{Ar}}$. In order to be able to compare the metrics μ_{can} and μ_{Ar}, we define the C^∞-function ϕ_{Ar} on X by the equation

$$\mu_{\mathrm{Ar}} = e^{\phi_{\mathrm{Ar}}}\mu_{\mathrm{hyp}}. \tag{1}$$

2.3 Heat kernels and heat traces

The heat kernel $K_{\mathbb{H}}(t;z,w)$ on $\mathbb{H}$ ($t \in \mathbb{R}_{>0}$; $z,w \in \mathbb{H}$) is given by the formula

$$K_{\mathbb{H}}(t;z,w) = K_{\mathbb{H}}(t;\rho) = \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}}\int_\rho^\infty \frac{ue^{-u^2/4t}}{\sqrt{\cosh(u)-\cosh(\rho)}}\,du, \tag{2}$$

where $\rho = d_{\mathbb{H}}(z,w)$ denotes the hyperbolic distance between z and w. If $z=w$, the previous formula can be shown to be equal to

$$K_{\mathbb{H}}(t;z,z) = K_{\mathbb{H}}(t;0) = \frac{1}{2\pi}\int_0^\infty e^{-(r^2+1/4)t}\,r\tanh(\pi r)\,dr.$$

The heat kernel $K_X(t;z,w)$ associated to X ($t \in \mathbb{R}_{>0}$; $z,w \in X$), respectively the hyperbolic heat kernel $HK_X(t;z,w)$ associated to X ($t \in \mathbb{R}_{>0}$; $z,w \in X$) is defined by averaging over the elements of Γ, respectively the elements of Γ different from the identity, namely

$$K_X(t;z,w) = \sum_{\gamma\in\Gamma} K_{\mathbb{H}}(t;z,\gamma w),$$

$$HK_X(t;z,w) = \sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}} K_{\mathbb{H}}(t;z,\gamma w),$$

respectively. The heat kernel $K_X(t;z,w)$ admits the following spectral representation. Let $\{\lambda_{X,n}\}$ denote the set of eigenvalues of the hyperbolic Laplacian Δ_X, which acts on the space of smooth functions on X with associated orthonormal eigenfunctions $\{\varphi_{X,n}\}$. Then, for all $z,w \in X$, we have

$$K_X(t;z,w) = \sum_n \varphi_{X,n}(z)\varphi_{X,n}(w)e^{-\lambda_{X,n}t}.$$

The convergence of this series is uniform and absolute (see [Cha84, p. 112]). Recall that the eigenfunctions can be taken to be real-valued, so there is no need for a complex conjugate over one of the terms.

If $z=w$, we write $K_X(t;z)$ instead of $K_X(t;z,z)$ and $HK_X(t;z)$ instead of $HK_X(t;z,z)$. The hyperbolic heat trace $H\,\mathrm{Tr}\,K_X(t)$ ($t\in\mathbb{R}_{>0}$) is now given by

$$H\,\mathrm{Tr}\,K_X(t) = \int_X HK_X(t;z)\,\mu_{\mathrm{hyp}}(z).$$

We note that the hyperbolic Green's function $g_{\mathrm{hyp}}(z,w)$ ($z,w\in X$; $z\neq w$) relates in the following way to the heat kernel

$$g_{\mathrm{hyp}}(z,w) = 4\pi\int_0^\infty \left(K_X(t;z,w) - \frac{1}{v_X}\right)dt. \tag{3}$$

The hyperbolic Green's function on $\mathbb{H}$ can be defined using the hyperbolic heat kernel, namely through the formula

$$g_{\mathbb{H}}(z,w) = 4\pi\int_0^\infty K_{\mathbb{H}}(t;z,w)\,dt.$$

As stated in the introduction, explicit formulas were given evaluating $g_{\mathbb{H}}(z,w)$, namely

$$g_{\mathbb{H}}(z,w) = -\log\left(\left|\frac{z-w}{z-\bar{w}}\right|^2\right),$$

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as well as

$$g_{\mathbb{H}}(z,w) = -\log(\tanh^2(d_{\mathbb{H}}(z,w)/2))$$

with $d_{\mathbb{H}}(z,w)$ denoting the hyperbolic distance from z to w (see [Hej83, p. 31], and [Bea95, p. 130]). Both identities will play a role in our work.

2.4 Selberg's zeta function

Let $H(\Gamma)$ denote a complete set of representatives of non-conjugate, primitive, hyperbolic elements in Γ. Denote by ℓ_γ the hyperbolic length of the closed geodesic determined by $\gamma \in H(\Gamma)$ on X; it is well known that the equality

$$|\mathrm{tr}(\gamma)| = 2\cosh(\ell_\gamma/2)$$

holds. For $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, the Selberg zeta function $Z_X(s)$ associated to X is defined via the Euler product expansion

$$Z_X(s) = \prod_{\gamma \in H(\Gamma)} Z_\gamma(s), \quad \text{where } Z_\gamma(s) = \prod_{n=0}^{\infty} (1 - e^{-(s+n)\ell_\gamma}).$$

The Selberg zeta function $Z_X(s)$ is known to have a meromorphic continuation to all of $\mathbb{C}$ and satisfies a functional equation. As in [JK01], we define the quantity

$$c_X = \lim_{s\to 1}\left(\frac{Z_X'}{Z_X}(s) - \frac{1}{s-1}\right),$$

which expresses c_X in terms of the hyperbolic heat kernel. From [JK01, Lemma 4.2], we recall the formula

$$c_X = 1 + \int_0^\infty (H\,\mathrm{Tr}\,K_X(t) - 1)\,dt = \int_0^\infty (H\,\mathrm{Tr}\,K_X(t) - 1 + e^{-t})\,dt. \tag{4}$$

The quantity c_X was studied in detail in [JK01]. Specifically, upper and lower bounds for c_X were obtained for a fixed hyperbolic Riemann surface X, and these bounds were also studied for surfaces X_1, which are finite-degree covers of a fixed hyperbolic Riemann surface X_0. The analysis of c_X was extended to the sequence $\{X_0(N)\}$ of hyperbolic modular surfaces in [JK05, § 5].

2.5 Heat kernel bounds

Directly from the integral formula (2) for $K_{\mathbb{H}}(t;\rho)$, one can prove the following two bounds. First, for any $0 < t_0 < 1$, there is a constant $c_0 > 0$ such that for $0 < t < t_0$, we have the upper bound

$$K_{\mathbb{H}}(t;\rho) \leqslant \frac{c_0}{4\pi t} e^{-\rho^2/(4t)}$$

for all $\rho \geqslant 0$. Second, there is a constant $c_\infty > 0$ such that, if $t \geqslant t_0$, then

$$K_{\mathbb{H}}(t;\rho) \leqslant c_\infty e^{-t/4}$$

for all $\rho \geqslant 0$. Continuing, one also has the bound

$$K_X(t;z,w) \leqslant \tfrac{1}{2}(K_X(t;z) + K_X(t;w)),$$

which holds for all $t > 0$ and all $z, w \in X$. To prove this inequality, observe that for each n, we have

$$\varphi_{X,n}(z)\varphi_{X,n}(w)e^{-\lambda_{X,n}t} \leqslant \tfrac{1}{2}(\varphi_{X,n}^2(z)e^{-\lambda_{X,n}t} + \varphi_{X,n}^2(w)e^{-\lambda_{X,n}t}),$$

from which the stated bound now follows by summing over all n.

