Exercises
Algebra II (Commutative Algebra)

Prof. Dr. J. Kramer

To be handed in on November 26th after the lecture

Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.

Exercise sheet 6 (40 points)

Exercise 1 (10 points)

Let A be a commutative ring with 1. Then, prove the following assertions:

(a) Let M and N be A-modules, and let $f : M \to N$ be an A-module homomorphism. Let

$$\mathbf{P} : \ \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

and

$$\mathbf{Q} : \ \ldots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow N \longrightarrow 0$$

be projective resolutions of M and N, respectively. Then, show that there is a morphism of chain complexes $\mathbf{f} : \mathbf{P} \to \mathbf{Q}$ extending f, i.e., for every $n \in \mathbb{Z}_{\geq 0}$, there exists an A-module homomorphism $f_n : P_n \to Q_n$ such that the diagram

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\ \ldots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

commutes. Furthermore, show that any two such extensions of f are chain homotopic.
(*Hint*: Given $n \in \mathbb{Z}_{\geq 0}$ such that for all $0 \leq i \leq n$ the maps $f_i : P_i \to Q_i$ exist, use the projectivity of P_{n+1} to construct a map $f_{n+1} : P_{n+1} \to Q_{n+1}$.)

(b) Let $\mathfrak{M}_A$ be the category of A-modules, $T : \mathfrak{M}_A \to \mathfrak{M}_A$ a covariant functor, and M an A-module. Then, using (a) show that the definition of the left-derived functor $LT(M)$ of T is independent of the projective resolution of M.

Exercise 2 (10 points)

Let A be a commutative ring with 1. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules. Then, show that there is a short exact sequence of projective resolutions of M', M, and M'', namely $\mathbf{P}'$, $\mathbf{P}$, and $\mathbf{P}''$, respectively, such that the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1' & \longrightarrow & P_1 & \longrightarrow & P_1'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0' & \longrightarrow & P_0 & \longrightarrow & P_0'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

commutes, and for every $n \in \mathbb{Z}_{\geq 0}$, the short exact sequence $0 \to P_n' \to P_n \to P_n'' \to 0$ is split.

Exercise 3 (10 points)

Let A be a commutative ring with 1, and let M, N be A-modules. Let $\mathrm{E}_A(M, N)$ denote the set of equivalent classes of extensions of M by N. Then, show that there is a bijection of sets

$$\mathrm{E}_A(M, N) \approx \mathrm{Ext}_A^1(M, N).$$

Exercise 4 (10 points)

Compute the $\mathbb{Z}$-module $\mathrm{Ext}_{\mathbb{Z}}^1\big((\mathbb{Z}/n\mathbb{Z})^2, \mathbb{Z}/n\mathbb{Z}\big)$, and give an interpretation of the elements as extensions of $(\mathbb{Z}/n\mathbb{Z})^2$ by $\mathbb{Z}/n\mathbb{Z}$ in the case of $n = 2$.