Towards the arithmetic degree of line bundles on abelian varieties

J. Jorgenson and J. Kramer *

Abstract

In this paper we analyze the integral of the star-product of $(n+1)$ Green currents associated to $(n+1)$ global sections of an ample line bundle equipped with a translation invariant metric over an n-dimensional, polarized abelian variety. The integral is shown to equal the logarithm of the Petersson norm of a certain Siegel modular form, which is explicitly described in terms of the given data. This result can be interpreted as evaluating an archimedean height on a family of polarized abelian varieties. The key ingredient to the proof of the main formula is a dd^c-variational formula for the integral under consideration. In the case of dimensions $n = 1, 2, 3$ explicit examples in terms of classical Riemann theta functions are given.

Mathematics Subject Classification (1991): 11F46, 11G10, 14K25, 31B05.

1 Introduction

1.1. A little known, yet important, formula in the study of theta functions of one variable is the following. Let

$$\vartheta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](\tau,z)=\sum_{m=-\infty}^{\infty}e^{\pi i(m+\alpha)^2\tau+2\pi i(m+\alpha)(z+\beta)}$$

be the theta function with characteristics $\alpha, \beta \in \mathbb{R}$ and variables $\tau \in \mathfrak{H}_1$, the upper half-plane, and $z \in \mathbb{C}$. Let $\Delta(\tau)$ be defined by

$$\Delta(\tau)=e^{2\pi i\tau}\prod_{m=1}^{\infty}\left(1-e^{2\pi i m\tau}\right)^{24},$$

which is (up to scaling) the unique cusp form of weight 12 with respect to the modular group $\mathrm{SL}_2(\mathbb{Z})$. Then, we have the formula

$$\int_0^1\int_0^1\log\left|\vartheta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](\tau,0)\right|^2 d\alpha\, d\beta=\frac{1}{24}\log|\Delta(\tau)|^2\,.$$

*The first author acknowledges support from NSF grant DMS-93-07023 and from a Sloan fellowship; the second author would like to express his thanks to G. Wüstholz for the invitation to the FIM at the ETH in Zurich (Switzerland), which was very inspiring for the preparation of this manuscript.

One proof of this result follows from the product expansion for the theta function, viewed as a function of one complex variable $z \in \mathbb{C}$ (cf. [20]). A second proof follows from an analysis of the integral: namely, one uses the differential operator $d_\tau d_\tau^c$ to show that the integral defines a harmonic function on the simply connected space $\mathfrak{H}_1$, hence is equal to the log-modulus square of some non-vanishing holomorphic function f on $\mathfrak{H}_1$. A further analysis shows that f^{24} is a cusp form of weight 12 with respect to the modular group $\mathrm{SL}_2(\mathbb{Z})$, so $f^{24} = c \cdot \Delta$ for some constant c, which then turns out to be given by $c = 1$ (cf. also [1] or [10]).

1.2. In this note a generalization of the above theta function relation to polarized abelian varieties of arbitrary dimension $n \geq 1$ is provided. Namely, in contrast to the 1-dimensional situation, the divisor of the theta function

$$\vartheta \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] (\tau, z) = \sum_{m \in \mathbb{Z}^n} e^{\pi i (m+\alpha)^t \tau (m+\alpha) + 2\pi i (m+\alpha)^t (z+\beta)} \tag{1}$$

moves in general, as one varies $\tau \in \mathfrak{H}_n$, the Siegel upper half-space. Therefore, the local variation $d_\tau d_\tau^c$ of the integral

$$\int_0^1 ... \int_0^1 \log \left| \vartheta \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] (\tau, 0) \right|^2 d\alpha_1 ... d\alpha_n d\beta_1 ... d\beta_n$$

does not vanish unless $n = 1$. Here these local variations for integrals of the above and of a similar type are calculated.

1.3. These local computations together with some global argument lead to the following results: Let $\vartheta(\tau, z)$ be the theta function (1) with characteristics $\alpha = \beta = 0$ and let $\mathcal{A}_{n,D,\Theta}$ denote the moduli space of n-dimensional abelian varieties of polarization type D equipped with the fixed divisor given by the theta function $\vartheta(\tau, z)$. Let $\mathcal{L}$ denote the line bundle on the universal abelian variety $A_{n,D,\Theta}$ over $\mathcal{A}_{n,D,\Theta}$ induced by the theta function $\vartheta(\tau, z)$ and being equipped with the smooth hermitian metric $\|\cdot\|$ having translation invariant curvature. Then, $(n+1)$ global sections $s_1, ..., s_{n+1}$ of $\mathcal{L}$ (eventually, $\mathcal{L}$ has to be replaced by some tensor-power of it multiplied by the pull-back of some line bundle on $\mathcal{A}_{n,D,\Theta}$) are constructed such that their restrictions $s_{1,\tau}, ..., s_{n+1,\tau}$ intersect properly on the abelian variety $A_\tau(\mathbb{C}) = \mathbb{C}^n / (\tau\mathbb{Z}^n \oplus D\mathbb{Z}^n)$ with τ ranging over the complement of a 1-codimensional subset of $\mathcal{A}_{n,D,\Theta}(\mathbb{C})$. In the domain under consideration the integral over the $(n+1)$-fold $*$-product of the Green currents $g_{j,\tau}(z) = -\log \|s_{j,\tau}(z)\|^2$ $(j = 1, ..., n+1)$ then turns out to be given by the formula

$$\int_{A_\tau(\mathbb{C})} g_{1,\tau}(z) * ... * g_{n+1,\tau}(z) = -\log |F(\tau)|^2 - k \cdot \log \det \mathrm{Im}\tau,$$

where F is a Siegel modular form of weight $k = \frac{1}{2} \cdot \det D \cdot (n+1)!$. A refinement of the proof of the above result leads to an explicit formula for F in terms of the given data. In particular, if we restrict ourselves in the domain under consideration to the set of those abelian varieties $A = A_\tau$, which are defined over some number field K and have semi-stable reduction at all the finite places of the ring of integers $\mathcal{O}_K$ of K, the analytic contribution $\deg_\infty(L, \|\cdot\|)$ to the arithmetic degree $\widehat{\deg}(L, \|\cdot\|)$ of the hermitian line bundle $(L, \|\cdot\|)$ (or rather, of its unique cubical extension $(\tilde{L}, \|\cdot\|)$ to the Néron model $\tilde{A}/\mathcal{O}_K$) induced by $(\mathcal{L}, \|\cdot\|)$ is essentially given by the above integral, namely we have the formula

$$\deg_\infty(L, \|\cdot\|) = -\frac{1}{2} \sum_{\sigma : K \hookrightarrow \mathbb{C}} \left(\log |F(\tau^{(\sigma)})|^2 + \frac{1}{2} \cdot \det D \cdot (n+1)! \cdot \log \det \mathrm{Im}\tau^{(\sigma)} \right),$$

the sum being taken over all embeddings from K to $\mathbb{C}$ and $\tau^{(\sigma)} \in \mathfrak{H}_n$ being defined by $A \times_\sigma \mathbb{C} = \mathbb{C}^n/(\tau^{(\sigma)}\mathbb{Z}^n \oplus D\mathbb{Z}^n)$. Of course, it has to be pointed out that the analytic part of the arithmetic degree depends on the choice of the sections under consideration, while the arithmetic degree itself is independent of that choice. In the special case of semi-stable elliptic curves $A/\mathbb{Q}$, a geometric computation together with the knowledge of $\deg_\infty(L, \|\cdot\|)$ leads to an explicit formula for the arithmetic degree $\widehat{\deg}(L, \|\cdot\|)$.

1.4. After a short review of some preliminaries in section 2, we perform in section 3 the computations of the local variations of the integrals mentioned in 1.2; the main result here is contained in Proposition 3.9. In section 4, in particular in Proposition 4.4, we provide the necessary algebraic geometric background in order to be able to globalize the results of section 3. In section 5, Theorem 5.2 and Corollary 5.4, we then prove the results mentioned in 1.3 concerning the computation of the integral over the $(n+1)$-fold $*$-product of Green currents. In section 6, we give some examples illustrating how Corollary 5.4 leads to explicit formulas.

1.5. We would like to thank J.-B. Bost and C. Soulé for their interest in the subject and very stimulating discussions. Furthermore, we point out to K. Köhler's preprint [9], which - from the point of view of the arithmetic Riemann-Roch Theorem - is complementary to the approach provided by this paper.

2 Preliminaries

2.1. Abelian varieties. We denote by A an n-dimensional abelian variety of polarization type $D = \operatorname{diag}(d_1, ..., d_n)$, where $d_1, ..., d_n$ are natural numbers satisfying $d_j | d_{j+1}$ $(j = 1, ..., n-1)$, defined over some field $K \subseteq \mathbb{C}$; we set $d := \det(D) = d_1 \cdot ... \cdot d_n$. For arithmetic applications we will restrict ourselves in parts of sections 5 and 6 to polarized abelian varieties A/K, K a number field, having semi-stable reduction at all the finite places of the ring of integers $\mathcal{O}_K$ of K.

The complex points $A(\mathbb{C})$ of A constitute the n-dimensional, complex torus $\mathbb{C}^n/(\tau\mathbb{Z}^n \oplus D\mathbb{Z}^n)$, where τ is an element of the Siegel upper half-space $\mathfrak{H}_n$ of degree n. In the sequel, the dependence of A (and objects related to A) on τ will be indicated by adding τ as an index, e.g., by writing A_τ for A, etc.. We note that the paramodular group $\Gamma_{n,D}$, defined by

$$\Gamma_{n,D} := \left\{ R = \begin{pmatrix} a_R & b_R \\ c_R & d_R \end{pmatrix} \in \mathrm{M}_{2n}(\mathbb{Z}) \,\middle|\, R \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} R^t = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \right\},$$

acts properly and discontinuously on $\mathfrak{H}_n$ by the formula

$$\tau \mapsto R(\tau) := (a_R\tau + b_R D)(D^{-1}c_R\tau + D^{-1}d_R D)^{-1},$$

and we have $A_\tau(\mathbb{C}) \cong A_{\tau'}(\mathbb{C})$, if and only, if $\tau' = R(\tau)$ for some $R \in \Gamma_{n,D}$ (cf. [7], chapter V, or [16], chapter 8).

We denote by $L = L_\tau$ the symmetric and ample line bundle on $A = A_\tau$ associated to the divisor of the theta function

$$\vartheta(\tau, z) := \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau, z) = \sum_{m \in \mathbb{Z}^n} e^{\pi i m^t \tau m + 2\pi i m^t z}.$$

We equip the complex line bundle $L_{\tau,\mathbb{C}} = L_\tau \otimes_K \mathbb{C}$ with the smooth hermitian metric $\|\cdot\|$, which has translation invariant curvature; it is unique up to scaling by a positive real number. The norm of a section s_τ of L_τ is explicitly given by the formula

$$\|s_\tau(z)\|^2 = |s_\tau(z)|^2 e^{-2\pi y^t \eta^{-1} y} \det \eta^{1/2};$$

here $\tau = \xi + i\eta \in \mathfrak{H}_n$ and $z = x + iy \in \mathbb{C}^n$. The Green current g_τ associated to the section s_τ then becomes $g_\tau(z) = -\log \|s_\tau(z)\|^2$, and, with $d_z^c = (4\pi i)^{-1}(\partial_z - \overline{\partial}_z)$, we easily compute $d_z d_z^c g_\tau + \delta_{\mathrm{div}(s_\tau)} = \omega$ with the $(1,1)$-form $\omega = \frac{i}{2}\, dz^t \cdot \eta^{-1} \wedge d\bar{z}$. We note that the n-fold wedge product $\Omega_n := \bigwedge^n \omega$ is the standard volume form on $A_\tau(\mathbb{C})$, up to a factor $n!$; namely, we have

$$\begin{aligned} \Omega_n &= (-1)^{n(n-1)/2}\, i^n\, 2^{-n}\, n!\, \det \eta^{-1}\, dz_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_n \\ &= n!\, \det \eta^{-1}\, dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n. \end{aligned}$$

2.2. Moduli spaces. We denote by $\mathcal{A}_{n,D}$ the moduli space of n-dimensional abelian varieties of polarization type D. Furthermore, we denote by $\mathcal{A}_{n,D,\Theta}$ the moduli space of n-dimensional abelian varieties of polarization type D equipped with the symmetric and ample line bundle associated to the divisor of the theta function $\vartheta(\tau, z)$; $\mathcal{A}_{n,D,\Theta}$ is a finite covering of $\mathcal{A}_{n,D}$. In general, $\mathcal{A}_{n,D,\Theta}$ is a smooth algebraic stack defined over $\mathbb{Q}$ (cf. [19], chapter 7; for the field of definition, cf. e.g., [7], chapters IV, V); by a Lemma of Serre, it is a smooth and quasi-projective scheme defined over $\mathbb{Q}$ provided $d_1 \geq 3$. In fact, it follows as in [3], chapter V, that $\mathcal{A}_{n,D,\Theta}$ can be defined over $\mathbb{Z}[1/d]$.

