HU Berlin Winter Term 2012/13

Exercises
Algebra II (Commutative Algebra)

Prof. Dr. J. Kramer

To be handed in on November 5th after the lecture

Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.

Exercise sheet 3 (50 points)

Exercise 1 (10 points)

Let A be a commutative ring with 1. Prove the following assertions:

(a) If every $x \in A$ satisfies $x^n = x$ for some integer $n \geq 2$, then show that every prime ideal in A is a maximal ideal.

(b) An element $x \in A$ is called an *idempotent*, if $x^2 = x$. Show that a local ring A has no idempotents different from 0 and 1.

Exercise 2 (10 points)

Let A be a commutative ring with 1. Let $A[X]$ be the ring of polynomials in X with coefficients in A and let $f = a_n X^n + \ldots + a_1 X + a_0 \in A[X]$. Prove the following assertions:

(a) f is a unit in $A[X]$ if and only if a_0 is a unit in A and $a_1, \ldots, a_n$ are nilpotent.

(b) f is nilpotent if and only if $a_0, a_1, \ldots, a_n$ are nilpotent.

(c) f is a zero-divisor if and only if there exists $a \in A$, $a \neq 0$, such that $af = 0$.

Exercise 3 (10 points)

Prove the following assertions:

(a) Show that the $\mathbb{Z}$-module $\mathbb{Q}$ is not free. Furthermore, prove that $\mathbb{Z}/n\mathbb{Z}$ is not a free $\mathbb{Z}$-module, but a free $\mathbb{Z}/n\mathbb{Z}$-module.

(b) Let A be a commutative ring with 1. Show that, if every finitely generated A-module M is free, then $A = \{0\}$ or A is a field.

(c) Find a commutative ring A with 1, an A-module M, and a set of generators $\{x_1, \ldots, x_n\}$ of M, such that $\{x_1, \ldots, x_n\}$ is not minimal, but such that $\{x_1, \ldots, x_{n-1}\}$ does not generate M anymore.

Please turn over !

Exercise 4 (10 points)

Let A be a commutative ring with 1. An ideal $\mathfrak{a} \subsetneq A$ is called an *idempotent ideal* if $\mathfrak{a}^2 = \mathfrak{a}$. Let $\mathfrak{a} \subsetneq A$ be a finitely generated idempotent ideal of A. Then, show that there exists an element $e \in \mathfrak{a}$ such that $e^2 = e$ and $\mathfrak{a} = eA$ (i.e., $\mathfrak{a}$ is a principal ideal generated by the element e).

Exercise 5 (10 points)

Let A be a commutative ring with 1. For $f \in A$, we define the *distinguished* or *basic set*

$$D(f) := \mathrm{Spec}(A) \setminus V(f)$$

to be the complement of $V(f)$ in $\mathrm{Spec}(A)$. Show that the sets $D(f)$ $(f \in A)$ are open and that they form a basis of open sets for the Zariski topology of $\mathrm{Spec}(A)$.
Furthermore, prove that for $f, g \in A$ we have:

(a) $D(f) \cap D(g) = D(f \cdot g)$.

(b) $D(f) = \emptyset \iff f$ is nilpotent.

(c) $D(f) = X \iff f \in A^\times$.

(d) $D(f) = D(g) \iff \mathfrak{r}(f) = \mathfrak{r}(g)$.
Here, $\mathfrak{r}(f)$ denotes the radical of the principal ideal (f).