More generally, one can use hyperbolic geometry in order to prove an upper bound for $K_X(t;z,w)$. For this, we follow [JL95, Lemma 2.3], in particular displayed formula (2.2) on p. 796, which we now recall in detail. Fix $0 < t_0 < 1$, and choose δ_0 sufficiently large such that $K_{\mathbb{H}}(t;\rho)$

is a monotone decreasing function of ρ for $\rho > \delta_0$ and all $0 < t < t_0$ (as with the above bounds for $K_{\mathbb{H}}(t;\rho)$, the verification of the existence of t_0 and δ_0 follows from the integral formula for $K_{\mathbb{H}}(t;\rho)$). Let r_X be any number less than or equal to the injectivity radius of X, meaning

$$r_X \leqslant \inf\{d_{\mathbb{H}}(z,\gamma z) \mid \gamma \in \Gamma, \gamma \neq \mathrm{id}, z \in X\}.$$

Since X is compact, one can choose $r_X > 0$. For $\delta > 0$ and fixed $z, w \in X$, we define the set

$$S_\Gamma(\delta; z, w) = \{\gamma \in \Gamma \mid d_{\mathbb{H}}(z,\gamma w) < \delta\}.$$

Then, as stated in [JL95, formula (2.2), p. 796], we have the bounds

$$\sum_{\gamma \in S_\Gamma(\delta;z,w)} K_{\mathbb{H}}(t; d_{\mathbb{H}}(z,\gamma w)) \leqslant K_X(t;z,w)$$

and, for all $0 < t < t_0$ and $\delta > \delta_0$, we have

$$\begin{aligned} K_X(t;z,w) \leqslant \sum_{\gamma \in S_\Gamma(\delta;z,w)} K_{\mathbb{H}}(t; d_{\mathbb{H}}(z,\gamma w)) + \frac{\sinh(r_X)\sinh(\delta)}{\sinh^2(r_X/2)} \cdot K_{\mathbb{H}}(t;\delta) \\ + \frac{1}{\sinh^2(r_X/2)} \int_{\delta - 4r_X}^{\infty} K_{\mathbb{H}}(t;\rho)\sinh(\rho + 2r_X)\, d\rho. \end{aligned}$$

The arguments proving these bounds are elementary and we refer the reader to [JL95] for details. We note here that the statement above is obtained through a slight refinement of that given in [JL95], coming from observing that the various hyperbolic discs whose volumes are used to estimate the number of lattice points can be taken to be centered at an orbit point of w. As a result, certain estimates above involve r_X rather than $2r_X$, as in [JL95]. This refinement is not critical for the analysis here; nonetheless, for the sake of precision, we do quote and then employ this refined result.

2.6 Certain hyperbolic-geometric invariants

For the convenience of the reader, we list here certain hyperbolic invariants that appear in our estimates.

The constants c_0, c_∞, t_0, and δ_0 appear in the upper bounds for $K_{\mathbb{H}}(t;\rho)$ and were defined in § 2.5. The constant r_X is any number less than or equal to the injectivity radius of X, and we take δ_X to be any number such that $\delta_X > \max\{\delta_0, 4r_X + 5\} > 0$. Given $0 < t_0 < 1$, we define

$$C_X^{HK} = \max_{z \in X} K_X(t_0; z),$$

which is finite, since X is compact. Following the arguments in [JL95], it can be shown that

$$\sup_{z,w \in X} \# S_\Gamma(\delta; z, w) \leqslant \frac{\sinh(\delta + r_X)}{\sinh(r_X)},$$

where the set $S_\Gamma(\delta; z, w)$ was defined in § 2.5. The smallest non-zero eigenvalue of the hyperbolic Laplacian on X is denoted by $\lambda_{X,1}$ and the length of the shortest non-zero closed geodesic on X is denoted by $\ell_{X,0}$. The constant c_X is the constant term in the Laurant expansion of the logarithmic derivative of the Selberg zeta function $Z_X(s)$ at $s = 1$, as defined in § 2.4. Finally, the sup-norm between the canonical and scaled hyperbolic volume forms is defined by

$$d_X = \sup_{z \in X} \left| \frac{\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{shyp}}(z)} \right|.$$

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3. Expressing canonical Green's function using hyperbolic data

In this section we obtain a closed-form expression for the canonical Green's function in terms of hyperbolic geometry. The main result of this section, Theorem 3.8, expresses $g_{\rm can}$ in terms of the hyperbolic Green's function $g_{\rm hyp}$ and analytic functions derived from the hyperbolic heat kernel. The steps in our proof are as follows. First, we derive a general expression relating $g_{\rm can}$ to $g_{\rm hyp}$ in terms of various integrals involving $\mu_{\rm can}$; see Lemma 3.1. Next, we prove an explicit relation between the canonical metric $\mu_{\rm can}$ and the hyperbolic metric $\mu_{\rm hyp}$ in terms of the hyperbolic heat kernel; see Theorem 3.4. We then substitute Theorem 3.4 into Lemma 3.1 in order to complete the proof of Theorem 3.8.

Lemma 3.1. *With the above notation, we have, for all $z, w \in X$, the formula*

$$\begin{aligned} g_{\rm hyp}(z,w) - g_{\rm can}(z,w) = &\int_X g_{\rm hyp}(z,\zeta)\mu_{\rm can}(\zeta) + \int_X g_{\rm hyp}(w,\zeta)\mu_{\rm can}(\zeta) \\ &- \int_X\int_X g_{\rm hyp}(\xi,\zeta)\mu_{\rm can}(\zeta)\mu_{\rm can}(\xi). \end{aligned}$$

Proof. Let $F_{\rm L}(z,w)$ (respectively $F_{\rm R}(z,w)$) denote the left-hand side (respectively right-hand side) of the stated identity. Using the characterizing properties of the Green's functions, one can show directly that we have, for fixed $w \in X$,

$$d_z d_z^c F_{\rm L}(z,w) = d_z d_z^c F_{\rm R}(z,w) = \mu_{\rm shyp}(z) - \mu_{\rm can}(z),$$

and

$$\int_X F_{\rm L}(z,w)\mu_{\rm can}(z) = \int_X F_R(z,w)\mu_{\rm can}(z) = \int_X g_{\rm hyp}(w,\zeta)\mu_{\rm can}(\zeta).$$

Consequently, $F_{\rm L}(z,w) = F_{\rm R}(z,w)$, again for fixed w. However, it is obvious that $F_{\rm L}$ and $F_{\rm R}$ are symmetric in z and w. This completes the proof of the lemma. □

Proposition 3.2. *With the above notation, we have, for all $z \in X$, the formula*

$$g_X\mu_{\rm can}(z) = \mu_{\rm shyp}(z) + \tfrac{1}{2}{\rm c}_1(\Omega_X^1, \|\cdot\|_{\rm hyp,res})(z)\,;$$

here Ω_X^1 denotes the canonical line bundle on X.

Proof. Let us rewrite the identity in Lemma 3.1 as

$$g_{\rm hyp}(z,w) - g_{\rm can}(z,w) = \phi(z) + \phi(w), \tag{5}$$

where

$$\phi(z) = \int_X g_{\rm hyp}(z,\zeta)\mu_{\rm can}(\zeta) - \frac{1}{2}\int_X\int_X g_{\rm hyp}(\xi,\zeta)\mu_{\rm can}(\zeta)\mu_{\rm can}(\xi).$$

Taking $d_z d_z^c$ in relation (5), we get the equation

$$\mu_{\rm shyp}(z) - \mu_{\rm can}(z) = d_z {\rm d}_z^c \phi(z). \tag{6}$$

On the other hand, we have by definition

$$\begin{aligned} \log\|dz\|^2_{\rm hyp,res} &= \lim_{w\to z}(g_{\rm hyp}(z,w) + \log|z-w|^2), \\ \log\|dz\|^2_{\rm can,res} &= \lim_{w\to z}(g_{\rm can}(z,w) + \log|z-w|^2). \end{aligned}$$

From this we deduce, again using (5),

$$\log\|dz\|^2_{\rm hyp,res} - \log\|dz\|^2_{\rm can,res} = \lim_{w\to z}(g_{\rm hyp}(z,w) - g_{\rm can}(z,w)) = 2\phi(z). \tag{7}$$

Now, taking $-d_z d_z^c$ of (7) yields

$${\rm c}_1(\Omega_X^1, \|\cdot\|_{\rm hyp,res})(z) - {\rm c}_1(\Omega_X^1, \|\cdot\|_{\rm can,res})(z) = -2 d_z d_z^c \phi(z). \tag{8}$$

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Combining (6) and (8) leads to

$$2(\mu_{\mathrm{shyp}}(z)-\mu_{\mathrm{can}}(z))=\mathrm{c}_1(\Omega^1_X,\|\cdot\|_{\mathrm{can,res}})(z)-\mathrm{c}_1(\Omega^1_X,\|\cdot\|_{\mathrm{hyp,res}})(z). \tag{9}$$