Let $\pi : A_{n,D,\Theta} \to \mathcal{A}_{n,D,\Theta}$ be the universal abelian scheme over $\mathcal{A}_{n,D,\Theta}$. If $4|d_1$, the divisor of the theta function $\vartheta(\tau, z)$ descends to give rise to a symmetric and relatively ample line bundle $\mathcal{L}$ on $A_{n,D,\Theta}$ restricting to the prescribed symmetric and ample line bundle on the abelian variety in question, i.e., for $x \in \mathcal{A}_{n,D,\Theta}$ and $A_\tau \cong \pi^{-1}(x)$, the restriction $\mathcal{L}|_{A_\tau}$ is equal to L_τ. From now on, we make the assumption $4|d_1$ throughout sections 2 to 5; in the last section we will relax this assumption slightly. As a further ingredient we also need the canonical line bundle $\mathcal{K}$ on $\mathcal{A}_{n,D,\Theta}$ given by the pull-back of the determinant of the relative cotangent bundle $\Omega^1_{A_{n,D,\Theta}/\mathcal{A}_{n,D,\Theta}}$ via the zero-section $e : \mathcal{A}_{n,D,\Theta} \to A_{n,D,\Theta}$.

By the theory of toroidal compactifications (cf. [3], chapters V, VI) there exist smooth compactifications of $\mathcal{A}_{n,D,\Theta}$, resp. $A_{n,D,\Theta}$, given by smooth and projective schemes $\overline{\mathcal{A}}_{n,D,\Theta}$, resp. $\overline{A}_{n,D,\Theta}$, together with a proper morphism $\overline{\pi} : \overline{A}_{n,D,\Theta} \to \overline{\mathcal{A}}_{n,D,\Theta}$ extending π, everything being defined over $\mathbb{Q}$ (or even $\mathbb{Z}[1/d]$). By the method of toroidal embeddings $\mathcal{K}$ extends to a line bundle $\overline{\mathcal{K}}$ on $\overline{\mathcal{A}}_{n,D,\Theta}$ and $\mathcal{L}$ extends to a relatively ample line bundle $\overline{\mathcal{L}}$ on $\overline{A}_{n,D,\Theta}$. As in [11], Theorem 2.12(ii), in the case of principal polarizations, using the results of [3], chapter V, it is possible to prove that the invertible sheaf $\overline{\mathcal{M}}(m_1, m_2) := \overline{\pi}^* \overline{\mathcal{K}}^{\otimes m_1} \otimes \overline{\mathcal{L}}^{\otimes m_2}$ is very ample provided $m_1 >> m_2 >> 0$. The local triviality of $\mathcal{K}$ on the base $\mathcal{A}_{n,D,\Theta}$ implies that, for $x \in \mathcal{A}_{n,D,\Theta}$ and $A_\tau \cong \pi^{-1}(x)$, the restriction $\overline{\mathcal{M}}(m_1, m_2)|_{A_\tau}$ is equal to $L_\tau^{\otimes m_2}$.

Analytically, $\mathcal{A}_{n,D,\Theta}$ has the following description (cf. [7], chapter V, or [16], chapter 8): $\mathcal{A}_{n,D,\Theta}(\mathbb{C}) = \Gamma_{n,D,\Theta} \backslash \mathfrak{H}_n$, where

$$\Gamma_{n,D,\Theta} := \left\{ R \in \Gamma_{n,D} \,\middle|\, R = \begin{pmatrix} 1 + Da & Db \\ Dc & 1 + Dd \end{pmatrix};\ \begin{array}{l} a, b, c, d \in \mathrm{M}_n(\mathbb{Z}) \\ b, c \text{ even diagonals} \end{array} \right\}.$$

3 Local computations

3.1. Notations. Throughout this section we make the following hypothesis: We let $A_0 := A_{\tau_0}$ be a fixed, n-dimensional abelian variety of polarization type D defined over some field $K \subseteq \mathbb{C}$

such that the complex line bundle $L_{0,\mathbb{C}} := L_{\tau_0,\mathbb{C}}$ under consideration (cf. 2.1) has $(n+1)$ global sections $s_{1,0} := s_{1,\tau_0}, ..., s_{n+1,0} := s_{n+1,\tau_0}$, whose divisors intersect properly on $A_0(\mathbb{C})$.

The condition of proper intersection being open, we know that there exists an open Hausdorff neighbourhood U_0 of $\tau_0 \in \mathfrak{H}_n$ such that the $(n+1)$ global sections $s_{k,0}$ extend for all $\tau \in U_0$ to $(n+1)$ global sections $s_{k,\tau}$ of $L_{\tau,\mathbb{C}}$ with properly intersecting divisors on $A_\tau(\mathbb{C})$. For $k = 1, ..., n+1$, we then put

$$\Theta_{k,\tau} := \mathrm{div}(s_{k,\tau}) \quad , \quad \Theta_{k,0} := \Theta_{k,\tau_0};$$
$$D_{k,\tau} := \Theta_{1,\tau} \cdot ... \cdot \Theta_{k,\tau} \quad , \quad D_{k,0} := D_{k,\tau_0}.$$

We make the convention $D_{0,\tau} := A_\tau(\mathbb{C})$ and $D_{0,0} := D_{0,\tau_0}$.

3.2. Lemma. *With the above notations and $m = 1, ..., n+1$, the m-fold $*$-product $g_{1,\tau} * ... * g_{m,\tau}$ of the Green currents $g_{k,\tau}(z) = -\log \|s_{k,\tau}(z)\|^2$ is given by the formula*

$$g_{1,\tau} * ... * g_{m,\tau} = \sum_{k=1}^{m} g_{k,\tau} \wedge \delta_{D_{k-1,\tau}} \wedge \Omega_{m-k}, \tag{2}$$

where Ω_k denotes the k-fold wedge product of the $(1,1)$-form ω with itself and $\Omega_0 := 1$.

Proof. We proceed by induction on m. For $m = 2$ we have $g_{1,\tau} * g_{2,\tau} = g_{1,\tau} \wedge \omega + g_{2,\tau} \wedge \delta_{\Theta_{1,\tau}}$, which coincides with (2). Therefore, we may assume that (2) is proven for $m \in \{1, ..., n\}$ and we have to establish the corresponding formula for $m+1$. By the definition of the $*$-product we find

$$(g_{1,\tau} * ... * g_{m,\tau}) * g_{m+1,\tau} = (g_{1,\tau} * ... * g_{m,\tau}) \wedge \omega + g_{m+1,\tau} \wedge \delta_{D_{m,\tau}} =$$
$$\sum_{k=1}^{m} g_{k,\tau} \wedge \delta_{D_{k-1,\tau}} \wedge \Omega_{m-k} \wedge \omega + g_{m+1,\tau} \wedge \delta_{D_{m,\tau}} = \sum_{k=1}^{m+1} g_{k,\tau} \wedge \delta_{D_{k-1,\tau}} \wedge \Omega_{m+1-k}.$$

□

3.3. Lemma. *With the above notations and $k = 1, ..., n+1$, the integrals*

$$c_{n,D;k,\tau} := \frac{1}{2} \int_{D_{k-1,\tau}} \Omega_{n+1-k}$$

depend only on n and the polarization type D, i.e., are independent of k and $\tau \in U_0$, and are given by

$$c_{n,D} := \frac{1}{2} \cdot d \cdot n! = \frac{1}{2} \cdot d_1 \cdot ... \cdot d_n \cdot n!.$$

Proof. By the de Rham Theorem we can interpret the integrals $c_{n,D;k,\tau}$ as the intersection numbers

$$c_{n,D;k,\tau} = \frac{1}{2} D_{k-1,\tau} \cdot \Theta_{1,\tau}^{n+1-k},$$

noting that Ω_{n+1-k} is the $(n+1-k)$-fold wedge product of the $(1,1)$-form ω and that the latter is the first Chern form of $L_{\tau,\mathbb{C}}$. By linear, hence numerical, equivalence, we then have

$$c_{n,D;k,\tau} = \frac{1}{2} \Theta_{1,\tau}^{k-1} \cdot \Theta_{1,\tau}^{n+1-k} = \frac{1}{2} \Theta_{1,\tau}^{n}.$$

The result now follows from [16], Corollary 10.5(d). □

3.4. Remark. For the subsequent considerations it will be very useful to replace the complex variable $z \in \mathbb{C}^n$ by the real variables $\alpha, \beta \in \mathbb{R}^n$, where $z = -\tau \cdot \alpha + \beta$. The $(1,1)$-form ω then simply equals $\omega' = d\alpha_1 \wedge d\beta_1 + ... + d\alpha_n \wedge d\beta_n = d\alpha^t \wedge d\beta$ and becomes therefore independent of τ. More generally, we denote the differential form corresponding to Ω_k under the above change of variables by Ω'_k; finally, we put $\Omega'_0 := 1$. In particular, Ω'_n is given by $\Omega'_n = n!\ d\alpha_1 \wedge d\beta_1 \wedge ... \wedge d\alpha_n \wedge d\beta_n$.

With $z = -\tau \cdot \alpha + \beta$ the $(n+1)$ sections $s_{k,\tau}(z)$ (being linear combinations of the theta functions with characteristics in $D^{-1}\mathbb{Z}^n/\mathbb{Z}^n$) give rise to smooth functions $s'_{k,\tau}(\alpha, \beta)$, defined through the equation

$$s'_{k,\tau}(\alpha,\beta) \cdot e^{-\pi i \alpha^t \tau \alpha + 2\pi i \alpha^t \beta} = s_{k,\tau}(z).$$

One easily checks that

$$\|s_{k,\tau}(z)\|^2 = |s'_{k,\tau}(\alpha,\beta)|^2 \det \eta^{1/2}.$$

We put $g'_{k,\tau}(\alpha,\beta) := -\log|s'_{k,\tau}(\alpha,\beta)|^2$ and note that the functions $g'_{k,\tau}(\alpha,\beta)$ are harmonic in τ away from $\Theta_{k,\tau}$. The Green currents $g_{k,\tau}(z)$ can now be written as $g_{k,\tau}(z) = -\frac{1}{2}\log\det\eta + g'_{k,\tau}(\alpha,\beta)$.

