HU Berlin Winter Term 2012/13

Exercises
Algebra II (Commutative Algebra)

Prof. Dr. J. Kramer

To be handed in on October 22nd after the lecture

Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.

Exercise sheet 1 (30 points)

Exercise 1 (10 points)

Let A be a commutative ring with 1. Prove the following assertions:

(a) The ideal $\mathfrak{p} \subseteq A$ is a prime ideal if and only if $A/\mathfrak{p}$ is an integral domain.

(b) The ideal $\mathfrak{m} \subseteq A$ is a maximal ideal if and only if $A/\mathfrak{m}$ is a field.

(c) Let $A \neq 0$. Then, there exists at least one maximal ideal $\mathfrak{m} \subseteq A$.

Exercise 2 (10 points)

Let A be a commutative ring with 1 and let $\mathfrak{a} \subseteq A$ be an ideal.

(a) Prove that the set

$$\mathfrak{N}(A) := \{a \in A \,|\, \exists\, n \in \mathbb{N}_{>0} : a^n = 0\}$$

of nilpotent elements of A is an ideal of A, the so-called *nilradical of* A.

(b) Prove that the set

$$\sqrt{\mathfrak{a}} := \mathfrak{r}(\mathfrak{a}) := \{a \in A \,|\, \exists\, n \in \mathbb{N}_{>0} : a^n \in \mathfrak{a}\}$$

is an ideal of A, the so-called *radical of* $\mathfrak{a}$.

(c) Let $\pi : A \longrightarrow A/\mathfrak{a}$ denote the canonical projection. Show that

$$\mathfrak{r}(\mathfrak{a}) = \pi^{-1}\big(\mathfrak{N}(A/\mathfrak{a})\big).$$

Exercise 3 (10 points)

Let A be a commutative ring with 1. Let $\mathfrak{a} \subseteq A$ be an ideal and let $\mathfrak{r}(\mathfrak{a})$ denote its radical. Prove that the following properties hold:

(a) $\mathfrak{r}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(\mathfrak{a})$.

(b) $\mathfrak{r}(\mathfrak{a} \cdot \mathfrak{b}) = \mathfrak{r}(\mathfrak{a} \cap \mathfrak{b}) = \mathfrak{r}(\mathfrak{a}) \cap \mathfrak{r}(\mathfrak{b})$.

(c) $\mathfrak{r}(\mathfrak{a}) = (1) \iff \mathfrak{a} = (1)$.

(d) $\mathfrak{r}(\mathfrak{a} + \mathfrak{b}) = \mathfrak{r}(\mathfrak{r}(\mathfrak{a}) + \mathfrak{r}(\mathfrak{b}))$.

(e) For $n \in \mathbb{N}_{>0}$ and every prime ideal $\mathfrak{p} \subseteq A$, we have $\mathfrak{r}(\mathfrak{p}^n) = \mathfrak{p}$.