Recalling

$$\mathrm{c}_1(\Omega^1_X,\|\cdot\|_{\mathrm{can,res}})(z)=(2g_X-2)\mu_{\mathrm{can}}(z),$$

from (9) we derive

$$\mu_{\mathrm{shyp}}(z)-\mu_{\mathrm{can}}(z)=\frac{2g_X-2}{2}\mu_{\mathrm{can}}(z)-\frac{1}{2}\mathrm{c}_1(\Omega^1_X,\|\cdot\|_{\mathrm{hyp,res}})(z),$$

which proves the proposition. □

PROPOSITION 3.3. *With the above notation, we have the following formula for the first Chern form of Ω^1_X with respect to $\|\cdot\|_{\mathrm{hyp,res}}$*

$$\mathrm{c}_1(\Omega^1_X,\|\cdot\|_{\mathrm{hyp,res}})(z)=\frac{1}{2\pi}\cdot\mu_{\mathrm{hyp}}(z)+\left(\int_0^\infty\Delta_X K_X(t;z)\,dt\right)\mu_{\mathrm{hyp}}(z).$$

Proof. By our definitions, for $z\in X$ we have

$$\begin{aligned}\mathrm{c}_1(\Omega^1_X,\|\cdot\|_{\mathrm{hyp,res}})(z)&=-d_zd_z^c\log\|dz\|^2_{\mathrm{hyp,res}}=-d_zd_z^c\lim_{w\to z}(g_{\mathrm{hyp}}(z,w)+\log|z-w|^2)\\&=-d_zd_z^c\lim_{w\to z}\left(4\pi\int_0^\infty\left(K_X(t;z,w)-\frac{1}{v_X}\right)dt+\log|z-w|^2\right)\\&=-d_zd_z^c\lim_{w\to z}\left(4\pi\int_0^\infty K_{\mathbb{H}}(t;z,w)\,dt+\log|z-w|^2\right)\\&\quad-d_zd_z^c\lim_{w\to z}\left(4\pi\int_0^\infty\left(\sum_{\substack{\gamma\in\Gamma\\\gamma\neq\mathrm{id}}}K_{\mathbb{H}}(t;z,\gamma w)-\frac{1}{v_X}\right)dt\right).\end{aligned}$$

Using the formula for the Green's function $g_{\mathbb{H}}(z,w)$ on $\mathbb{H}$, for the first summand in the latter sum we obtain

$$\begin{aligned}A&=-d_zd_z^c\lim_{w\to z}\left(4\pi\int_0^\infty K_{\mathbb{H}}(t;z,w)\,dt+\log|z-w|^2\right)\\&=-d_zd_z^c\lim_{w\to z}(g_{\mathbb{H}}(z,w)+\log|z-w|^2)\\&=-d_zd_z^c\log|z-\bar z|^2=-\frac{2i}{2\pi}\,\partial_z\,\bar\partial_z\log(z-\bar z)\\&=\frac{i}{\pi}\partial_z\frac{d\bar z}{z-\bar z}=-\frac{i}{\pi}\cdot\frac{dz\wedge d\bar z}{(z-\bar z)^2}\\&=-\frac{i}{\pi}\cdot\frac{dz\wedge d\bar z}{(2i\,\mathrm{Im}(z))^2}=\frac{1}{2\pi}\cdot\mu_{\mathrm{hyp}}(z).\end{aligned}$$

For the second summand we obtain

$$\begin{aligned}B&=-d_zd_z^c\lim_{w\to z}\left(4\pi\int_0^\infty\left(\sum_{\substack{\gamma\in\Gamma\\\gamma\neq\mathrm{id}}}K_{\mathbb{H}}(t;z,\gamma w)-\frac{1}{v_X}\right)dt\right)\\&=-4\pi d_zd_z^c\int_0^\infty\left(\sum_{\substack{\gamma\in\Gamma\\\gamma\neq\mathrm{id}}}K_{\mathbb{H}}(t;z,\gamma z)-\frac{1}{v_X}\right)dt.\end{aligned}$$

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Since the latter integral converges absolutely, as does the integral of derivatives of the integrand, we are allowed to interchange differentiation and integration; this gives

$$\begin{aligned} B &= -4\pi \int_0^\infty d_z d_z^c \Bigg(\sum_{\substack{\gamma\in\Gamma \\ \gamma\neq \mathrm{id}}} K_{\mathbb{H}}(t;z,\gamma z) - \frac{1}{v_X} \Bigg)\, dt \\ &= -4\pi \int_0^\infty \sum_{\substack{\gamma\in\Gamma \\ \gamma\neq \mathrm{id}}} d_z d_z^c\, K_{\mathbb{H}}(t;z,\gamma z)\, dt. \end{aligned}$$

The claimed formula then follows, since $K_{\mathbb{H}}(t;z,z)$ is independent of z, and recalling the identity

$$-4\pi d_z d_z^c f(z) = (\Delta_X f(z))\mu_{\mathrm{hyp}}(z),$$

for any smooth function f on X. □

Theorem 3.4. *With the above notation, we have, for all $z \in X$, the formula*

$$\mu_{\mathrm{can}}(z) = \mu_{\mathrm{shyp}}(z) + \frac{1}{2g_X}\bigg(\int_0^\infty \Delta_X K_X(t;z)\, dt \bigg) \mu_{\mathrm{hyp}}(z).$$

Proof. We simply have to combine Propositions 3.2 and 3.3, and to use that

$$\frac{1}{g_X} + \frac{v_X}{4\pi g_X} = 1.$$

□

Lemma 3.5. *For all $z \in X$, let $H(z)$ be defined by*

$$H(z) = \int_0^\infty \bigg(HK_X(t;z) - \frac{1}{v_X} \bigg)\, dt - \frac{c_X - 1}{v_X}.$$

Then, $H(z)$ is uniquely characterized by satisfying the integral formula

$$\int_X H(z)\mu_{\mathrm{hyp}}(z) = 0$$

and the differential equation

$$\Delta_X H(z) = \int_0^\infty \Delta_X K_X(t;z)\, dt.$$

Proof. Concerning the integral equation, note that, by interchanging the order of integration, we have

$$\begin{aligned} \int_X H(z)\mu_{\mathrm{hyp}}(z) &= \int_X \bigg(\int_0^\infty \bigg(HK_X(t;z) - \frac{1}{v_X} \bigg)\, dt - \frac{c_X - 1}{v_X} \bigg) \mu_{\mathrm{hyp}}(z) \\ &= \int_0^\infty (H\,\mathrm{Tr}\, K_X(t) - 1)\, dt - (c_X - 1) = 0, \end{aligned}$$

where the last equality follows from formula (4), given in § 2.4. As for the differential equation, note that for any $z \in X$, we have

$$HK_X(t;z) = K_X(t;z) - K_{\mathbb{H}}(t,0).$$

Since $K_{\mathbb{H}}(t,0)$ and $(c_X - 1)/v_X$ are annihilated by Δ_X, the result follows. □

Lemma 3.6. *With the above notation, we have, for all $z \in X$, the formula*

$$\int_X g_{\mathrm{hyp}}(z,\zeta)\mu_{\mathrm{can}}(\zeta) = \frac{2\pi}{g_X} H(z).$$

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Proof. Using Theorem 3.4, we have

$$\begin{aligned}\int_X g_{\text{hyp}}(z,\zeta)\mu_{\text{can}}(\zeta) &= \int_X g_{\text{hyp}}(z,\zeta)\Big(\mu_{\text{shyp}}(\zeta) + \frac{1}{2g_X}\Big(\int_0^\infty \Delta_X K_X(t;\zeta)\,dt\Big)\mu_{\text{hyp}}(\zeta)\Big) \\ &= \frac{1}{2g_X}\int_X g_{\text{hyp}}(z,\zeta)\Big(\int_0^\infty \Delta_X K_X(t;\zeta)\,dt\Big)\mu_{\text{hyp}}(\zeta) \\ &= \frac{1}{2g_X}\int_X g_{\text{hyp}}(z,\zeta)\Delta_X H(\zeta)\mu_{\text{hyp}}(\zeta),\end{aligned}$$

where the last equality follows from Lemma 3.5. Using the integral formula in Lemma 3.5, the assertion is proved by using that g_{hyp} inverts the operator $-dd^c$ on the space of functions whose integral is zero. □

Lemma 3.7. *With the above notation, we have the formula*

$$\int_X\int_X g_{\text{hyp}}(\xi,\zeta)\mu_{\text{can}}(\zeta)\mu_{\text{can}}(\xi) = \frac{\pi}{g_X^2}\int_X H(\xi)\Delta_X H(\xi)\mu_{\text{hyp}}(\xi).$$

Proof. Using Lemma 3.6, we have

$$\int_X\int_X g_{\text{hyp}}(\xi,\zeta)\mu_{\text{can}}(\zeta)\mu_{\text{can}}(\xi) = \frac{2\pi}{g_X}\int_X H(\xi)\mu_{\text{can}}(\xi).$$

