Exercises

Algebra II (Commutative Algebra)

Prof. Dr. J. Kramer

To be handed in on December 17th after the lecture

Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.

Exercise sheet 9 (40 + 10 points)

Exercise 1 (10 points)

Let A be a commutative ring with 1 and S a multiplicatively closed subset of A. Let M' and M'' be A-submodules of an A-module M. Then, prove the following assertions:

(a) $S^{-1}(M' + M'') = S^{-1}M' + S^{-1}M''$.

(b) $S^{-1}(M' \cap M'') = S^{-1}M' \cap S^{-1}M''$.

(c) $S^{-1}(M/M') \cong (S^{-1}M)/(S^{-1}M')$.

(d) $S^{-1}A \otimes_A M \cong S^{-1}M$.

Exercise 2 (10 points)

Prove the following assertions:

(a) Let A be a commutative local ring with 1 and $\mathfrak{m}$ its unique maximal ideal. Then, show that $A_\mathfrak{m} \cong A$.

(b) Let $A = \mathbb{Z}[X]/(X^2 - 1)$ and $\mathfrak{p}$ the ideal generated by the element $(X + 1)$ in A. Then, $\mathfrak{p}$ is a prime ideal and we have $A_\mathfrak{p} \cong \mathbb{Q}$.

Exercise 3 (10 points)

Let A be a commutative ring with 1, and let M, N be A-modules, and $f : M \to N$ an A-module homomorphism. A property $\mathcal{P}$ of M (or f) is said to be a *local property* of M (or f), if the following holds: M (or f) has $\mathcal{P}$ if and only if $M_\mathfrak{p}$ (or the induced $A_\mathfrak{p}$-module homomorphism $f_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$) has $\mathcal{P}$ for every $\mathfrak{p} \in \operatorname{Spec}(A)$.

(a) An A-module M is called *flat* if tensoring with M is an exact functor. Prove that flatness is a local property of M.

(b) Let $f : M \to N$ be an A-module homomorphism. Prove that the injectivity of f is a local property of f.

(c) Let $f : M \to N$ be an A-module homomorphism. Prove that the surjectivity of f is a local property of f.

Exercise 4 (10 points)

Let A be a Noetherian ring with 1. Prove that the ring $A[[X]]$ of formal power series in the variable X over A is also a Noetherian ring.

Exercise 5* (10 points)

Let A be a commutative ring with 1.

(a) Show that every surjective endomorphism of a Noetherian A-module is an isomorphism.

(b) Give an example of an injective endomorphism of a Noetherian A-module which is not an isomorphism.