3.5. Further notations. Eventually, after a resolution of singularities, $A_0(\mathbb{C})$ can locally be identified with $\mathbb{C}^n$, with coordinates $w = (w_1, ..., w_n)$, such that $\Theta_{k,0}$ is described by the equation $w_k = 0$ and $D_{k,0}$ by the equations $w_1 = ... = w_k = 0$; by abuse of notation, we denote the latter subsets of $\mathbb{C}^n$ again by $\Theta_{k,0}$, resp. $D_{k,0}$ $(k = 1, ..., n)$. For suitable $\varepsilon > 0$, we define the following tubular neighbourhoods $\Theta_k^{(\varepsilon)}$ of $\Theta_{k,0}$, resp. $D_k^{(\varepsilon)}$ of $D_{k,0}$ $(k = 1, ..., n)$:

$$\Theta_k^{(\varepsilon)} := \{w = (w_1, ..., w_n) \in \mathbb{C}^n |\ |w_k|^2 < \varepsilon^2\}, \text{ resp.}$$
$$D_k^{(\varepsilon)} := \{w = (w_1, ..., w_n) \in \mathbb{C}^n |\ |w_1|^2 + ... + |w_k|^2 < \varepsilon^2\}.$$

We denote the differential form on $\mathbb{C}^n$ induced by Ω'_k under the above identification by $\tilde{\Omega}_k$ $(k = 0, ..., n)$.

Following [5], p. 80, we then define for the cycles $\Theta_{k,0}$, resp. $D_{k,0}$, the functions $f_{\Theta_{k,0}}$ on $\Theta_k^{(\varepsilon)}$, resp. $f_{D_{k,0}}$ on $D_k^{(\varepsilon)}$, by

$$f_{\Theta_{k,0}}(w) := \log|w_k|^2 \ \ (k = 1, ..., n), \text{ resp.}$$
$$f_{D_{k,0}}(w) := -\frac{4\pi}{(2k-2)\gamma_{2k}} \cdot (|w_1|^2 + ... + |w_k|^2)^{1-k} \ \ (k = 2, ..., n),$$

where γ_{2k} is the volume of the unit sphere in $\mathbb{C}^k$, i.e., $\gamma_{2k} = 2\pi^k/(k-1)!$; we further put $f_{D_{1,0}} := f_{\Theta_{1,0}}$. These functions have the property that for any smooth function h on $\Theta_k^{(\varepsilon)}$, resp. $D_k^{(\varepsilon)}$, we have

$$\int_{\Theta_k^{(\varepsilon)}} h(w) \wedge \delta_{\Theta_{k,0}} \wedge \tilde{\Omega}_{n-1} = \int_{\Theta_k^{(\varepsilon)}} h(w) \wedge d_w d_w^c f_{\Theta_{k,0}}(w) \wedge \tilde{\Omega}_{n-1} \ \ (k = 1, ..., n), \text{ resp.}$$
$$\int_{D_k^{(\varepsilon)}} h(w) \wedge \delta_{D_{k,0}} \wedge \tilde{\Omega}_{n-k} = \int_{D_k^{(\varepsilon)}} h(w) \wedge d_w d_w^c f_{D_{k,0}}(w) \wedge \tilde{\Omega}_{n-1} \ \ (k = 2, ..., n).$$

For $\tau \in U_0$, the above local identification of $A_0(\mathbb{C})$ with $\mathbb{C}^n$ leads to a local identification of $A_\tau(\mathbb{C})$ with (the same) $\mathbb{C}^n$ such that $\Theta_{k,\tau}$ is described by the equation $w_k = p_k(\tau)$ and $D_{k,\tau}$

by the equations $w_1 = p_1(\tau), ..., w_k = p_k(\tau)$ for certain holomorphic functions $p_1(\tau), ..., p_k(\tau)$ satisfying $p_1(\tau_0) = ... = p_k(\tau_0) = 0$; by abuse of notation, we denote the latter subsets of $\mathbb{C}^n$ again by $\Theta_{k,\tau}$, resp. $D_{k,\tau}$ $(k = 1, ..., n)$. By choosing the neighbourhood U_0 and $\varepsilon > 0$ suitably, we may assume that $\Theta_{k,\tau}$, resp. $D_{k,\tau}$, vary within the thickenings $\Theta_k^{(\varepsilon)}$, resp. $D_k^{(\varepsilon)}$.

As before, we can now define distribution functions $f_{\Theta_{k,\tau}}$ on $\Theta_k^{(\varepsilon)}$ for $\Theta_{k,\tau}$, resp. $f_{D_{k,\tau}}$ on $D_k^{(\varepsilon)}$ for $D_{k,\tau}$, when τ varies in the neighbourhood U_0, namely

$$f_{\Theta_{k,\tau}}(w) := \log |w_k - p_k(\tau)|^2 \;\; (k = 1, ..., n), \text{ resp.}$$

$$f_{D_{k,\tau}}(w) := -\frac{4\pi}{(2k-2)\gamma_{2k}} \cdot (|w_1 - p_1(\tau)|^2 + ... + |w_k - p_k(\tau)|^2)^{1-k} \;\; (k = 2, ..., n);$$

again we put $f_{D_{1,\tau}} := f_{\Theta_{1,\tau}}$. The functions $f_{\Theta_{k,\tau}}$, resp. $f_{D_{k,\tau}}$, satisfy the same type of integration formulae as the functions $f_{\Theta_{k,0}}$, resp. $f_{D_{k,0}}$, with $\delta_{\Theta_{k,0}}$, resp. $\delta_{D_{k,0}}$, replaced by $\delta_{\Theta_{k,\tau}}$, resp. $\delta_{D_{k,\tau}}$. It is important to observe that the functions $f_{\Theta_{k,\tau}}$, resp. $f_{D_{k,\tau}}$, are defined on the fixed sets $\Theta_k^{(\varepsilon)}$, resp. $D_k^{(\varepsilon)}$, which are independent of $\tau \in U_0$.

3.6. Remark. By means of the local identification of $A_\tau(\mathbb{C})$ with $\mathbb{C}^n$ made in 3.5, the $(n+1)$ functions $s'_{k,\tau}(\alpha, \beta)$, resp. $g'_{k,\tau}(\alpha, \beta)$, of Remark 3.4 give rise to functions $\tilde{s}_{k,\tau}(w)$, resp. $\tilde{g}_{k,\tau}(w)$, on $\mathbb{C}^n$. Again, we note that the functions $\tilde{g}_{k,\tau}(w)$ are harmonic in τ away from $\Theta_{k,\tau}$ and that the Green currents $g_{k,\tau}(z)$ can be written as $g_{k,\tau}(z) = -\frac{1}{2} \log \det \eta + \tilde{g}_{k,\tau}(w)$.

3.7. Lemma. *With the above notations, we have for* $k = 2, ..., n+1$

$$\int_{D_{k-1}^{(\varepsilon)}} \tilde{g}_{k,\tau}(w) \wedge \delta_{D_{k-1,\tau}} \wedge \tilde{\Omega}_{n+1-k} = \int_{D_{k-1}^{(\varepsilon)}} \tilde{g}_{k,\tau}(w) \wedge d_w d_w^c f_{D_{k-1,\tau}}(w) \wedge \tilde{\Omega}_{n-1}.$$

Proof. For large T, define $g_{k,\tau}^{(T)}$ to be a smoothening of the function $\min\{\tilde{g}_{k,\tau}, T\}$ such that for each $w \in D_{k-1}^{(\varepsilon)}$, we have

$$g_{k,\tau}^{(T_1)}(w) \leq g_{k,\tau}^{(T_2)}(w),$$

if $T_1 \leq T_2$. By the definition of the function $f_{D_{k-1,\tau}}$, we have

$$\int_{D_{k-1}^{(\varepsilon)}} g_{k,\tau}^{(T)}(w) \wedge \delta_{D_{k-1,\tau}} \wedge \tilde{\Omega}_{n+1-k} = \int_{D_{k-1}^{(\varepsilon)}} g_{k,\tau}^{(T)}(w) \wedge d_w d_w^c f_{D_{k-1,\tau}}(w) \wedge \tilde{\Omega}_{n-1}.$$

We now apply the monotone convergence theorem to conclude the stated result. □

3.8. Lemma. *With the above notations, we have for* $k = 2, ..., n$

$$d_\tau d_\tau^c \int_{D_{k-1}^{(\varepsilon)}} f_{D_{k-1,\tau}}(w) \wedge \delta_{\Theta_{k,\tau}} \wedge \tilde{\Omega}_{n-1} = d_\tau d_\tau^c \int_{D_k^{(\varepsilon)}} f_{D_{k,\tau}}(w) \wedge \tilde{\Omega}_n. \tag{3}$$

Proof. Assume first $k > 2$. We begin by rewriting the right hand side integral of the claimed formula (3): For this we first integrate with respect to the variable w_k, keeping all the other variables fixed. The corresponding region of integration is a disk of the form $\Delta_k = \{w_k \in \mathbb{C} \,|\, |w_k| < \varepsilon_k\}$ $(\varepsilon_k^2 = \varepsilon^2 - \sum_{j=1}^{k-1} |w_j|^2)$, which we parametrize by the polar coordinates $w_k -$

$p_k(\tau) = \rho e^{i\varphi}$; note that the radius ρ is a function of the angle φ. With this notation we need to evaluate the integral

$$-\frac{4\pi}{(2k-2)\gamma_{2k}} \int_0^{2\pi} \int_0^{\rho(\varphi)} \frac{\rho \, d\rho d\varphi}{(|w_1 - p_1(\tau)|^2 + ... + |w_{k-1} - p_{k-1}(\tau)|^2 + \rho^2)^{k-1}}.$$

Integrating with respect to ρ, this integral becomes

$$-\frac{4\pi}{(2k-2)\gamma_{2k}} \cdot \frac{1}{2(-k+2)} \int_0^{2\pi} (|w_1 - p_1(\tau)|^2 + ... + |w_{k-1} - p_{k-1}(\tau)|^2 + \rho^2)^{-k+2}\Big|_0^{\rho(\varphi)} d\varphi;$$

using the recursion formula $\gamma_{2k}/\gamma_{2k-2} = \pi/(k-1)$, the above integral can be rewritten as

$$\frac{2}{(2k-4)\gamma_{2k-2}} \int_0^{2\pi} (|w_1 - p_1(\tau)|^2 + ... + |w_{k-1} - p_{k-1}(\tau)|^2 + \rho(\varphi)^2)^{2-k} d\varphi -$$

$$\frac{4\pi}{(2k-4)\gamma_{2k-2}} \cdot (|w_1 - p_1(\tau)|^2 + ... + |w_{k-1} - p_{k-1}(\tau)|^2)^{2-k}.$$

To complete the computation of the integral in question, we now have to integrate the above two summands with respect to the variables $w_1, ..., w_{k-1}$ together with the additional variables $w_{k+1}, ..., w_n$. It turns out that the corresponding integral over the second summand equals

$$\int_{D_{k-1}^{(\varepsilon)}} f_{D_{k-1,\tau}}(w) \wedge \delta_{\Theta_{k,\tau}} \wedge \tilde{\Omega}_{n-1},$$

which is precisely the left hand side integral of the claimed formula (3). To complete the proof of the Lemma, we therefore need to show that the corresponding integral over the first summand is harmonic in τ.

Recall that the thickening $D_k^{(\varepsilon)}$ is by construction independent of $\tau \in U_0$. For any angle φ, the quantity $\rho(\varphi)$ is the distance from $p_k(\tau)$ to a point at the boundary of Δ_k, which is in fact a point on the boundary of $D_k^{(\varepsilon)}$; we can write $\rho(\varphi) = |a(\varphi, \varepsilon) - p_k(\tau)|$, where $a(\varphi, \varepsilon)$ is independent of τ. Therefore, for every φ, the integrand in question is bounded, hence harmonic for $\tau \in U_0$, with range of integration being independent of τ. Consequently, we can interchange the operator $d_\tau d_\tau^c$ with the corresponding integration and conclude that the integral over the first summand is harmonic in τ as claimed.