We now employ Theorem 3.4, which gives

$$\begin{aligned}\int_X H(\xi)\mu_{\text{can}}(\xi) &= \int_X H(\xi)\Big(\mu_{\text{shyp}}(\xi) + \frac{1}{2g_X}\Big(\int_0^\infty \Delta_X K_X(t;\xi)\,dt\Big)\mu_{\text{hyp}}(\xi)\Big) \\ &= \frac{1}{2g_X}\int_X H(\xi)\Big(\int_0^\infty \Delta_X K_X(t;\xi)\,dt\Big)\mu_{\text{hyp}}(\xi),\end{aligned}$$

where we have used the integral equation from Lemma 3.5 to obtain the last equality. The result follows by using the differential equation from Lemma 3.5. □

Theorem 3.8. *With the above notation, we have the formula*

$$g_{\text{can}}(z,w) - g_{\text{hyp}}(z,w) = \phi_X(z) + \phi_X(w),$$

where

$$\phi_X(z) = \frac{2\pi}{g_X}H(z) - \frac{\pi}{2g_X^2}\int_X H(\xi)\Delta_X H(\xi)\mu_{\text{hyp}}(\xi).$$

Proof. The proof is obtained by combining Lemmas 3.1, 3.6 and 3.7. □

Remark 3.9. Recall from § 2.3 that the hyperbolic Green's function g_{hyp} is simply expressed in terms of the hyperbolic heat kernel. Together with the definition of $H(z)$ given in Lemma 3.5, the main result in Theorem 3.8 then states a closed form expression for the canonical Green's function g_{can} using the hyperbolic heat kernel. By comparison, note that the analysis in [Jor90] relied on an evaluation of the canonical Green's function in terms of the classical Riemann theta function; see [Jor90], in particular Proposition 2.4 and the preceding computations. Consequently, we now have a complete, closed-form expression for the Riemann theta function in terms of the hyperbolic heat kernel. A potentially fascinating study would be to explore this relation further, either from the point of view of obtaining results in hyperbolic geometry from the algebraic geometry of the theta function, or conversely.

4. Bounds of various hyperbolic data

We now work from Theorem 3.8 and obtain bounds for the canonical Green's function for a fixed surface X. First, we study bounds for the hyperbolic Green's function, which we derive using the heat

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kernel bound stated in § 2.5; these bounds are given in Theorem 4.5. Next, we estimate the function ϕ_X in Theorem 3.8; these estimates are given in Corollary 4.6 and Proposition 4.7. After this, the bounds we seek for the canonical Green's function are immediate and are stated in Theorems 4.8 and 4.9. As we will see in the next section, the explicit nature of these bounds are such that we can easily determine the behavior of the estimates through covers and for sequences of hyperbolic modular curves.

Lemma 4.1. *Let t_0 and C_X^{HK} be as in §§ 2.5 and 2.6. For any $\varepsilon > 0$ and $z, w \in X$, we then have the following estimate involving the eigenfunctions $\varphi_{X,n}$ of the hyperbolic Laplacian*

$$\sum_{0 \leqslant \lambda_{X,n} < \varepsilon} |\varphi_{X,n}(z)\varphi_{X,n}(w)| \leqslant C_X^{HK} \cdot e^{\varepsilon t_0}.$$

Proof. First observe that for each n, we have

$$|\varphi_{X,n}(z)\varphi_{X,n}(w)| \leqslant \tfrac{1}{2}(\varphi_{X,n}^2(z) + \varphi_{X,n}^2(w));$$

hence, it suffices to prove the claim when $z = w$. For this, we note that $e^{-\lambda_{X,n}t_0} \cdot e^{\varepsilon t_0} \geqslant 1$, provided that $\lambda_{X,n} < \varepsilon$. Therefore, we find

$$\sum_{0 \leqslant \lambda_{X,n} < \varepsilon} \varphi_{X,n}^2(z) \leqslant \sum_{0 \leqslant \lambda_{X,n} < \varepsilon} \varphi_{X,n}^2(z) e^{-\lambda_{X,n}t_0} \cdot e^{\varepsilon t_0} \leqslant e^{\varepsilon t_0} \cdot K_X(t_0; z) \leqslant C_X^{HK} \cdot e^{\varepsilon t_0},$$

which proves the claim. □

Lemma 4.2. *Let $c_0, c_\infty, t_0, r_X, \delta_X$, and C_X^{HK} be as in §§ 2.5 and 2.6. For any $\delta \geqslant \delta_X$, $\varepsilon > 0$, and $z, w \in X$, let*

$$K_X^{\varepsilon,\delta}(t; z, w) = K_X(t; z, w) - \sum_{0 \leqslant \lambda_{X,n} < \varepsilon} \varphi_{X,n}(z)\varphi_{X,n}(w) e^{-\lambda_{X,n}t} - \sum_{\gamma \in S_\Gamma(\delta; z, w)} K_{\mathbb{H}}(t; d_{\mathbb{H}}(z, \gamma w)).$$

Then, we have the following bounds:

(a) *if $0 < t < t_0$, then*

$$|K_X^{\varepsilon,\delta}(t; z, w)| \leqslant C_X^{HK} \cdot e^{\varepsilon t_0} + \frac{c_0 \sinh(r_X)\sinh(\delta)}{8\delta^2 \sinh^2(r_X/2)} + \frac{c_0 e^{2r_X}}{2\pi \sinh^2(r_X/2)};$$

(b) *if $t \geqslant t_0$, then*

$$|K_X^{\varepsilon,\delta}(t; z, w)| \leqslant C_X^{HK} \cdot e^{-\varepsilon(t - t_0)} + \frac{c_\infty \sinh(\delta + r_X)}{\sinh(r_X)} e^{-t/4}.$$

Proof. To prove part (a), we first use the triangle inequality to write

$$|K_X^{\varepsilon,\delta}(t; z, w)| \leqslant \sum_{0 \leqslant \lambda_{X,n} < \varepsilon} |\varphi_{X,n}(z)\varphi_{X,n}(w)| e^{-\lambda_{X,n}t} + \sum_{\gamma \notin S_\Gamma(\delta; z, w)} K_{\mathbb{H}}(t; d_{\mathbb{H}}(z, \gamma w)).$$

By Lemma 4.1, the first summand is bounded by $C_X^{HK} \cdot e^{\varepsilon t_0}$. As for the second summand, we proceed by using the heat kernel estimates from § 2.5, namely the bounds

$$\sum_{\gamma \notin S_\Gamma(\delta; z, w)} K_{\mathbb{H}}(t; d_{\mathbb{H}}(z, \gamma w)) \leqslant \frac{\sinh(r_X)\sinh(\delta)}{\sinh^2(r_X/2)} \cdot K_{\mathbb{H}}(t; \delta) + \frac{1}{\sinh^2(r_X/2)} \int_{\delta - 4r_X}^{\infty} K_{\mathbb{H}}(t; \rho) \sinh(\rho + 2r_X)\, d\rho.$$

Trivially, the lower bound for the sum in question is zero, since each term in the series is positive. Since $0 < t < t_0 < 1$, we can use the bound

$$K_{\mathbb{H}}(t; \delta) \leqslant \frac{c_0}{4\pi t} e^{-\delta^2/(4t)},$$

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which gives

$$\frac{\sinh(r_X)\sinh(\delta)}{\sinh^2(r_X/2)} \cdot K_{\mathbb{H}}(t;\delta) \leqslant \frac{c_0 \sinh(r_X)\sinh(\delta)}{\sinh^2(r_X/2)} \cdot \frac{1}{4\pi t} e^{-\delta^2/(4t)}.$$

It is elementary to compute that the maximum of $e^{-a/t}/t$, as a function of $t > 0$ and fixed $a > 0$, occurs when $t = a$, yielding the maximum value of e^{-1}/a. Therefore, taking $a = \delta^2/4$, we get

$$\frac{c_0 \sinh(r_X)\sinh(\delta)}{\sinh^2(r_X/2)} \cdot \frac{1}{4\pi t} e^{-\delta^2/(4t)} \leqslant \frac{c_0 \sinh(r_X)\sinh(\delta)}{\sinh^2(r_X/2)} \cdot \frac{1}{\pi\delta^2} e^{-1} \leqslant \frac{c_0 \sinh(r_X)\sinh(\delta)}{8\delta^2 \sinh^2(r_X/2)},$$

using that $\pi e > 8$; thus, we have computed the second term in the stated upper bound. For the last term, we use the stated upper bound for $K_{\mathbb{H}}(t;\rho)$ together with the trivial estimate $\sinh(x) \leqslant e^x/2$ in order to write

$$\frac{1}{\sinh^2(r_X/2)} \int_{\delta-4r_X}^{\infty} K_{\mathbb{H}}(t;\rho)\sinh(\rho+2r_X)\,d\rho \leqslant \frac{c_0 e^{2r_X}}{8\pi t \cdot \sinh^2(r_X/2)} \int_{\delta-4r_X}^{\infty} e^{-\rho^2/(4t)+\rho}\,d\rho.$$

