HU Berlin Winter Term 2012/13

Exercises
Algebra II (Commutative Algebra)

Prof. Dr. J. Kramer

To be handed in on October 29th after the lecture

Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.

Exercise sheet 2 (40 points)

Exercise 1 (10 points)

Let A, B be commutative rings with 1 and let $f : A \to B$ be a ring homomorphism. Let $\mathfrak{a} \subseteq A$ and $\mathfrak{b} \subseteq B$ be ideals.

(a) Show that $f^{-1}(\mathfrak{b})$ is an ideal of A, called the *contraction* $\mathfrak{b}^{\mathfrak{c}}$ of $\mathfrak{b}$. Show that $\mathfrak{b}^{\mathfrak{c}}$ is a prime ideal, if $\mathfrak{b}$ is a prime ideal.

(b) Show that $f(\mathfrak{a})$ is not necessarily an ideal of B. We define the *extension* $\mathfrak{a}^{\mathfrak{e}}$ of $\mathfrak{a}$ to be the ideal generated by $f(\mathfrak{a})$ in B. Show that $\mathfrak{a}^{\mathfrak{e}}$ need not be a prime ideal, if $\mathfrak{a}$ is a prime ideal.

(c) Show that $\mathfrak{a} \subseteq \mathfrak{a}^{\mathfrak{ec}}$ and $\mathfrak{b} \supseteq \mathfrak{b}^{\mathfrak{ce}}$. Further, prove that $\mathfrak{a}^{\mathfrak{e}} = \mathfrak{a}^{\mathfrak{ece}}$ and $\mathfrak{b}^{\mathfrak{c}} = \mathfrak{b}^{\mathfrak{cec}}$.

Exercise 2 (Prime Avoidance Lemma) (10 points)

Let A be a commutative ring with 1.

(a) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ ($n \in \mathbb{N}$, $n \geq 1$) be prime ideals of A and let $\mathfrak{a}$ be an ideal contained in $\bigcup_{j=1}^n \mathfrak{p}_j$. Prove that $\mathfrak{a} \subseteq \mathfrak{p}_j$ for at least one j.

(b) Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ ($n \in \mathbb{N}$, $n \geq 1$) be ideals of A and let $\mathfrak{p}$ be a prime ideal containing $\bigcap_{j=1}^n \mathfrak{a}_j$. Prove that $\mathfrak{a}_j \subseteq \mathfrak{p}$ for at least one j. Further, if $\mathfrak{p} = \bigcap_{j=1}^n \mathfrak{a}_j$, then $\mathfrak{p} = \mathfrak{a}_j$ for some j.

Exercise 3 (10 points)

Let $A := C([0,1], \mathbb{R})$ denote the ring of continuous, real-valued functions on $[0,1]$.

(a) Let $x \in [0,1]$. Show that the set

$$\mathfrak{m}_x := \{\, f \in A \mid f(x) = 0\}$$

is a maximal ideal in A.

(b) Let $f_1, \ldots, f_n \in A$ ($n \in \mathbb{N}$, $n > 1$) be functions without any common zero. Show that $f_1^2 + \cdots + f_n^2$ is a unit in A.

(c) Let $\mathfrak{a} \subset A$ be a proper ideal and let

$$V(\mathfrak{a}) := \{a \in [0,1] \,|\, f(a) = 0 \text{ for every } f \in \mathfrak{a}\}$$

be the set of common zeros of all functions in $\mathfrak{a}$. Prove that $V(\mathfrak{a}) \neq \emptyset$.

(d) Deduce using (c) that $\mathrm{Max}(A) = \{\mathfrak{m}_x \,|\, x \in [0,1]\}$ and show that the map

$$[0,1] \to \mathrm{Max}(A),$$

given by the assignment $x \mapsto \mathfrak{m}_x$, is a bijection.

Exercise 4 (10 points)
Let A be a commutative ring with 1. We consider the set

$$\mathrm{Spec}(A) = \{\mathfrak{p} \subset A \,|\, \mathfrak{p} \text{ prime ideal}\}$$

of all prime ideals of A. For a subset $S \subseteq A$, let

$$V(S) := \{\mathfrak{p} \in \mathrm{Spec}(A) \,|\, \mathfrak{p} \supseteq S\}$$

denote the set of all prime ideals of A which contain S.

(a) Prove that $V(S) = V(\mathfrak{a}) = V(\mathfrak{r}(\mathfrak{a}))$, where $\mathfrak{a}$ denotes the ideal generated by S in A.

(b) Show that the sets $V(S)$, where S ranges over the subsets of A, satisfy the axioms for closed sets in $\mathrm{Spec}(A)$. Conclude that $\mathrm{Spec}(A)$ is a topological space.
The resulting topology on $\mathrm{Spec}(A)$ is called the *Zariski topology*; the topological space $\mathrm{Spec}(A)$ is called the *prime spectrum of* A.