The case $k = 2$ is treated in an analogous way by using the explicit formulae for $f_{D_{1,\tau}} = f_{\Theta_{1,\tau}}$ and $f_{D_{2,\tau}}$. □

3.9. Proposition. *Let $\{U_m\}_{m=1}^M$ be a complete set of subcubes of $A_0(\mathbb{C})$ determined by torsion points such that within each subcube U_m the local identifications of 3.5 are valid. Then, with the above notations, we have the following variational formulae for $\tau \in U_0$ (recall $c_{n,D} = \frac{1}{2} \cdot d \cdot n!$):*
(a)

$$d_\tau d_\tau^c \int_{A_\tau(\mathbb{C})} g_{1,\tau}(z) \wedge \Omega_n + c_{n,D} \cdot d_\tau d_\tau^c \log \det \eta = -d_\tau d_\tau^c \sum_{m=1}^M \int_{\Theta_1^{(\varepsilon)} \cap U_m} f_{\Theta_{1,\tau}}(w) \wedge \tilde{\Omega}_n.$$

(b) For $k = 2, ..., n$ we have

$$d_\tau d_\tau^c \int_{A_\tau(\mathbb{C})} g_{k,\tau}(z) \wedge \delta_{D_{k-1,\tau}} \wedge \Omega_{n+1-k} + c_{n,D} \cdot d_\tau d_\tau^c \log \det \eta =$$

$$d_\tau d_\tau^c \sum_{m=1}^M \int_{D_{k-1}^{(\varepsilon)} \cap U_m} f_{D_{k-1,\tau}}(w) \wedge \tilde{\Omega}_n - d_\tau d_\tau^c \sum_{m=1}^M \int_{D_k^{(\varepsilon)} \cap U_m} f_{D_{k,\tau}}(w) \wedge \tilde{\Omega}_n + H_{k,\tau}(\varepsilon),$$

where $\lim_{\varepsilon\to 0} H_{k,\tau}(\varepsilon) = 0$.
(c)

$$d_\tau d_\tau^c \int_{A_\tau(\mathbb{C})} g_{n+1,\tau}(z) \wedge \delta_{D_{n,\tau}} \wedge \Omega_0 + c_{n,D} \cdot d_\tau d_\tau^c \log\det\eta =$$

$$d_\tau d_\tau^c \sum_{m=1}^{M} \int_{D_n^{(\varepsilon)} \cap U_m} f_{D_{n,\tau}}(w) \wedge \tilde{\Omega}_n + H_{n+1,\tau}(\varepsilon),$$

where $\lim_{\varepsilon\to 0} H_{n+1,\tau}(\varepsilon) = 0$.

Proof. (a) By Remark 3.4 we have $g_{1,\tau}(z) = -\frac{1}{2}\log\det\eta + g'_{1,\tau}(\alpha,\beta)$, hence

$$d_\tau d_\tau^c \int_{A_\tau(\mathbb{C})} g_{1,\tau}(z) \wedge \Omega_n =$$

$$-c_{n,D} \cdot d_\tau d_\tau^c \log\det\eta + d_\tau d_\tau^c \int_{\mathbb{R}^n/\mathbb{Z}^n \oplus \mathbb{R}^n/D\mathbb{Z}^n} g'_{1,\tau}(\alpha,\beta) \wedge \Omega'_n.$$

To compute the variation of the latter integral, we decompose it as follows, using Remark 3.6

$$d_\tau d_\tau^c \int_{\mathbb{R}^n/\mathbb{Z}^n \oplus \mathbb{R}^n/D\mathbb{Z}^n} g'_{1,\tau}(\alpha,\beta) \wedge \Omega'_n =$$

$$d_\tau d_\tau^c \sum_{m=1}^{M} \int_{U_m \setminus (\Theta_1^{(\varepsilon)} \cap U_m)} \tilde{g}_{1,\tau}(w) \wedge \tilde{\Omega}_n + d_\tau d_\tau^c \sum_{m=1}^{M} \int_{\Theta_1^{(\varepsilon)} \cap U_m} \tilde{g}_{1,\tau}(w) \wedge \tilde{\Omega}_n.$$

The variation of the integral over the region $U_m \setminus (\Theta_1^{(\varepsilon)} \cap U_m)$ is zero, since the region of integration is independent of $\tau \in U_0$ and the integrand is harmonic in τ away from $\Theta_{1,\tau}$. Therefore, we obtain

$$d_\tau d_\tau^c \int_{A_\tau(\mathbb{C})} g_{1,\tau}(z) \wedge \Omega_n + c_{n,D} \cdot d_\tau d_\tau^c \log\det\eta = d_\tau d_\tau^c \sum_{m=1}^{M} \int_{\Theta_1^{(\varepsilon)} \cap U_m} \tilde{g}_{1,\tau}(w) \wedge \tilde{\Omega}_n =$$

$$d_\tau d_\tau^c \sum_{m=1}^{M} \int_{\Theta_1^{(\varepsilon)} \cap U_m} \big(\tilde{g}_{1,\tau}(w) + f_{\Theta_{1,\tau}}(w)\big) \wedge \tilde{\Omega}_n - d_\tau d_\tau^c \sum_{m=1}^{M} \int_{\Theta_1^{(\varepsilon)} \cap U_m} f_{\Theta_{1,\tau}}(w) \wedge \tilde{\Omega}_n.$$

Again, the variation of the first integral is zero, since the region of integration is independent of $\tau \in U_0$ and the integrand is harmonic in τ. Hence, we arrive at the formula

$$d_\tau d_\tau^c \int_{A_\tau(\mathbb{C})} g_{1,\tau}(z) \wedge \Omega_n + c_{n,D} \cdot d_\tau d_\tau^c \log\det\eta = -d_\tau d_\tau^c \sum_{m=1}^{M} \int_{\Theta_1^{(\varepsilon)} \cap U_m} f_{\Theta_{1,\tau}}(w) \wedge \tilde{\Omega}_n,$$

as claimed.

(b) For $k = 2, ..., n$ we set

$$I_{k,\tau} := \int_{A_\tau(\mathbb{C})} g_{k,\tau}(z) \wedge \delta_{D_{k-1,\tau}} \wedge \Omega_{n+1-k}.$$

Writing $I_{k,\tau}$ as a sum of integrals over the subcubes under consideration and substituting $g_{k,\tau}(z) = -\frac{1}{2}\log\det\eta + \tilde{g}_{k,\tau}(w)$ as in Remark 3.6, we find, using Lemma 3.3

$$I_{k,\tau} = -c_{n,D}\cdot\log\det\eta + \sum_{m=1}^{M}\int_{D^{(\varepsilon)}_{k-1}\cap U_m}\tilde{g}_{k,\tau}(w)\wedge\delta_{D_{k-1,\tau}}\wedge\tilde{\Omega}_{n+1-k}.$$

By applying Lemma 3.7, we obtain

$$I_{k,\tau} = -c_{n,D}\cdot\log\det\eta + \sum_{m=1}^{M}\int_{D^{(\varepsilon)}_{k-1}\cap U_m}\tilde{g}_{k,\tau}(w)\wedge d_w d_w^c f_{D_{k-1,\tau}}(w)\wedge\tilde{\Omega}_{n-1}.$$

Next we integrate the above integral by parts twice in order to obtain the formula

$$I_{k,\tau} = -c_{n,D}\cdot\log\det\eta - \sum_{m=1}^{M}\int_{\partial(D^{(\varepsilon)}_{k-1}\cap U_m)}\tilde{g}_{k,\tau}(w)\wedge d_w^c f_{D_{k-1,\tau}}(w)\wedge\tilde{\Omega}_{n-1}+$$
$$\sum_{m=1}^{M}\int_{\partial(D^{(\varepsilon)}_{k-1}\cap U_m)} d_w^c\tilde{g}_{k,\tau}(w)\wedge f_{D_{k-1,\tau}}(w)\wedge\tilde{\Omega}_{n-1}+$$
$$\sum_{m=1}^{M}\int_{D^{(\varepsilon)}_{k-1}\cap U_m} d_w d_w^c\tilde{g}_{k,\tau}(w)\wedge f_{D_{k-1,\tau}}(w)\wedge\tilde{\Omega}_{n-1}.$$

Let us now examine the three integrals in the above expression, beginning with the two boundary integrals over $\partial(D^{(\varepsilon)}_{k-1}\cap U_m) = (\partial D^{(\varepsilon)}_{k-1}\cap U_m)\cup(D^{(\varepsilon)}_{k-1}\cap\partial U_m)$. First, we note that both integrals over the boundary piece $D^{(\varepsilon)}_{k-1}\cap\partial U_m$ will vanish after summing over $m = 1,\ldots,M$, since all normal vectors involved appear in pairs with opposite directions. Hence, we are left to consider the two integrals over the boundary piece $\partial D^{(\varepsilon)}_{k-1}\cap U_m$. For this we set

$$\begin{aligned} H_{k,\tau}(\varepsilon) \quad := \quad & -d_\tau d_\tau^c\sum_{m=1}^{M}\int_{\partial D^{(\varepsilon)}_{k-1}\cap U_m}\tilde{g}_{k,\tau}(w)\wedge d_w^c f_{D_{k-1,\tau}}(w)\wedge\tilde{\Omega}_{n-1} \\ & +d_\tau d_\tau^c\sum_{m=1}^{M}\int_{\partial D^{(\varepsilon)}_{k-1}\cap U_m} d_w^c\tilde{g}_{k,\tau}(w)\wedge f_{D_{k-1,\tau}}(w)\wedge\tilde{\Omega}_{n-1}. \end{aligned}$$

Since the region of integration in the above two integrals is independent of τ, let us interchange the differentiation $d_\tau d_\tau^c$ with the integration over $\partial D^{(\varepsilon)}_{k-1}\cap U_m$. Expanding $\tilde{g}_{k,\tau}(w)$ in the form

$$\tilde{g}_{k,\tau}(w) = -\log|w_k - p_k(\tau)|^2 + \log|h_{k,\tau}(w)|^2$$

with $h_{k,\tau}(w)$ a non-vanishing, holomorphic function on $D^{(\varepsilon)}_{k-1}\cap U_m$, we observe that replacing $\tilde{g}_{k,\tau}(w)$ by $-\log|w_k - p_k(\tau)|^2$, changes $H_{k,\tau}(\varepsilon)$ only by $O_\tau(\varepsilon)$. Therefore, in studying $H_{k,\tau}(\varepsilon)$ up to the order $O_\tau(\varepsilon)$, we may substitute $\tilde{g}_{k,\tau}(w)$ by $-\log|w_k - p_k(\tau)|^2$. Using then Leibniz' rule twice in order to compute $d_\tau d_\tau^c$ and setting $\tau = \tau_0$, we obtain eight integrals over $\partial D^{(\varepsilon)}_{k-1}\cap U_m$. Observing that $-\log|w_k - p_k(\tau_0)|^2 = -\log|w_k|^2$ and $f_{D_{k-1,0}}(w)$ are even functions and that each of the eight integrands in question contains exactly three derivatives, we conclude that in all eight cases the integral vanishes as an integral of an odd function over a symmetric region.