Over the specified limits of integration, we have that $\rho^2 \geqslant \rho(\delta - 4r_X)$, so then

$$e^{-\rho^2/(4t)+\rho} \leqslant e^{-\rho(\delta-4r_X)/(4t)+\rho} = e^{-\rho(\delta-4r_X-4t)/(4t)}.$$

By assumption, $\delta \geqslant \delta_X > 4r_X + 5$, so then for $0 < t < t_0 < 1$, we have that $\delta - 4r_X - 4t > 1$, hence the exponential functions $e^{-\rho^2/(4t)+\rho}$ are integrable for all $0 < t < t_0$ near infinity. With this, we then have

$$\int_{\delta-4r_X}^{\infty} e^{-\rho^2/(4t)+\rho}\,d\rho \leqslant \int_{\delta-4r_X}^{\infty} e^{-\rho(\delta-4r_X-4t)/(4t)}\,d\rho = \frac{4t}{\delta-4r_X-4t} e^{-(\delta-4r_X)(\delta-4r_X-4t)/(4t)}.$$

Since $\delta - 4r_X - 4t > 1$, we have $\delta - 4r_X > 1$, so then

$$\frac{4t}{\delta-4r_X-4t} e^{-(\delta-4r_X)(\delta-4r_X-4t)/(4t)} \leqslant 4t \cdot e^{-(\delta-4r_X)(\delta-4r_X-4t)/(4t)} \leqslant 4t.$$

Summing up, we find

$$\frac{1}{\sinh^2(r_X/2)} \int_{\delta-4r_X}^{\infty} K_{\mathbb{H}}(t;\rho)\sinh(\rho+2r_X)\,d\rho \leqslant \frac{c_0 e^{2r_X}}{2\pi \sinh^2(r_X/2)},$$

which completes the proof of part (a).

We now prove part (b). To begin, we use the spectral decomposition of the heat kernel and the triangle inequality to get

$$|K_X^{\varepsilon,\delta}(t;z,w)| \leqslant \sum_{\lambda_{X,n}\geqslant\varepsilon} |\varphi_{X,n}(z)\varphi_{X,n}(w)| e^{-\lambda_{X,n}t} + \sum_{\gamma\in S_\Gamma(\delta;z,w)} K_{\mathbb{H}}(t;d_{\mathbb{H}}(z,\gamma w)).$$

From § 2.6, we then have

$$\sum_{\gamma\in S_\Gamma(\delta;z,w)} K_{\mathbb{H}}(t;d_{\mathbb{H}}(z,\gamma w)) \leqslant \#S_\Gamma(\delta;z,w) \cdot \sup_{\eta\in[0,\delta]} K_{\mathbb{H}}(t;\eta) \leqslant \frac{c_\infty \sinh(\delta+r_X)}{\sinh(r_X)} e^{-t/4},$$

which yields one of the terms in the stated upper bound. For the other term, we note that

$$\sum_{\lambda_{X,n}\geqslant\varepsilon} |\varphi_{X,n}(z)\varphi_{X,n}(w)| e^{-\lambda_{X,n}t} \leqslant \frac{1}{2}\Bigg(\sum_{\lambda_{X,n}\geqslant\varepsilon} \varphi_{X,n}^2(z) e^{-\lambda_{X,n}t} + \sum_{\lambda_{X,n}\geqslant\varepsilon} \varphi_{X,n}^2(w) e^{-\lambda_{X,n}t} \Bigg),$$

so it suffices to prove that

$$\sum_{\lambda_{X,n}\geqslant\varepsilon} \varphi_{X,n}^2(z) e^{-\lambda_{X,n}t} \leqslant C_X^{HK} \cdot e^{-\varepsilon(t-t_0)}.$$

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For this, we consider the function

$$h(t;z) = e^{\varepsilon t} \cdot \sum_{\lambda_{X,n} \geqslant \varepsilon} \varphi^2_{X,n}(z) e^{-\lambda_{X,n} t}.$$

For fixed $z \in X$, the function $h(t;z)$ is monotone decreasing in t for all $t > 0$. In particular, we then have

$$h(t;z) \leqslant h(t_0;z) = e^{\varepsilon t_0} \cdot \sum_{\lambda_{X,n} \geqslant \varepsilon} \varphi^2_{X,n}(z) e^{-\lambda_{X,n} t_0} \leqslant e^{\varepsilon t_0} \cdot K_X(t_0;z) \leqslant C_X^{HK} \cdot e^{\varepsilon t_0}.$$

Therefore, we end up with

$$0 \leqslant \sum_{\lambda_{X,n} \geqslant \varepsilon} \varphi^2_{X,n}(z) e^{-\lambda_{X,n} t} = e^{-\varepsilon t} \cdot h(t;z) \leqslant e^{-\varepsilon t} \cdot C_X^{HK} \cdot e^{\varepsilon t_0} = C_X^{HK} \cdot e^{-\varepsilon (t - t_0)}.$$

With all this, part (b) is proved. □

Remark 4.3. If required, the estimates in Lemma 4.2 could be enhanced to reflect the role played by δ. For example, the estimates for $0 < t < t_0$ can be easily improved so that the upper bound approaches zero as δ increases. However, rather than weigh down the above estimates any further, we choose to underplay the role of δ solely because further bounds are not needed in the present article.

Lemma 4.4. *For any $z, w \in \mathbb{H}$ with $d_{\mathbb{H}}(z,w) \in [a,b]$, we have the estimate*

$$|g_{\mathbb{H}}(z,w)| \leqslant \max\{|\log(\tanh^2(a/2))|, |\log(\tanh^2(b/2))|\}.$$

Proof. From [Bea95, p. 130], we have

$$g_{\mathbb{H}}(z,w) = -\log(\tanh^2(d_{\mathbb{H}}(z,w)/2)).$$

The function $\tanh(u)$ is monotone increasing for $u > 0$, so its maximum and minimum for $u \in [a,b]$ occur at the boundary, from which the lemma follows. □

Theorem 4.5. *Let $c_0, c_\infty, t_0, r_X, \delta_X$, and C_X^{HK} be as in §§ 2.5 and 2.6. For any $\delta > 0$, $\varepsilon > 0$, and $z, w \in X$, we then have the estimate*

$$\left| g_{\text{hyp}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w) - \sum_{0 < \lambda_{X,n} < \varepsilon} \frac{4\pi}{\lambda_{X,n}} \varphi_{X,n}(z) \varphi_{X,n}(w) \right| \leqslant B_{X,\varepsilon,\delta},$$

where

$$B_{X,\varepsilon,\delta} = \begin{cases} 4\pi \left(C_X^{HK} \cdot e^{\varepsilon t_0} + \dfrac{c_0 \sinh(r_X)\sinh(\delta)}{8\delta^2 \sinh^2(r_X/2)} + \dfrac{c_0 e^{2r_X}}{2\pi \sinh^2(r_X/2)} + \dfrac{4c_\infty \sinh(\delta + r_X)}{\sinh(r_X)} + \dfrac{C_X^{HK}}{\varepsilon} \right), \\ \quad \text{if } \delta > \delta_X, \\ 4\pi \left(C_X^{HK} \cdot e^{\varepsilon t_0} + \dfrac{c_0 \sinh(r_X)\sinh(\delta_X)}{8\delta_X^2 \sinh^2(r_X/2)} + \dfrac{c_0 e^{2r_X}}{2\pi \sinh^2(r_X/2)} + \dfrac{4c_\infty \sinh(\delta_X + r_X)}{\sinh(r_X)} + \dfrac{C_X^{HK}}{\varepsilon} \right) \\ \quad + \dfrac{\sinh(\delta_X + r_X)}{\sinh(r_X)} \max\{|\log(\tanh^2(\delta/2))|, |\log(\tanh^2(\delta_X/2))|\}, \quad \text{if } \delta \leqslant \delta_X. \end{cases}$$

