HU Berlin Winter Term 2012/13

Exercises
Algebra II (Commutative Algebra)

Prof. Dr. J. Kramer

To be handed in on November 19th after the lecture

Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.

Exercise sheet 5 (40 points)

Exercise 1 (10 points)

Let A be a commutative ring with 1. Let $\mathbf{C}$ and $\mathbf{D}$ be chain complexes of A-modules, and let $\mathbf{f}$ and $\mathbf{g}$ be morphisms between them. The morphisms $\mathbf{f}$ and $\mathbf{g}$ are said to be *chain homotopic*, if for each $n \in \mathbb{Z}$, there is an A-module homomorphism

$$s_n : C_n \to D_{n+1} \text{ such that } f_n - g_n = \partial^{\mathbf{D}}_{n+1} s_n + s_{n-1} \partial^{\mathbf{C}}_n,$$

where $\partial^{\mathbf{C}}$ and $\partial^{\mathbf{D}}$ are the boundary maps of the chain complexes $\mathbf{C}$ and $\mathbf{D}$, respectively. If $\mathbf{f}$ and $\mathbf{g}$ are chain homotopic, then for each $n \in \mathbb{Z}$, show that the A-module homomorphism

$$f_{n*} - g_{n*} : H_n(\mathbf{C}) \to H_n(\mathbf{D})$$

is the zero map, i.e., the morphisms $\mathbf{f}$ and $\mathbf{g}$ induce the same A-module homomorphisms $H_n(\mathbf{C}) \to H_n(\mathbf{D})$, for all $n \in \mathbb{Z}$.

Exercise 2 (10 points)

Does the following statement hold true?

Let $\mathbf{C}$ and $\mathbf{D}$ be chain complexes such that $H_n(\mathbf{C}) \cong H_n(\mathbf{D})$ for all $n \in \mathbb{Z}$. Then there exists either a morphism of chain complexes $\mathbf{f} : \mathbf{C} \longrightarrow \mathbf{D}$ or a morphism of chain complexes $\mathbf{f} : \mathbf{D} \longrightarrow \mathbf{C}$ such that $H_n(\mathbf{f})$ is an isomorphism for all $n \in \mathbb{Z}$.

Give a proof or a counter-example.

Exercise 3 (10 points)

Denote by $C^0 := C^\infty(\mathbb{R}^n, \mathbb{R})$ the ring of smooth real-valued functions on $\mathbb{R}^n$. A differential form ω on $\mathbb{R}^n$ of degree r is given by

$$\omega = \sum_{\{i_1,\ldots,i_r\} \subseteq \{1,\ldots,n\}} f_{i_1,\ldots,i_r}(x_1,\ldots,x_n)\, \mathrm{d}x_{i_1} \ldots \mathrm{d}x_{i_r} \quad (f_{i_1,\ldots,i_r} \in C^0),$$

where the differentials $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ are subject to the relation

$$\mathrm{d}x_j \mathrm{d}x_k = -\mathrm{d}x_k \mathrm{d}x_j \quad (j, k \in \{1, \ldots, n\}).$$

Therefore, ω can be rewritten as

$$\omega = \sum_{1 \le i_1 < \cdots < i_r \le n} g_{i_1,\ldots,i_r}(x_1, \ldots, x_n)\, \mathrm{d}x_{i_1} \ldots \mathrm{d}x_{i_r} \quad (g_{i_1,\ldots,i_r} \in C^0).$$

Consider the set

$$C^r := \{\omega \mid \omega \text{ is a differential form of degree } r\}.$$

Furthermore, consider the $\mathbb{R}$-linear map $\mathrm{d} : C^r \longrightarrow C^{r+1}$ given by

$$\mathrm{d}\omega := \sum_{1 \le i_1 < \cdots < i_r \le n} \sum_{j=1}^{n} \frac{\partial g_{i_1,\ldots,i_r}}{\partial x_j} \mathrm{d}x_j \mathrm{d}x_{i_1} \ldots \mathrm{d}x_{i_r}.$$

(a) Show that
$$\mathbf{C} : 0 \longrightarrow C^0 \xrightarrow{\mathrm{d}} C^1 \xrightarrow{\mathrm{d}} C^2 \xrightarrow{\mathrm{d}} \ldots$$
is a cochain complex (of $\mathbb{R}$-vector spaces), i.e., $\mathrm{d}^2 = \mathrm{d} \circ \mathrm{d} = 0$.

(b) Compute the cohomology groups $H^r(\mathbf{C})$ in the case $n = 2$, for $r \in \mathbb{Z}$.

Exercise 4 (10 points)

Let X be a non-empty topological space. The space X is called *irreducible* if for any closed subsets Y_1 and Y_2 of X, the equality $X = Y_1 \cup Y_2$ implies $X = Y_1$ or $X = Y_2$.

(a) Show that X is irreducible if and only if every pair of non-empty open sets in X has a non-empty intersection. Furthermore, show that X is irreducible if and only if every non-empty open subset of X is dense in X.

(b) Let A be a commutative ring with 1. Show that $\mathrm{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.