Finally, we obtain $H_{k,\tau_0}(\varepsilon) = O_{\tau_0}(\varepsilon)$, and hence $H_{k,\tau}(\varepsilon) = O_\tau(\varepsilon)$ by continuity with respect to τ, i.e., $\lim_{\varepsilon\to 0} H_{k,\tau}(\varepsilon) = 0$. As for the integral over $D_{k-1}^{(\varepsilon)} \cap U_m$, we use the differential equation for $g_{k,\tau}(z)$, resp. $\tilde{g}_{k,\tau}(w)$, to write

$$\sum_{m=1}^{M} \int_{D_{k-1}^{(\varepsilon)} \cap U_m} d_w d_w^c \tilde{g}_{k,\tau}(w) \wedge f_{D_{k-1,\tau}}(w) \wedge \tilde{\Omega}_{n-1} =$$

$$\sum_{m=1}^{M} \int_{D_{k-1}^{(\varepsilon)} \cap U_m} f_{D_{k-1,\tau}}(w) \wedge \tilde{\Omega}_n - \sum_{m=1}^{M} \int_{D_{k-1}^{(\varepsilon)} \cap U_m} f_{D_{k-1,\tau}}(w) \wedge \delta_{\Theta_{k,\tau}} \wedge \tilde{\Omega}_{n-1}.$$

Summing up and taking $d_\tau d_\tau^c$, we get the equation

$$d_\tau d_\tau^c \int_{A_\tau(\mathbb{C})} g_{k,\tau}(z) \wedge \delta_{D_{k-1,\tau}} \wedge \Omega_{n+1-k} + c_{n,D} \cdot d_\tau d_\tau^c \log\det\eta =$$

$$d_\tau d_\tau^c \sum_{m=1}^{M} \int_{D_{k-1}^{(\varepsilon)} \cap U_m} f_{D_{k-1,\tau}}(w) \wedge \tilde{\Omega}_n - d_\tau d_\tau^c \sum_{m=1}^{M} \int_{D_{k-1}^{(\varepsilon)} \cap U_m} f_{D_{k-1,\tau}}(w) \wedge \delta_{\Theta_{k,\tau}} \wedge \tilde{\Omega}_{n-1} + H_{k,\tau}(\varepsilon).$$

By applying Lemma 3.8, which is easily verified to hold true with the domain of integration $D_{k-1}^{(\varepsilon)}$ replaced by the domain $D_{k-1}^{(\varepsilon)} \cap U_m$, the proof of (b) is then complete.

(c) To prove the last part of the Proposition, we proceed as in (b) with $k = n+1$. Observing that the Dirac current $\delta_{\Theta_{n+1,\tau}}$ vanishes on $D_n^{(\varepsilon)} \cap U_m$ for sufficiently small ε, we obtain from the last formula in the proof of part (b)

$$d_\tau d_\tau^c \int_{A_\tau(\mathbb{C})} g_{n+1,\tau}(z) \wedge \delta_{D_{n,\tau}} \wedge \Omega_0 + c_{n,D} \cdot d_\tau d_\tau^c \log\det\eta =$$

$$d_\tau d_\tau^c \sum_{m=1}^{M} \int_{D_n^{(\varepsilon)} \cap U_m} f_{D_{n,\tau}}(w) \wedge \tilde{\Omega}_n + H_{n+1,\tau}(\varepsilon),$$

where $H_{n+1,\tau}(\varepsilon)$ is defined as in part (b) with $k = n+1$. This concludes the proof of the Proposition. □

3.10. Corollary. *Let $A_0 = A_{\tau_0}$ be an n-dimensional abelian variety of polarization type D defined over some field $K \subseteq \mathbb{C}$ such that the complex line bundle $L_{0,\mathbb{C}} = L_{\tau_0,\mathbb{C}}$ has $(n+1)$ global sections $s_{1,0} = s_{1,\tau_0}, ..., s_{n+1,0} = s_{n+1,\tau_0}$, whose divisors intersect properly on $A_0(\mathbb{C})$. Then, there exists an open Hausdorff neighbourhood U_0 of $\tau_0 \in \mathfrak{H}_n$ such that the given situation extends to U_0, and the following variational formula holds for all $\tau \in U_0$:*

$$d_\tau d_\tau^c \left(\int_{A_\tau(\mathbb{C})} g_{1,\tau}(z) * ... * g_{n+1,\tau}(z) + C_{n,D} \cdot \log\det\eta \right) = 0,$$

where $C_{n,D} := (n+1) \cdot c_{n,D} = \frac{1}{2} \cdot d \cdot (n+1)!$.

Proof. The existence of the claimed open Hausdorff neighbourhood U_0 of $\tau_0 \in \mathfrak{H}_n$ is evident by 3.1. Then, Lemma 3.2 applies in order to obtain the formula

$$d_\tau d_\tau^c\left(\int_{A_\tau(\mathbb{C})} g_{1,\tau}(z)*...*g_{n+1,\tau}(z)+C_{n,D}\cdot\log\det\eta\right)=$$

$$\sum_{k=1}^{n+1}\left(d_\tau d_\tau^c\int_{A_\tau(\mathbb{C})} g_{k,\tau}(z)\wedge\delta_{D_{k-1,\tau}}\wedge\Omega_{n+1-k}+c_{n,D}\cdot d_\tau d_\tau^c\log\det\eta\right).$$

Eventually by shrinking U_0 suitably, the corollary now becomes an immediate consequence of Proposition 3.9 by letting ε tend to zero. □

4 Global arguments

4.1. Lemma. *Let F be a field of infinite cardinality, X/F a geometrically irreducible, projective scheme of dimension $d\geq 2$ and $\mathcal{N}$ a very ample line bundle on X. Then, for any integer $\nu>>0$ and any integer $k\in\{1,...,d-1\}$, there exists a non-empty Zariski open subset consisting of $(k+1)$-tuples of global sections $(s_1,...,s_{k+1})\in\Gamma(X,\mathcal{N}^{\otimes\nu})^{k+1}$ satisfying the following two properties:*
(a') $\operatorname{div}(s_1),...,\operatorname{div}(s_{k+1})$ intersect properly on X,
(b') $\operatorname{div}(s_{i_1})\cap...\cap\operatorname{div}(s_{i_k})$ is geometrically irreducible for any choice of k indices $1\leq i_1<...<i_k\leq k+1$.

Proof. Since $\mathcal{N}$ is very ample, there exists for any integer $\nu>>0$ an embedding $\varphi: X\hookrightarrow\mathbb{P}_F^N$ for some N, defined over F, such that $\varphi^*\mathcal{O}_{\mathbb{P}_F^N}(1)=\mathcal{N}^{\otimes\nu}$ and such that the restriction map $\varphi^*:\Gamma(\mathbb{P}_F^N,\mathcal{O}_{\mathbb{P}_F^N}(1))\to\Gamma(X,\mathcal{N}^{\otimes\nu})$ is surjective; we fix such an embedding in the sequel.

Let $\mathbf{Grass}(l,N)$ denote the Grassmannian variety parametrizing the l-codimensional linear subvarieties of $\mathbb{P}_F^N$. Furthermore, let U_l be the subset of those l-tuples of global sections in $\Gamma(\mathbb{P}_F^N,\mathcal{O}_{\mathbb{P}_F^N}(1))^l$, whose divisors constitute l properly intersecting hyperplanes in $\mathbb{P}_F^N$. We note that U_l is a non-empty Zariski open, hence dense, subset of $\Gamma(\mathbb{P}_F^N,\mathcal{O}_{\mathbb{P}_F^N}(1))^l$ and that there is a natural, surjective morphism $p_l: U_l\longrightarrow\mathbf{Grass}(l,N)(F)$.

By [8], Corollaire 6.11.1 and the assumption $k+1\leq d$, there exists a non-empty Zariski open, hence dense, subset $V'_{k+1}\subseteq\mathbf{Grass}(k+1,N)(F)$ with the property that $\operatorname{codim}(L\cap X)=k+1$ for all $L\in V'_{k+1}$. We define $V_{k+1}:=p_{k+1}^{-1}(V'_{k+1})$; we note that V_{k+1} is a non-empty Zariski open, hence dense, subset of U_{k+1}, whence of $\Gamma(\mathbb{P}_F^N,\mathcal{O}_{\mathbb{P}_F^N}(1))^{k+1}$, consisting of $(k+1)$-tuples of global sections $(s'_1,...,s'_{k+1})\in\Gamma(\mathbb{P}_F^N,\mathcal{O}_{\mathbb{P}_F^N}(1))^{k+1}$ with the property that

$$\operatorname{codim}(\operatorname{div}(s'_1)\cap...\cap\operatorname{div}(s'_{k+1})\cap X)=k+1.$$

Again, using [8], now Corollaire 6.11.3 and the assumption $k+1\leq d$, one proves the existence of a non-empty Zariski open, hence dense, subset $W'_k\subseteq\mathbf{Grass}(k,N)(F)$ such that the intersection $L\cap X$ is geometrically irreducible for all $L\in W'_k$. We define $W_k:=p_k^{-1}(W'_k)$; we note that W_k is a non-empty Zariski open, hence dense, subset of U_k, whence of $\Gamma(\mathbb{P}_F^N,\mathcal{O}_{\mathbb{P}_F^N}(1))^k$, consisting of k-tuples of global sections $(s'_1,...,s'_k)\in\Gamma(\mathbb{P}_F^N,\mathcal{O}_{\mathbb{P}_F^N}(1))^k$ such that $\operatorname{div}(s'_1)\cap...\cap\operatorname{div}(s'_k)\cap X$ is geometrically irreducible.

For any choice of k indices $1\leq i_1<...<i_k\leq k+1$ consider the projections

$$q_{i_1,...,i_k}:\Gamma(\mathbb{P}_F^N,\mathcal{O}_{\mathbb{P}_F^N}(1))^{k+1}\longrightarrow\Gamma(\mathbb{P}_F^N,\mathcal{O}_{\mathbb{P}_F^N}(1))^k$$

given by mapping $(s'_1,...,s'_{k+1})$ to $(s'_{i_1},...,s'_{i_k})$. Define the subset

$$\mathcal{U}_k := V_{k+1} \cap \bigcap_{1\leq i_1<...<i_k\leq k+1} q^{-1}_{i_1,...,i_k}(W_k);$$

it is a non-empty Zariski open, hence dense, subset of $\Gamma(\mathbb{P}^N_F, \mathcal{O}_{\mathbb{P}^N_F}(1))^{k+1}$. By the surjectivity of the restriction map $\varphi^* : \Gamma(\mathbb{P}^N_F, \mathcal{O}_{\mathbb{P}^N_F}(1))^{k+1} \to \Gamma(X, \mathcal{N}^{\otimes\nu})^{k+1}$ the set $\varphi^*\mathcal{U}_k$ is a non-empty Zariski open subset of $\Gamma(X, \mathcal{N}^{\otimes\nu})^{k+1}$ consisting of $(k+1)$-tuples of global sections $(s_1,...,s_{k+1})$ satisfying the two properties (a'), (b'). □

4.2. Remark. Assuming that F is of characteristic zero and that the scheme X/F is smooth, it is easily seen by using [8], Corollaire 6.11.2, that Lemma 4.1 holds true with property (a') replaced by the property that $\mathrm{div}(s_1),...,\mathrm{div}(s_{k+1})$ intersect properly and smoothly on X.