Proof. By the definition of $K_X^{\varepsilon,\delta}(t;z,w)$ given in Lemma 4.2, we have

$$g_{\text{hyp}}(z,w) - \sum_{0 < \lambda_{X,n} < \varepsilon} \frac{4\pi}{\lambda_{X,n}} \varphi_{X,n}(z)\varphi_{X,n}(w) - \sum_{\gamma \in S_\Gamma(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w) = 4\pi \int_0^\infty K_X^{\varepsilon,\delta}(t;z,w)\, dt.$$

If $\delta > \delta_X$, the result immediately follows from integrating the bounds from Lemma 4.2, taking into account the decomposition

$$\int_0^\infty |K_X^{\varepsilon,\delta}(t;z,w)|\,dt = \int_0^{t_0} |K_X^{\varepsilon,\delta}(t;z,w)|\,dt + \int_{t_0}^\infty |K_X^{\varepsilon,\delta}(t;z,w)|\,dt.$$

On the other hand, if $\delta \leqslant \delta_X$, we simply write

$$K_X^{\varepsilon,\delta}(t;z,w) = K_X^{\varepsilon,\delta_X}(t;z,w) + \sum_{\gamma\in S_\Gamma(\delta_X;z,w)\setminus S_\Gamma(\delta;z,w)} K_{\mathbb{H}}(t;d_{\mathbb{H}}(z,\gamma w)).$$

Then, taking absolute values and using the triangle inequality, the integral over $|K_X^{\varepsilon,\delta_X}(t;z,w)|$ is estimated as in the previous case using Lemma 4.2, but with δ replaced by δ_X, while the remaining sum is estimated using Lemma 4.3 together with the bound

$$\#(S_\Gamma(\delta_X;z,w)\setminus S_\Gamma(\delta;z,w)) \leqslant \#S_\Gamma(\delta_X;z,w) \leqslant \frac{\sinh(\delta_X + r_X)}{\sinh(r_X)}.$$

The proof of the theorem is now complete. □

Corollary 4.6. *Let $\lambda_{X,1}$ and $\ell_{X,0}$ be as in § 2.6, and put*

$$F(z) = \int_0^\infty \left(HK_X(t;z) - \frac{1}{v_X}\right) dt \quad (z\in X).$$

For any $\varepsilon \in (0,\lambda_{X,1})$ and $\delta\in(0,\ell_{X,0})$, we then have the estimate

$$\sup_{z\in X} |F(z)| \leqslant \frac{B_{X,\varepsilon,\delta}}{4\pi},$$

where $B_{X,\varepsilon,\delta}$ is as in Theorem 4.5.

Proof. The result follows immediately from the argument given in the proof of Theorem 4.5, taking into account that for the stated choices of ε and δ, we have

$$HK_X(t;z) - \frac{1}{v_X} = K_X^{\varepsilon,\delta}(t;z,z).$$ □

Proposition 4.7. *Let $\lambda_{X,1}$ and d_X be as in § 2.6, and $H(z)$ as in Lemma 3.5. For any Riemann surface X of genus $g_X > 1$, we then have the estimate*

$$0 \leqslant \frac{\pi}{2g_X^2}\int_X H(z)\Delta_X H(z)\mu_{\mathrm{hyp}}(z) \leqslant \frac{\pi(d_X+1)^2 v_X}{2g_X^2\lambda_{X,1}}.$$

Proof. With $H(z)$ as in Lemma 3.5, we have as in Corollary 4.6

$$F(z) = \int_0^\infty \left(HK_X(t;z) - \frac{1}{v_X}\right) dt = H(z) + \frac{c_X - 1}{v_X}.$$

It is elementary to show that

$$\int_X H(z)\Delta_X H(z)\mu_{\mathrm{hyp}}(z) = \int_X F(z)\Delta_X F(z)\mu_{\mathrm{hyp}}(z),$$

since

$$\Delta_X F(z) = \Delta_X H(z) \quad \text{and} \quad \int_X \Delta_X F(z)\mu_{\mathrm{hyp}}(z) = 0.$$

Therefore, it suffices to prove that

$$0 \leqslant \int_X F(z)\Delta_X F(z)\mu_{\mathrm{hyp}}(z) \leqslant \frac{(d_X+1)^2 v_X}{\lambda_{X,1}},$$

which is precisely the statement from [JK05, Proposition 4.1], which we refer to for further details. □

Theorem 4.8. *Let $\lambda_{X,1}$, $\ell_{X,0}$, c_X, and d_X be as in § 2.6. For any $\varepsilon \in (0, \lambda_{X,1})$, $\delta \in (0, \ell_{X,0})$, and $z, w \in X$, we then have the estimate*

$$|g_{\mathrm{can}}(z,w) - g_{\mathrm{hyp}}(z,w)| \leqslant \frac{B_{X,\varepsilon,\delta}}{g_X} + \frac{4\pi|c_X - 1|}{g_X v_X} + \frac{\pi(d_X+1)^2 v_X}{g_X^2 \lambda_{X,1}},$$

where $B_{X,\varepsilon,\delta}$ is as in Theorem 4.5.

Proof. By combining Theorem 3.8 and Proposition 4.7, we get

$$|g_{\mathrm{can}}(z,w) - g_{\mathrm{hyp}}(z,w)| \leqslant \frac{4\pi}{g_X} \sup_{z \in X} |H(z)| + \frac{\pi(d_X+1)^2 v_X}{g_X^2 \lambda_{X,1}}.$$

By the definition of $H(z)$ and $F(z)$, we now derive from Corollary 4.6

$$\sup_{z\in X} |H(z)| \leqslant \sup_{z \in X} |F(z)| + \frac{|c_X - 1|}{v_X} \leqslant \frac{B_{X,\varepsilon,\delta}}{4\pi} + \frac{|c_X - 1|}{v_X}.$$

By combining the above estimates, the theorem is proved. □

Theorem 4.9. *Let $\lambda_{X,1}$, $\ell_{X,0}$, c_X, and d_X be as in § 2.6. For any $\varepsilon \in (0, \lambda_{X,1})$, $\delta \in (0, \ell_{X,0})$, and $z, w \in X$, we then have the estimate*

$$\Bigg| g_{\mathrm{can}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta; z,w)} g_{\mathbb{H}}(z, \gamma w) \Bigg| \leqslant A_{X,\varepsilon,\delta},$$

where

$$A_{X,\varepsilon,\delta} = B_{X,\varepsilon,\delta} + \frac{B_{X,\varepsilon,\delta}}{g_X} + \frac{4\pi|c_X - 1|}{g_X v_X} + \frac{\pi(d_X+1)^2 v_X}{g_X^2 \lambda_{X,1}}$$

with $B_{X,\varepsilon,\delta}$ as in Theorem 4.5.

Proof. Since

$$\Bigg| g_{\mathrm{can}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta; z,w)} g_{\mathbb{H}}(z, \gamma w) \Bigg| \leqslant |g_{\mathrm{can}}(z,w) - g_{\mathrm{hyp}}(z,w)| + \Bigg| g_{\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta; z,w)} g_{\mathbb{H}}(z, \gamma w) \Bigg|,$$

the claim follows immediately by combining the bounds in Theorems 4.5 and 4.8. □

Remark 4.10. Note that Theorem 4.5 follows from elementary considerations in hyperbolic geometry. In order to prove Theorem 4.8, we needed the representation of the canonical Green's function in terms of the hyperbolic Green's function, which we proved in Theorem 3.8. All quantities from hyperbolic geometry that appear in the definition for $A_{X,\varepsilon,\delta}$ are well-known invariants except for c_X. However, it has been recognized for some time that either c_X or $Z'_X(1)$ are global hyperbolic invariants, which determine the complexity of the Riemann surface X.

5. Uniform bounds for families of Riemann surfaces

In this section, we study the upper bounds obtained in Theorems 4.5, 4.8, and 4.9 for certain sequences of compact Riemann surfaces. For the purpose of notational convenience, we use the following definition.

