# Exercises

# Algebra II (Commutative Algebra)

Prof. Dr. J. Kramer

To be handed in on November 26th after the lecture

Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.

## Exercise sheet 6 (40 points)

### Exercise 1 (10 points)

Let A be a commutative ring with 1. Then, prove the following assertions:

(a) Let M and N be A-modules, and let  $f:M\to N$  be an A-module homomorphism. Let

$$\mathbf{P}: \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

and

$$\mathbf{Q}: \ldots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow N \longrightarrow 0$$

be projective resolutions of M and N, respectively. Then, show that there is a morphism of chain complexes  $\mathbf{f}: \mathbf{P} \to \mathbf{Q}$  extending f, i.e., for every  $n \in \mathbb{Z}_{\geq 0}$ , there exists an A-module homomorphism  $f_n: P_n \to Q_n$  such that the diagram

commutes. Furthermore, show that any two such extensions of f are chain homotopic.

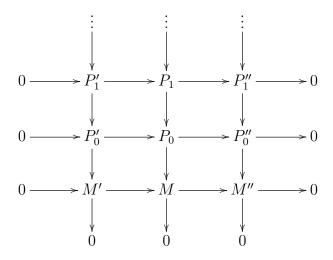
(*Hint*: Given  $n \in \mathbb{Z}_{\geq 0}$  such that for all  $0 \leq i \leq n$  the maps  $f_i : P_i \to Q_i$  exist, use the projectivity of  $P_{n+1}$  to construct a map  $f_{n+1} : P_{n+1} \to Q_{n+1}$ .)

(b) Let  $\mathfrak{M}_A$  be the category of A-modules,  $T:\mathfrak{M}_A\to\mathfrak{M}_A$  a covariant functor, and M an A-module. Then, using (a) show that the definition of the left-derived functor LT(M) of T is independent of the projective resolution of M.

#### Exercise 2 (10 points)

Let A be a commutative ring with 1. Let  $0 \to M' \to M \to M'' \to 0$  be a short exact sequence of A-modules. Then, show that there is a short exact sequence of projective

resolutions of M', M, and M'', namely P', P, and P'', respectively, such that the diagram



commutes, and for every  $n \in \mathbb{Z}_{\geq 0}$ , the short exact sequence  $0 \to P'_n \to P_n \to P''_n \to 0$  is split.

#### Exercise 3 (10 points)

Let A be a commutative ring with 1, and let M, N be A-modules. Let  $E_A(M, N)$  denote the set of equivalent classes of extensions of M by N. Then, show that there is a bijection of sets

$$E_A(M, N) \approx Ext_A^1(M, N).$$

## Exercise 4 (10 points)

Compute the  $\mathbb{Z}$ -module  $\operatorname{Ext}^1_{\mathbb{Z}}((\mathbb{Z}/n\mathbb{Z})^2,\mathbb{Z}/n\mathbb{Z})$ , and give an interpretation of the elements as extensions of  $(\mathbb{Z}/n\mathbb{Z})^2$  by  $\mathbb{Z}/n\mathbb{Z}$  in the case of n=2.