4.3. Notation. Any k global sections $s_1,...,s_k$ of $\overline{\mathcal{M}}(m_1,m_2)$ induce k global sections of the complex line bundle $\overline{\mathcal{M}}(m_1,m_2)\otimes_{\mathbb{Q}}\mathbb{C}$ via the canonical embedding of $\mathbb{Q}$ into $\mathbb{C}$; these sections will again be denoted by $s_1,...,s_k$. With this in mind, we define for any k global sections $s_1,...,s_k \in \Gamma\left(\overline{A}_{n,D,\Theta}, \overline{\mathcal{M}}(m_1,m_2)\right)$ the following two subsets of $\overline{\mathcal{A}}_{n,D,\Theta}(\mathbb{C})$:

$$\begin{aligned}
\mathcal{S}(s_1,...,s_k) &:= \{x\in\overline{\mathcal{A}}_{n,D,\Theta}(\mathbb{C}) \mid \mathrm{div}(s_1|_{\overline{\pi}^{-1}(x)}),...,\mathrm{div}(s_k|_{\overline{\pi}^{-1}(x)}) \\
&\qquad \text{intersect properly on } \overline{\pi}^{-1}(x)\},\\
\mathcal{T}(s_1,...,s_k) &:= \overline{\mathcal{A}}_{n,D,\Theta}(\mathbb{C}) \setminus \mathcal{S}(s_1,...,s_k).
\end{aligned}$$

We note that $\mathcal{T}(s_1,...,s_k)$ is a Zariski closed subset of $\overline{\mathcal{A}}_{n,D,\Theta}(\mathbb{C})$.

4.4. Proposition. *With the above notations we have the following result: For $m_1 >> m_2 >> 0$ there exists a non-empty Zariski open subset consisting of $(n+1)$-tuples of global sections $(s_1,...,s_{n+1}) \in \Gamma\left(\overline{A}_{n,D,\Theta}, \overline{\mathcal{M}}(m_1,m_2)\right)^{n+1}$ satisfying the following two properties:*
(a) $\mathrm{codim}(\mathcal{T}(s_1,...,s_{n+1})) \geq 1$,
(b) $\mathrm{codim}(\mathcal{T}(s_{i_1},...,s_{i_n})) \geq 2$ *for any choice of n indices* $1\leq i_1<...<i_n\leq n+1$.

Proof. By applying Lemma 4.1 with $F=\mathbb{Q}$, $X=\overline{A}_{n,D,\Theta}$, $\mathcal{N}=\overline{\mathcal{M}}(m_1,m_2)$ (i.e., choose $m_1 >> m_2 >> 0$ in particular such that $\overline{\mathcal{M}}(m_1,m_2)$ is very ample) and $k=n$, we find for $\nu >> 0$ a non-empty Zariski open subset consisting of $(n+1)$-tuples of global sections $(s_1,...,s_{n+1}) \in \Gamma\left(\overline{A}_{n,D,\Theta}, \overline{\mathcal{M}}(\nu m_1,\nu m_2)\right)^{n+1}$ such that the corresponding $(n+1)$-tuples of holomorphic global sections, which we denote again by $(s_1,...,s_{n+1})$, satisfy the following two properties
(a') $\mathrm{div}(s_1),...,\mathrm{div}(s_{n+1})$ intersect properly on $\overline{A}_{n,D,\Theta}(\mathbb{C})$,
(b') $\mathrm{div}(s_{i_1})\cap...\cap\mathrm{div}(s_{i_n}) \subseteq \overline{A}_{n,D,\Theta}(\mathbb{C})$ is irreducible for any choice of n indices $1\leq i_1<...<i_n\leq n+1$.

It remains to show that the global sections under consideration satisfy properties (a) and (b); to do this we write m_j instead of νm_j $(j=1,2)$. To check property (a) we now put

$$X_{1,...,n+1} := \mathrm{div}(s_1)\cap...\cap\mathrm{div}(s_{n+1}),$$

which is by construction an $(n+1)$-codimensional cycle in $\overline{A}_{n,D,\Theta}(\mathbb{C})$. Furthermore, let $\rho := \overline{\pi}|_{X_{1,...,n+1}} : X_{1,...,n+1} \to \overline{\mathcal{A}}_{n,D,\Theta}(\mathbb{C})$ denote the restriction of $\overline{\pi}$ to $X_{1,...,n+1}$ and put $Y :=$

$\rho(X_{1,...,n+1}) \subseteq \overline{\mathcal{A}}_{n,D,\Theta}(\mathbb{C})$. Since ρ is proper, the subset Y is Zariski closed and we have $\mathrm{codim}(Y) \geq 1$. On the other hand, we have by definition

$$\mathcal{T}(s_1, ..., s_{n+1}) = \{x \in Y \mid \dim(\rho^{-1}(x)) \geq 0\};$$

hence, we arrive at $\mathrm{codim}(\mathcal{T}(s_1, ..., s_{n+1})) \geq \mathrm{codim}(Y) \geq 1$.

To check property (b), we note that for any choice of n indices $1 \leq i_1 < ... < i_n \leq n+1$, the intersections

$$X_{i_1,...,i_n} := \mathrm{div}(s_{i_1}) \cap ... \cap \mathrm{div}(s_{i_n})$$

are irreducible subschemes of $\overline{A}_{n,D,\Theta}(\mathbb{C})$ of codimension n. Let us fix a set of indices $1 \leq i_1 < ... < i_n \leq n+1$ and denote by $\rho' := \overline{\pi}|_{X_{i_1,...,i_n}} : X_{i_1,...,i_n} \to \overline{\mathcal{A}}_{n,D,\Theta}(\mathbb{C})$ the restriction of $\overline{\pi}$ to $X_{i_1,...,i_n}$. Because the restrictions of the n global sections $s_{i_1}, ..., s_{i_n}$ to any fibre of $\overline{\pi}$ intersect in at least one point, the morphism ρ' is surjective. Hence, we have by definition

$$\mathcal{T}(s_{i_1}, ..., s_{i_n}) = \{x \in \overline{\mathcal{A}}_{n,D,\Theta}(\mathbb{C}) \mid \dim(\rho'^{-1}(x)) \geq 1\}.$$

We note that the Zariski closed set $\mathcal{T}(s_{i_1}, ..., s_{i_n})$ is properly contained in $\overline{\mathcal{A}}_{n,D,\Theta}(\mathbb{C})$. If we now had $\mathrm{codim}(\mathcal{T}(s_{i_1}, ..., s_{i_n})) = 1$, we could deduce the inequality $\dim(\rho'^{-1}\mathcal{T}(s_{i_1}, ..., s_{i_n})) \geq \big(\dim(\overline{\mathcal{A}}_{n,D,\Theta}(\mathbb{C})) - 1\big) + 1$, i.e., $\mathrm{codim}(\rho'^{-1}\mathcal{T}(s_{i_1}, ..., s_{i_n})) \leq n$; the irreducibility of $X_{i_1,...,i_n}$ would then imply $\rho'^{-1}\mathcal{T}(s_{i_1}, ..., s_{i_n}) = X_{i_1,...,i_n}$, which contradicts the fact that $\mathcal{T}(s_{i_1}, ..., s_{i_n})$ is properly contained in $\overline{\mathcal{A}}_{n,D,\Theta}(\mathbb{C})$. Hence, we deduce $\mathrm{codim}(\mathcal{T}(s_{i_1}, ..., s_{i_n})) \geq 2$, as claimed. □

4.5. Remark. Denote by $\pi_{n,D,\Theta}$ the canonical projection $\pi_{n,D,\Theta} : \mathfrak{H}_n \longrightarrow \mathcal{A}_{n,D,\Theta}(\mathbb{C})$ and, with the notation 4.3, set

$$\begin{aligned}
\mathcal{S}'(s_1, ..., s_k) &:= \pi^{-1}_{n,D,\Theta}(\mathcal{S}(s_1, ..., s_k)) \subseteq \mathfrak{H}_n, \\
\mathcal{T}'(s_1, ..., s_k) &:= \pi^{-1}_{n,D,\Theta}(\mathcal{T}(s_1, ..., s_k)) \subseteq \mathfrak{H}_n.
\end{aligned}$$

We then call $(n+1)$ global sections $s_1, ..., s_{n+1} \in \Gamma\left(\overline{A}_{n,D,\Theta}, \overline{\mathcal{M}}(m_1, m_2)\right)$ to be *in general position at* $\tau_0 \in \mathfrak{H}_n$, if the following properties are satisfied:

(a) $\mathrm{codim}(\mathcal{T}'(s_1, ..., s_{n+1})) \geq 1$,
(b) $\mathrm{codim}(\mathcal{T}'(s_{i_1}, ..., s_{i_n})) \geq 2$ for any choice of n indices $1 \leq i_1 < ... < i_n \leq n+1$,
(c) $\tau_0 \in \mathcal{S}'(s_1, ..., s_{n+1})$.

The fact that for m_2 sufficiently large there exists a non-empty Zariski open subset of $(n+1)$-tuples of global sections $(s_{1,0}, ..., s_{n+1,0}) \in \Gamma\left(A_0, L_0^{\otimes m_2}\right)^{n+1}$, whose divisors intersect properly on A_0, together with the statement of Proposition 4.4 implies that there exist $(n+1)$ global sections $s_1, ..., s_{n+1} \in \Gamma\left(\overline{A}_{n,D,\Theta}, \overline{\mathcal{M}}(m_1, m_2)\right)$, which are in general position at $\tau_0 \in \mathfrak{H}_n$, provided $m_1 >> m_2 >> 0$. Furthermore, we note that the above construction shows that the global sections $s_1, ..., s_{n+1}$ can be chosen as rational, hence integral linear combinations of products of theta functions and Thetanullwerte with characteristics in $D^{-1}\mathbb{Z}^n/\mathbb{Z}^n$.

4.6. Remark. Assume that there are $(n+1)$ global sections $s_1, ..., s_{n+1} \in \Gamma(\overline{A}_{n,D,\Theta}, \overline{\mathcal{L}})$, which are in general position at $\tau_0 \in \mathfrak{H}_n$. Then, by associating

$$x \mapsto P = P(x) \in \mathrm{div}(s_1|_{\pi^{-1}(x)}) \cdot ... \cdot \mathrm{div}(s_n|_{\pi^{-1}(x)}),$$

we obtain a holomorphic section $P : \mathcal{A}_{n,D,\Theta} \longrightarrow A_{n,D,\Theta}$. By [25] such a section is necessarily a torsion section, whence the α, β-coordinates of $P(x)$ are constant, i.e., independent of $x \in$

$\mathcal{A}_{n,D,\Theta}$, resp. $\tau \in \mathfrak{H}_n$, mod. $\mathbb{Z}^n$ and rational. With regard to Remark 3.4, we therefore have for all $\tau \in \mathcal{S}'(s_1, ..., s_{n+1})$

$$d_\tau d_\tau^c \left(\sum_{P \in D_{n,\tau}} g_{n+1,\tau}(P) + c_{n,D} \cdot \log \det \eta \right) = 0.$$

Hence, by Corollary 3.10, we obtain *locally* for all $\tau \in \mathcal{D}$, an embedded unit disc satisfying $\mathcal{D} \subseteq \mathcal{S}'(s_1, ..., s_n)$ and $\mathcal{D}^* \subseteq \mathcal{S}'(s_1, ..., s_{n+1})$ (here $\mathcal{D}^* = \mathcal{D} \setminus \{0\}$),

$$d_\tau d_\tau^c \left(\int_{A_\tau(\mathbb{C})} g_{1,\tau}(z) * ... * g_{n+1,\tau}(z) - \sum_{P \in D_{n,\tau}} g_{n+1,\tau}(P) + n \cdot c_{n,D} \cdot \log \det \eta \right) = 0.$$

5 Main results

5.1. Definition. A holomorphic function f on $\mathfrak{H}_n$, $n \geq 2$, is called a *Siegel modular form of weight k with respect to the subgroup $\Gamma_{n,D,\Theta}$ of the paramodular group $\Gamma_{n,D}$ and some finite character $\chi : \Gamma_{n,D,\Theta} \to \mathbb{C}^*$*, if it satisfies (cf. 2.1)

$$F\,(R(\tau))\,\det(D^{-1}c_R\tau + D^{-1}d_R D)^{-k} = \chi(R)\,F(\tau)$$

for all

$$R = \begin{pmatrix} a_R & b_R \\ c_R & d_R \end{pmatrix} \in \Gamma_{n,D,\Theta}.$$

The $\mathbb{C}$-vector space spanned by such functions will be denoted by $M_k(\Gamma_{n,D,\Theta}, \chi)$.

