HU Berlin Winter Term 2012/13

Exercises BMS Basic Course
Algebra II (Commutative Algebra)

Prof. Dr. J. Kramer

To be handed in on January 21st after the lecture

Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.

Exercise sheet 12 (40 points)

Exercise 1 (10 points)

In the ring $\mathbb{Z}[\sqrt{-5}] := \{a + b\sqrt{-5} \,|\, a, b \in \mathbb{Z}\}$, the *norm map* $N : \mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}$ is given by $N(a + b\sqrt{-5}) := a^2 + 5b^2$.

(a) Show that $N(\alpha \cdot \beta) = N(\alpha) \cdot N(\beta)$ for $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]$.

(b) Using the norm map, find the units of the ring $\mathbb{Z}[\sqrt{-5}]$.

(c) Show that the element 6 has two distinct decompositions into irreducible elements in the ring $\mathbb{Z}[\sqrt{-5}]$.

(d) Compute the prime ideal decomposition of the principal ideal (6) in the ring $\mathbb{Z}[\sqrt{-5}]$.

Exercise 2 (10 points)

Let $\mathfrak{p} := (X, Y)$, $\mathfrak{q} := (X, Z)$, and $\mathfrak{m} := (X, Y, Z)$ be three ideals in the ring $k[X, Y, Z]$, where k is a field. Let $\mathfrak{a} := \mathfrak{p}\mathfrak{q}$. Then, show that $\mathfrak{a} = \mathfrak{p} \cap \mathfrak{q} \cap \mathfrak{m}^2$ is a minimal primary decomposition of $\mathfrak{a}$ and determine the isolated and embedded prime ideals.

Exercise 3 (10 points)

Let A denote the ring of continuous real-valued functions on $[0, 1]$. Then, show that the zero ideal in A is not decomposable.

Exercise 4 (10 points)

Let $\mathcal{F}$ be a presheaf of rings on a topological space X and let $x \in X$. The *stalk* $\mathcal{F}_x$ *of* $\mathcal{F}$ *at* x is defined as

$$\mathcal{F}_x := \varinjlim_{\substack{U \subseteq X \text{ open} \\ U \ni x}} \mathcal{F}(U),$$

where the direct limit is taken over all neighborhoods U of x, via the restriction maps of the presheaf $\mathcal{F}$.

Now, consider the presheaf $\mathcal{F}$ of real-valued differentiable functions on the open unit disc $\{z \in \mathbb{C} \,|\, |z| < 1\}$ (in the classical topology). Show that the stalk of $\mathcal{F}$ at the origin is a local ring.