Definition 5.1. Let $\{X_N\}$, indexed by $N \in \mathcal{N} \subseteq \mathbb{N}$, be a sequence of compact Riemann surfaces of genus $g_{X_N} > 1$ equipped with the hyperbolic metric μ_{hyp}. We say that the sequence is *admissible*, if it is of one of the following two types:

(i) $\mathcal{N} = \mathbb{N}$, and for each $N \in \mathcal{N}$, the compact Riemann surface X_{N+1} is a finite degree cover of X_N;

(ii) the sequence is the subsequence of one of the families of modular curves $\{X_0(N)\}$, $\{X_1(N)\}$, or $\{X(N)\}$ consisting of those modular curves having genus bigger than one.

Denote by $p_0 \in \mathcal{N}$ the minimal element in case (i), that is, $p_0 = 0$, and the smallest prime in $\mathcal{N}$ in case (ii).

Remark 5.2. In this section, we study the bounds stated in Theorems 4.5, 4.8, and 4.9 for admissible sequences of compact Riemann surfaces. The purpose is to determine the extent to which the derived bounds are uniform for all elements in the admissible sequence. We denote any bound by O_{p_0}, which signifies an implied constant being universal for all Riemann surfaces in the admissible sequence $\{X_N\}_{N\in\mathcal{N}}$ under consideration. Similar notation is used to denote constants, say $c(p_0)$, whose dependence is universal for all elements in the admissible sequence.

Lemma 5.3. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of compact Riemann surfaces. Then, the hyperbolic invariants defined in § 2.6 satisfy the following bounds:*

(a) *there is a constant $C_1 = C_1(p_0) > 0$ such that for all $N \in \mathcal{N}$, we have $\ell_{X_N,0} \geqslant C_1$;*

(b) *there is a constant $C_2 = C_2(p_0) > 0$ such that for all $N \in \mathcal{N}$, we can take $r_{X_N} = C_2$;*

(c) *there is a constant $C_3 = C_3(p_0) > 0$ such that for all $N \in \mathcal{N}$, we have $d_{X_N} \leqslant C_3$;*

(d) *there is a constant $C_4 = C_4(p_0) > 0$ such that for all $N \in \mathcal{N}$, we have $C^{HK}_{X_N} \leqslant C_4$;*

(e) *there is a constant $C_5 = C_5(p_0) > 0$ such that for all $N \in \mathcal{N}$, we have $c_{X_N} \leqslant C_5 \cdot g_{X_N}/\lambda_{X_N,1}$.*

Proof. Let us first prove the results for an admissible sequence of compact Riemann surfaces of type (i) and then consider the case of an admissible sequence of type (ii), that is, the sequences of modular curves. In order to prove the lemma for an admissible sequence of compact Riemann surfaces of type (i), we have to consider the pair of compact Riemann surfaces X_N ($N \in \mathbb{N}$) and X_0, where X_N is a finite degree cover of X_0.

By taking $C_1 = \ell_{X_0,0}$, part (a) follows from the observation that $\ell_{X_N,0} \geqslant \ell_{X_0,0}$. Since the only requirement on r_{X_N} is that $r_{X_N} \in (0, \ell_{X_N,0})$, part (b) follows from part (a) by choosing, for example, $C_2 = C_1/2$. The bound in part (c) is stated as the main theorem in [Don96] (see also [JK04]). For part (d), we argue as follows. As usual, we have $X_N = \Gamma_N\backslash\mathbb{H}$ and $X_0 = \Gamma_0\backslash\mathbb{H}$ for suitable subgroups Γ_N and Γ_0 in $\mathrm{PSL}_2(\mathbb{R})$. Since Γ_N is a subgroup of Γ_0, we have the trivial bound $K_{X_N}(t;z) \leqslant K_{X_0}(t;z)$, from which part (d) follows by taking $C_4 = C^{HK}_{X_0}$. Finally, for part (e), we refer to the main results in [JK01], where upper and lower bounds for c_{X_N} are proved. The upper bound stated here comes from the proof of Theorem 4.7 in [JK01]. In particular, one has to use the top displayed line on p. 21 of [JK01] with $\delta = 5$ and $\varepsilon \in (0, \alpha)$, $\alpha = \min\{7/64, \lambda_{X_N,1}\}$. From this point on, one then uses the following bounds: the number of small eigenvalues less than ε is one, namely the zero eigenvalue; the number of elements in $H(\Gamma_N)$ of length at most five is bounded by $O_{p_0}(g_{X_N})$, as argued in the proof of Theorem 4.11 in [JK01]; and the constant $C_{X_N,\varepsilon}$ defined on p. 20 in [JK01] is bounded by $O_{p_0}(g_{X_N})$, which is proved by combining the main result in [JK02] and the well-known estimate that the number of eigenvalues less than $\frac{1}{4}$ is $O(g_{X_N})$, with an implied constant that is universal. We also refer to [JK05, Proposition 4.2], for a proof of part (e).

Let us now consider the stated assertions for the admissible sequences of modular curves. For this, complete proofs of parts (a), (c), and (e) are given in [JK05, Proposition 5.3] for the sequence of modular curves $\{X_0(N)\}_{N\in\mathcal{N}}$, while part (b) again follows directly from part (a). The proof of all parts of Proposition 5.3 in [JK05] extend with only notational changes to the other sequences of modular curves $\{X_1(N)\}_{N\in\mathcal{N}}$ (respectively $\{X(N)\}_{N\in\mathcal{N}}$); one only has to observe that, if p is a prime in $\mathcal{N}$, then $\deg(X_1(p_0p)/X_1(p_0)) = O(g_{X_1(p)})$ (respectively $\deg(X(p_0p)/X(p_0)) = O(g_{X(p)})$), with implied constants that are universal. The verification of the latter claim follows directly from known formulas (see, e.g., [Shi94]).

Finally, it remains to prove part (d) for the sequences of modular curves. We give a proof of part (d) for the sequence of modular curves $\{X_0(N)\}_{N\in\mathcal{N}}$. For a prime $p > p_0$ in $\mathcal{N}$, consider the finite-degree cover $X_0(p_0p) \longrightarrow X_0(p)$. Since

$$K_{X_0(p)}(t;z,w) = \sum_{\gamma\in\Gamma_0(p_0p)\backslash\Gamma_0(p)} K_{X_0(p_0p)}(t;z,\gamma w)$$

by the existence and uniqueness of heat kernels, we find

$$K_{X_0(p)}(t;z) \leqslant \frac{1}{2} \sum_{\gamma\in\Gamma_0(p_0p)\backslash\Gamma_0(p)} (K_{X_0(p_0p)}(t;z) + K_{X_0(p_0p)}(t;\gamma z)).$$

This shows

$$C^{HK}_{X_0(p)} \leqslant (p_0+1)\cdot C^{HK}_{X_0(p_0p)}.$$

Using the trivial inequality $C^{HK}_{X_0(p_0p)} \leqslant C^{HK}_{X_0(p_0)}$, we get $C^{HK}_{X_0(p)} \leqslant (p_0+1)\cdot C^{HK}_{X_0(p_0)}$ for all primes $p\in\mathcal{N}$. The claimed bound for $C^{HK}_{X_0(N)}$ now follows by the same principle as used in the proof of Proposition 5.3 in [JK05]. The proof for the other sequences of modular curves $\{X_1(N)\}_{N\in\mathcal{N}}$ (respectively $\{X(N)\}_{N\in\mathcal{N}}$) is analogous. □

Remark 5.4. The proofs of parts (a), (b), (c), and (d) in Lemma 5.3 are elementary and follow from standard arguments in hyperbolic geometry and analysis. Part (e) is considerably more involved. As can be seen from [JK01, JK05], the bound stated in part (e) ultimately reduces to two bounds: the number of eigenvalues less than $\frac{1}{4}$ and the implied constant in the error term of the prime geodesic theorem. The latter constant is the focus of study in [JK02].

Theorem 5.5. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of compact Riemann surfaces. For any $\delta>0$, $\varepsilon>0$ and $N\in\mathcal{N}$, we then have the estimate*

$$g_{\mathrm{hyp},X_N}(z,w) - \sum_{\gamma\in S_{\Gamma_N}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w) - \sum_{0<\lambda_{X_N,n}<\varepsilon} \frac{4\pi}{\lambda_{X_N,n}}\varphi_{X_N,n}(z)\varphi_{X_N,n}(w) = O_{p_0,\varepsilon,\delta}(1).$$

Here, we have written $g_{\mathrm{hyp},X_N}(z,w)$ instead of $g_{\mathrm{hyp}}(z,w)$ for the hyperbolic Green's function on $X_N=\Gamma_N\backslash\mathbb{H}$ in order to emphasize the dependence on X_N.