5.2. Theorem. *Let $A_0 = A_{\tau_0}$ be an n-dimensional abelian variety, $n \geq 2$, of polarization type D together with the line bundle $L_0 = L_{\tau_0}$, everything being defined over some field $K \subseteq \mathbb{C}$. Furthermore, assume that there exist $(n+1)$ global sections $s_1, ..., s_{n+1} \in \Gamma\left(\overline{\mathcal{A}}_{n,D,\Theta}, \overline{\mathcal{M}}(m_1, m_2)\right)$, which are in general position at $\tau_0 \in \mathfrak{H}_n$. Then, there exists a Siegel modular form $F \in M_{C_{n,D;m_1,m_2}}(\Gamma_{n,D,\Theta}, \chi)$ of weight $C_{n,D;m_1,m_2} := (m_1 + m_2/2) \cdot m_2^n \cdot d \cdot (n+1)!$ and some finite character χ such that the equality*

$$\int_{A_\tau(\mathbb{C})} g_{1,\tau}(z) * ... * g_{n+1,\tau}(z) = -\log|F(\tau)|^2 - C_{n,D;m_1,m_2} \cdot \log \det \eta \tag{4}$$

holds for all $\tau \in \mathcal{S}'(s_1, ..., s_{n+1})$.

Proof. Noting that the global sections in question now have weight $(m_1 + m_2/2)$ instead of $1/2$ and that L_0 has been replaced by $L_0^{\otimes m_2}$, Corollary 3.10 asserts that for any $\tau' \in \mathcal{S}'(s_1, ..., s_{n+1})$

$$d_\tau d_\tau^c \left(\int_{A_\tau(\mathbb{C})} g_{1,\tau}(z) * ... * g_{n+1,\tau}(z) + C_{n,D;m_1,m_2} \cdot \log \det \eta \right) = 0 \tag{5}$$

for all τ sufficiently close to τ'. We conclude that for any open and simply connected subset $U \subseteq \mathcal{S}'(s_1, ..., s_{n+1})$ there exists a non-vanishing, holomorphic function F_U satisfying

$$\int_{A_\tau(\mathbb{C})} g_{1,\tau}(z) * ... * g_{n+1,\tau}(z) + C_{n,D;m_1,m_2} \cdot \log \det \eta + \log|F_U(\tau)|^2 = 0$$

for all $\tau \in U$; by the commutativity of the integral over the $(n+1)$-fold $*$-product of Green currents, the preceding formula holds true with the factors $g_{1,\tau}(z), ..., g_{n+1,\tau}(z)$ permuted in an arbitrary way. We will show now that F_U extends to a holomorphic function on all of $\mathfrak{H}_n$. If we have $\mathrm{codim}(\mathcal{T}'(s_1, ..., s_{n+1})) \geq 2$, F_U extends to a holomorphic function F defined on all of $\mathfrak{H}_n$ by the Riemann removable singularity Theorem (cf. [12], p. 262) and the simple connectivity of $\mathfrak{H}_n$. On the other hand, if the codimension in question is one, we proceed as follows: Define

$$\mathcal{U} := \bigcup_{1 \leq i_1 < ... < i_n \leq (n+1)} \mathcal{S}'(s_{i_1}, ..., s_{i_n}) \subseteq \mathfrak{H}_n\,;$$

further, let $\mathcal{D} \subseteq \mathfrak{H}_n$ be an embedded unit disk, let $\mathcal{D}^*$ be $\mathcal{D}$ minus the image of the origin of that unit disk and assume

(i) $\mathcal{D}^* \subseteq \mathcal{S}'(s_1, ..., s_{n+1})$,
(ii) $\mathcal{D} \subseteq \mathcal{S}'(s_{i_1}, ..., s_{i_n})$ for some choice of n indices $1 \leq i_1 < ... < i_n \leq n+1$.

Then, for the choice of indices $1 \leq i_1 < ... < i_n \leq n+1$ made in (ii), denote by $D_{i_1,...,i_n;\tau}$ the intersection of the n divisors $\Theta_{i_1,\tau}, ..., \Theta_{i_n,\tau}$; finally, let $i_{n+1} \in \{1, ..., n+1\}$ be the index different from $i_1, ..., i_n$. We now derive from Remark 4.6 that the equality (note again that the global sections in question have weight $(m_1 + m_2/2)$ instead of $1/2$, L_0 has been replaced by $L_0^{\otimes m_2}$ and that the set of indices $\{1, ..., n+1\}$ has been replaced by $\{i_1, ..., i_{n+1}\}$)

$$d_\tau d_\tau^c \left(\int_{A_\tau(\mathbb{C})} g_{i_1,\tau}(z) * ... * g_{i_{n+1},\tau}(z) + n \cdot (m_1 + m_2/2) \cdot m_2^n \cdot d \cdot n! \log\det\eta - \right.$$
$$\left. \sum_{P \in D_{i_1,...,i_n;\tau}} g_{i_{n+1},\tau}(P) \right) = 0$$

holds for all $\tau \in \mathcal{D}$. Therefore, there is a non-vanishing, holomorphic function $f_{\mathcal{D}}^{(1)}$ on $\mathcal{D}$ such that the equality

$$\int_{A_\tau(\mathbb{C})} g_{i_1,\tau}(z) * ... * g_{i_{n+1},\tau}(z) + n \cdot (m_1 + m_2/2) \cdot m_2^n \cdot d \cdot n! \log\det\eta -$$
$$\sum_{P \in D_{i_1,...,i_n;\tau}} g_{i_{n+1},\tau}(P) + \log|f_{\mathcal{D}}^{(1)}(\tau)|^2 = 0 \tag{6}$$

holds for all $\tau \in \mathcal{D}$. Let now t be a local coordinate on $\mathcal{D}$ and let ν denote the number of points in the intersection of $D_{i_1,...,i_n;\tau''}$ with $\Theta_{i_{n+1},\tau''}$ at the point $\tau'' \in \mathcal{D}$ corresponding to $t = 0$. Then, it again follows from Remark 4.6 that there is a non-vanishing, holomorphic function $f_{\mathcal{D}}^{(2)}$ on $\mathcal{D}$ such that the equality

$$\sum_{P \in D_{i_1,...,i_n;\tau}} g_{i_{n+1},\tau}(P) + (m_1 + m_2/2) \cdot m_2^n \cdot d \cdot n! \log\det\eta +$$
$$\nu \log|t|^2 + \log|f_{\mathcal{D}}^{(2)}(t)|^2 = 0 \tag{7}$$

holds for all $\tau \in \mathcal{D}$. Adding (6) to (7) defines a holomorphic function $F_{\mathcal{D}}$ on $\mathcal{D}$, which vanishes only at the point $\tau'' \in \mathcal{D}$ corresponding to $t = 0$, namely to the positive integral order ν, such that we have for all $\tau \in \mathcal{D}$

$$\int_{A_\tau(\mathbb{C})} g_{i_1,\tau}(z) * ... * g_{i_{n+1},\tau}(z) + C_{n,D;m_1,m_2} \cdot \log\det\eta + \log|F_{\mathcal{D}}(\tau)|^2 = 0.$$

The above argument implies that for any open and simply connected subset $U \subseteq \mathcal{U}$ there exists a holomorphic, not necessarily non-vanishing function F_U such that

$$\int_{A_\tau(\mathbb{C})} g_{1,\tau}(z) * ... * g_{n+1,\tau}(z) + C_{n,D;m_1,m_2} \cdot \log \det \eta + \log |F_U(\tau)|^2 = 0$$

for all $\tau \in U$. By assumption we have $\mathrm{codim}(\mathfrak{H}_n \setminus \mathcal{U}) \geq 2$; hence, again by the Riemann removable singularity Theorem and the simple connectivity of $\mathfrak{H}_n$, F_U extends to a holomorphic function F defined on all of $\mathfrak{H}_n$.

Since the integral

$$\int_{A_\tau(\mathbb{C})} g_{1,\tau}(z) * ... * g_{n+1,\tau}(z)$$

is $\Gamma_{n,D,\Theta}$-invariant, so is

$$|F(\tau)|^2 \det \eta^{C_{n,D;m_1,m_2}} = \exp\left(\log |F(\tau)|^2 + C_{n,D;m_1,m_2} \cdot \log \det \eta\right).$$

Therefore, F is a Siegel modular form of weight $C_{n,D;m_1,m_2}$ with respect to $\Gamma_{n,D,\Theta}$ and some character χ. By [22], p. 109, the commutator subgroup $[\Gamma_{n,D,\Theta}, \Gamma_{n,D,\Theta}]$ is of finite index in $\Gamma_{n,D,\Theta}$, which shows that the character χ is finite. We have now constructed $F \in M_{C_{n,D;m_1,m_2}}(\Gamma_{n,D,\Theta}, \chi)$ such that the claimed formula (4) holds for all $\tau \in \mathcal{S}'(s_1, ..., s_{n+1})$. This finishes the proof of the theorem. □

5.3. Remark. As we shall see in the next corollary, a variation of the proof of Theorem 5.2 leads to an explicit description of the modular form $F \in M_{C_{n,D;m_1,m_2}}(\Gamma_{n,D,\Theta}, \chi)$. If one is only interested in proving formula (4),

$$\int_{A_\tau(\mathbb{C})} g_{1,\tau}(z) * ... * g_{n+1,\tau}(z) = -\log |F(\tau)|^2 - C_{n,D;m_1,m_2} \cdot \log \det \eta,$$

without any further knowledge about the modular form F in question, J.-B. Bost pointed out to the authors the following argument: To simplify the exposition we assume that there exist $(n+1)$ global sections $s_1, ..., s_{n+1} \in \Gamma(A_{n,D,\Theta}, \mathcal{L})$ having properly intersecting divisors, i.e., we assume $m_1 = 0, m_2 = 1$. Then, $g_1 * ... * g_{n+1}$ is a well-defined Green current of type (n, n) on $A_{n,D,\Theta}(\mathbb{C})$ satisfying

$$dd^c(g_1 * ... * g_{n+1}) + \delta_{D_{n+1}} = c_1(\mathcal{L}, \|\cdot\|)^{n+1}.$$

Taking the direct image of this equation with respect to the proper map $\pi : A_{n,D,\Theta} \longrightarrow \mathcal{A}_{n,D,\Theta}$ and observing that π_* commutes with dd^c and δ, we derive

$$d_\tau d^c_\tau \left(\pi_*(g_1 * ... * g_{n+1})\right) + \delta_{\pi_* D_{n+1}} = \pi_* \left(c_1(\mathcal{L}, \|\cdot\|)^{n+1}\right).$$

Now, it is clear that the class of $e^*\mathcal{L}$ equals half of the class of $\mathcal{K}$ in $\mathrm{Pic}(\mathcal{A}_{n,D,\Theta})_{\mathbb{Q}}$ and, furthermore, it follows from [3], chapter I, or [17], appendice 2, that the class of $\det \pi_*(\mathcal{L} \otimes \pi^* e^* \mathcal{L}^{\otimes -1})$ equals $-d/2$ times the class of $\mathcal{K}$, again in $\mathrm{Pic}(\mathcal{A}_{n,D,\Theta})_{\mathbb{Q}}$; this implies that the class of $\det \pi_* \mathcal{L}$ is trivial in $\mathrm{Pic}(\mathcal{A}_{n,D,\Theta})_{\mathbb{Q}}$. We also note that [18] shows that all of the above is compatible with the hermitian metrics in question, $\mathcal{K}$ being equipped with the Petersson metric $\|\cdot\|_{Pet}$. A short calculation, using the Hirzebruch-Riemann-Roch Theorem, then gives

$$\pi_* \left(c_1(\mathcal{L}, \|\cdot\|)^{n+1}\right) = \frac{1}{2} \cdot d \cdot (n+1)! \cdot c_1(\mathcal{K}, \|\cdot\|_{Pet}).$$

Therefore, the function $\pi_*(g_1 * ... * g_{n+1})$ in question is given as

$$-\log\left(|F(\tau)|^2 \cdot \det \eta^{\frac{1}{2}\cdot d\cdot(n+1)!}\right),$$

where $F \in \Gamma(\mathcal{A}_{n,D,\Theta}, \mathcal{K}^{\otimes \frac{1}{2}\cdot d\cdot(n+1)!})$ (with divisor $\pi_* D_{n+1}$), i.e., F is a modular form of weight $k = \frac{1}{2}\cdot d \cdot (n+1)!$ with respect to $\Gamma_{n,D,\Theta}$.