Proof. The bound follows directly by combining Theorem 4.5 with parts (b) and (d) of Lemma 5.3, as well as the definition of δ_X in terms of r_X, e.g., by simply taking $\delta_X = \max\{\delta_0, 4r_X+5\}+1$ (see § 2.6). □

Theorem 5.6. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of compact Riemann surfaces. For any $N\in\mathcal{N}$, we then have the estimate*

$$g_{\mathrm{can},X_N}(z,w) - g_{\mathrm{hyp},X_N}(z,w) = O_{p_0}\left(\frac{1}{g_{X_N}}\left(1+\frac{1}{\lambda_{X_N,1}}\right)\right).$$

Here, we have written $g_{\mathrm{can},X_N}(z,w)$ instead of $g_{\mathrm{can}}(z,w)$ for the canonical Green's function on X_N.

Proof. Taking $\varepsilon<1$, using parts (b) and (d) of Lemma 5.3, and choosing $\delta = C_1/2$ with the constant C_1 of Lemma 5.3(a), we derive from the explicit formula for $B_{X_N,\varepsilon,\delta}$ as stated in Theorem 4.5 that

$$B_{X_N,\varepsilon,\delta} = O_{p_0}\left(1+\frac{1}{\varepsilon}\right).$$

Now we turn to the bound given in Theorem 4.8. Then, by taking $\varepsilon = \min\{\frac{1}{2}, \lambda_{X_N,1}/2\}$, and using parts (c), (e) of Lemma 5.3, the result follows. □

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Corollary 5.7. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of compact Riemann surfaces. For any $\delta>0$ and $N\in\mathcal{N}$, we then have the estimate*

$$g_{\mathrm{can},X_N}(z,w)-\sum_{\gamma\in S_{\Gamma_N}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)=O_{p_0,\delta}\left(1+\frac{1}{\lambda_{X_N,1}}\right);$$

again, we have written $g_{\mathrm{can},X_N}(z,w)$ instead of $g_{\mathrm{can}}(z,w)$ for the canonical Green's function on $X_N=\Gamma_N\backslash\mathbb{H}$.

Proof. The claim follows by combining Theorem 5.5 with $\varepsilon=\min\{\frac{1}{2},\lambda_{X_N,1}/2\}$ with Theorem 5.6 after having used the triangle inequality. □

Corollary 5.8. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of compact Riemann surfaces. For any $N\in\mathcal{N}$, we then have the estimate*

$$\max_{z\in X_N}|\phi_{\mathrm{Ar}}(z)|=O_{p_0}\left(1+\frac{1}{\lambda_{X_N,1}}\right);$$

here the C^∞-function ϕ_{Ar} has been introduced in (1) in § 2.2.

Proof. Using the known formula for $g_{\mathbb{H}}(z,w)$, as stated in § 2.3, we can write

$$g_{\mathrm{can},X_N}(z,w)-g_{\mathbb{H}}(z,w)=g_{\mathrm{can},X_N}(z,w)+\log|z-w|^2-\log|z-\bar{w}|^2.$$

Therefore, when using the definition of the residual metrics as given in § 2.2, we then have

$$\begin{aligned}\lim_{w\to z}(g_{\mathrm{can},X_N}(z,w)-g_{\mathbb{H}}(z,w))&=\log\|dz\|^2_{\mathrm{can,res}}-\log(2\,\mathrm{Im}(z))^2\\&=\log\left(\frac{\|dz\|^2_{\mathrm{can,res}}}{\mathrm{Im}^2(z)}\right)-\log(4)\\&=\log\left(\frac{\mu_{\mathrm{hyp}}(z)}{\mu_{\mathrm{Ar}}(z)}\right)-\log(4)=-\phi_{\mathrm{Ar}}(z)-\log(4).\end{aligned}$$

From this, the asserted result follows directly from Corollary 5.7 by taking $\delta=C_1/2$ (see Lemma 5.3(a)). □

Lemma 5.9. *Let X be any of the modular curves $X_0(N)$, $X_1(N)$, or $X(N)$ having genus bigger than one. Then, there is a constant $c>0$ satisfying $\lambda_{X,1}\geqslant c$.*

Proof. We recall from [Bro99, Theorem 3.1], that

$$\liminf_{N\to\infty}\lambda_{X(N),1}\geqslant\frac{5}{36}.$$

Hence, there is a constant $c>0$, independent of N, such that $\lambda_{X(N),1}\geqslant c$ for all $N>N_0$, for some N_0, thus, the claim holds for the modular curves $X(N)$ of genus bigger than one. Since $X(N)$ is a cover of $X_0(N)$ (respectively $X_1(N)$), the Raleigh quotient method for estimating eigenvalues, which shows that the smallest eigenvalue decreases through covers, now implies that $\lambda_{X(N),1}\leqslant\lambda_{X_0(N),1}$ (respectively $\lambda_{X(N),1}\leqslant\lambda_{X_1(N),1}$), which completes the proof. □

Corollary 5.10. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of compact Riemann surfaces of type (ii), that is, of modular curves. For any $N\in\mathcal{N}$, we then have the following estimates:*

(a)

$$\max_{z,w\in X_N}|g_{\mathrm{can},X_N}(z,w)-g_{\mathrm{hyp},X_N}(z,w)|=O_{p_0}\left(\frac{1}{g_{X_N}}\right);$$

(b)

$$\max_{z,w\in X_N}\left|g_{\mathrm{can},X_N}(z,w)-\sum_{\gamma\in S_{\Gamma_N}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\right| = O_{p_0,\delta}(1) \quad (\delta>0);$$

(c)

$$\max_{z\in X_N}|\phi_{\mathrm{Ar}}(z)| = O_{p_0}(1).$$

Proof. Combine Lemma 5.9 with the previous results, namely Theorem 5.6 for part (a), Corollary 5.7 for part (b), and Corollary 5.8 for part (c). □

Remark 5.11. It is immediate from Theorem 5.6 and Corollaries 5.7 and 5.8 that Corollary 5.10 holds for any admissible sequence, which admits a universal non-zero arbitrary cover X_1 of X_0, we claim that

$$\frac{1}{\lambda_{X_1,1}} = O_{X_0}(g_{X_1}^2).$$

For this, one applies [Cha84, Theorem 14, p. 112], which reduces the problem to that of bounding an isoperimetric constant associated to X_1 as a function of the degree $\deg(X_1/X_0)$, and the bound needed to prove this claim follows immediately from the definition of the isoperimetric constant in question (see also [Cha84, Theorem 12, p. 111 and Definition 5, p. 110]).

Remark 5.12. As stated in the introduction, this paper was motivated by a question from Edixhoven who asked for bounds for the canonical Green's function on $X_1(N)$. Recall that, as stated in the proof of Lemma 4.4, the hyperbolic Green's function $g_{\mathbb{H}}(z,w)$ $(z,w\in\mathbb{H})$ is expressible in terms of elementary functions. Combining this expression with Corollary 5.10(b) provides the upper and lower bounds sought by Edixhoven.

Remark 5.13. In a slightly more general situation, one can restrict attention to arbitrary compact subsets of X_N, and consider admissible sequences of non-compact hyperbolic surfaces. Beginning with Lemma 4.2, the constant r_{X_N} would then be bounded away from zero with a lower bound that depends on the subset of X_N under consideration. The resulting bound for hyperbolic heat kernels and hyperbolic Green's functions then can be applied throughout the subsequent calculations. By doing so, one can address the problem of understanding the asymptotic behavior of the canonical Green's function for a degenerating family of algebraic curves approaching the Deligne–Mumford boundary of the moduli space of stable curves of a fixed positive genus, as first studied in [Jor90].

Remark 5.14. In his recent work [Küh05], Kühn used the analysis of the present paper and from [JK04] to derive bounds for the arithmetic self-intersection number of the relative dualizing sheaf on an arithmetic surface. By revisiting the analytic component of the computations in [AU97], he is able to both simplify the method of proof given in [AU97] and to provide a technique which extends to the modular curves $X_1(N)$ and $X(N)$.

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Annotations from cmat0199.pdf

Please reply to these questions on the relevant page of the proof; please do not write on this page.

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Annotation 1

Author: line 44.

Spelling `Laurant' OK here or should this be `Laurent'?

Page 22

Annotation 1

Author: ref. Fal84.

Please cite in text or delete from the reference list.

Annotation 2

Aurthor: refs JK05 and Kuh05.

Please update if possible.