5.4. Corollary. *Assume that the hypotheses of Theorem 5.2 hold. Then, the modular form $F \in M_{C_{n,D;m_1,m_2}}(\Gamma_{n,D,\Theta}, \chi)$ of Theorem 5.2 is given by the formula*

$$F(\tau) = \zeta \prod_{1\leq i_1 < ... < i_n \leq n+1} \prod_{P \in D_{i_1,...,i_n;\tau}} s_{i_{n+1},\tau}(P) \cdot e^{\pi i m_2 \alpha_P^t \tau \alpha_P},$$

where ζ is a non-zero constant (unique up to multiplication by a complex number of absolute value one), $i_{n+1} \in \{1, ..., n+1\}$ denotes the index different from $i_1, ..., i_n$ and $D_{i_1,...,i_n;\tau}$ the intersection of the n divisors $\Theta_{i_1,\tau}, ..., \Theta_{i_n,\tau}$ taken for $\tau \in \mathcal{S}'(s_{i_1}, ..., s_{i_n})$ and $P \in D_{i_1,...,i_n;\tau}$ is written as $P = -\tau \cdot \alpha_P + \beta_P$.

Proof. We will vary the proof of Theorem 5.2 slightly: We consider the difference

$$\int_{A_\tau(\mathbb{C})} g_{1,\tau}(z) * ... * g_{n+1,\tau}(z) - \sum_{1\leq i_1<...<i_n\leq n+1} \sum_{P\in D_{i_1,...,i_n;\tau}} g_{i_{n+1},\tau}(P); \tag{8}$$

by Corollary 3.10 and Remark 4.6 it is harmonic for $\tau \in \mathcal{S}'(s_1, ..., s_{n+1})$. Let

$$\mathcal{U} := \bigcap_{1\leq i_1<...<i_n\leq n+1} \mathcal{S}'(s_{i_1}, ..., s_{i_n}) \subseteq \mathfrak{H}_n$$

and $\mathcal{D} \subseteq \mathfrak{H}_n$ be an embedded unit disk satisfying

(i) $\mathcal{D}^* \subseteq \mathcal{S}'(s_1, ..., s_{n+1})$,
(ii) $\mathcal{D} \subseteq \mathcal{U}$.

Then, we rewrite (8) as

$$\int_{A_\tau(\mathbb{C})} g_{1,\tau}(z) * ... * g_{n+1,\tau}(z) \quad - \sum_{P\in D_{n,\tau}} g_{n+1,\tau}(P)$$

$$- \sum_{i_{n+1}\neq n+1} \sum_{P\in D_{i_1,...,i_n;\tau}} g_{i_{n+1},\tau}(P), \tag{9}$$

where the sum over $i_{n+1} \neq n+1$ is a shorthand for the sum over all n-tuples $1 \leq i_1 < ... < i_n \leq n+1$ different from the n-tuple $1, ..., n$. By considering the difference between the above integral and the first sum and then the double sum individually, we conclude as in the proof of Theorem 5.2 using Remark 4.6 that the difference (8) can be expressed as the logarithm of the absolute value square of some holomorphic function on $\mathcal{D}$, hence on the whole of $\mathcal{U}$. Since $\operatorname{codim}(\mathfrak{H}_n \setminus \mathcal{U}) \geq 2$, this function extends to a holomorphic function H on all of $\mathfrak{H}_n$ satisfying

$$\int_{A_\tau(\mathbb{C})} g_{1,\tau}(z) * ... * g_{n+1,\tau}(z) - \sum_{1\leq i_1<...<i_n\leq n+1} \sum_{P\in D_{i_1,...,i_n;\tau}} g_{i_{n+1},\tau}(P) = \log|H(\tau)|^2.$$

Obviously, H must be a modular form of weight 0, i.e., H is a holomorphic modular function, hence a constant on all of $\mathfrak{H}_n$. This constant must be non-zero, since $H(\tau)$ does not vanish for $\tau \in \mathcal{S}'(s_1, ..., s_{n+1})$. Now the proof can be easily completed. $\square$

5.5. Remark. The quantity ζ appearing in Corollary 5.4 is by construction a constant with respect to $\tau \in \mathfrak{H}_n$, which depends of course on the choice of the global sections $s_1, ..., s_{n+1}$, i.e., $s_{1,0}, ..., s_{n+1,0}$ under consideration. Here this dependence is determined: First, ζ is a symmetric function in $s_{1,0}, ..., s_{n+1,0}$; hence, it suffices to study ζ as a function of the single variable $s = s_{n+1,0}$, while fixing $s_{1,0}, ..., s_{n,0}$ such that their divisors intersect properly. Secondly, the definition of ζ in the proof of Corollary 5.4 together with formula (9) shows that ζ is independent of the scale of s; hence, ζ can be viewed as a function on the projective space $\mathbb{P} := \mathbb{P}\Gamma(A_0(\mathbb{C}), L_{0,\mathbb{C}}^{\otimes m_2})$. Let $\mathbb{P}^\vee$ denote the dual projective space of $\mathbb{P}$. Since we may assume without loss of generality that the line bundle under consideration is very ample, we have an embedding $A_0(\mathbb{C}) \hookrightarrow \mathbb{P}^\vee$ given by mapping $P \in A_0(\mathbb{C})$ to the hyperplane $E_P \in \mathbb{P}^\vee$ determined by the set of those sections $s \in \mathbb{P}$ vanishing at P.

By arguing as in section 4, it can be shown that the intersection of $\mathrm{div}(s)$ with $D_{n,0}$ is proper, i.e., empty, for all $s \in \mathbb{P}$ away from a 1-codimensional subset $E \subset \mathbb{P}$, and that the intersections $D_{i_1,...,i_n;\tau_0}$ are proper for all n-tuples $1 \leq i_1 < ... < i_n \leq n+1$ different from the n-tuple $1, ..., n$ for all $s \in \mathbb{P}$ away from a 2-codimensional subset $E' \subset E \subset \mathbb{P}$. For $s \in \mathbb{P} \setminus E$, the definition of ζ together with formula (9) leads to the following differential equation

$$-d_s d_s^c \log |\zeta|^2 = d_s d_s^c \sum_{i_{n+1} \neq n+1} \sum_{P \in D_{i_1,...,i_n;\tau_0}} g_{i_{n+1},\tau_0}(P).$$

For $s \in \mathbb{P} \setminus E'$, let N denote the cardinality of the disjoint union of the proper intersections $D_{i_1,...,i_n;\tau_0}$ (counting multiplicities) for all n-tuples $1 \leq i_1 < ... < i_n \leq n+1$ different from the n-tuple $1, ..., n$, namely $N = n \cdot m_2^n \cdot d \cdot n!$. Furthermore, let $\mathbb{A} := \mathrm{Sym}_N(A_0(\mathbb{C}))$ be the N-fold symmetric product of $A_0(\mathbb{C})$ with itself. Then, we obtain a morphism $f : \mathbb{P} \setminus E' \longrightarrow \mathbb{A}$, given by associating to $s \in \mathbb{P} \setminus E'$ the points in the disjoint union of the proper intersections

$$\dot{\bigcup_{i_{n+1} \neq n+1}} D_{i_1,...,i_n;\tau_0} \subset \mathbb{A},$$

again taking into account multiplicities. Since $\mathrm{codim}(E') \geq 2$, this morphism extends to a morphism from $\mathbb{P}$ to $\mathbb{A}$, which is again denoted by f. Representing the points $P \in f(s)$ by $z(s) = x(s) + iy(s) \in \mathbb{C}^n$ (the universal covering of $A_0(\mathbb{C})$) and using the definition of the Green current $g_{i_{n+1},\tau_0}(z(s))$, the above differential equation becomes

$$-d_s d_s^c \log |\zeta|^2 + n \cdot m_2 \cdot \sum_{P \in D_{n,0}} \delta_{E_P} = m_2 \cdot \mu_{\mathbb{P}}$$

with the $(1,1)$-form

$$\mu_{\mathbb{P}} = d_s d_s^c \sum_{P \in f(s)} 2\pi y(s)^t \eta_0^{-1} y(s);$$

here, as usual, $\tau_0 = \xi_0 + i\eta_0 \in \mathfrak{H}_n$. A direct computation yields $\mu_{\mathbb{P}} = f^* \mu_{\mathbb{A}}$, where

$$\mu_{\mathbb{A}} = \sum_{j=1}^{N} p_j^* \left(d_z d_z^c (2\pi y^t \eta_0^{-1} y) \right)$$

with $p_j : \prod_{j=1}^N A_0(\mathbb{C}) \to A_0(\mathbb{C})$ denoting the projection onto the j-th factor ($j = 1, ..., N$). The translation invariance of the $(1,1)$-form $d_z d_z^c(2\pi y^t \eta_0^{-1} y)$ on $A_0(\mathbb{C})$ then implies the invariance of the $(1,1)$-forms $(p_j \circ f)^*(d_z d_z^c(2\pi y^t \eta_0^{-1} y))$ with respect to the action of the unitary group operating on $\mathbb{P}$; by symmetry, we therefore conclude $\mu_{\mathbb{P}} = N \cdot c'(L_0) \cdot \mu_{FS}$, where $c'(L_0)$ is a constant depending only on the line bundle L_0 and μ_{FS} denotes the first Chern form of $\mathcal{O}_{\mathbb{P}}(1)$ with respect to the Fubini-Study metric. Hence, we obtain the differential equation

$$-d_s d_s^c \log|\zeta|^2 + n \cdot m_2 \cdot \sum_{P \in D_{n,0}} \delta_{E_P} = N \cdot m_2 \cdot c'(L_0) \cdot \mu_{FS}.$$

In particular, by taking cohomology classes on both sides of the above equation, we derive $c'(L_0) = 1$. Now, this differential equation can be solved explicitly in terms of the normalized Green's function $g_{\mathbb{P}}(\,\cdot\,,\,\cdot\,)$ relative to μ_{FS} viewed as a function on $\mathbb{P} \times \mathbb{P}^\vee$ (cf. [15], p. 26). Using the symmetry in $s_{1,0}, ..., s_{n+1,0}$, we finally arrive at the formula

$$-\log|\zeta|^2 = m_2 \cdot \sum_{1 \leq i_1 < ... < i_n \leq n+1} \sum_{P \in D_{i_1,...,i_n;\tau_0}} g_{\mathbb{P}}(s_{i_{n+1},0}, P) + m_2^n \cdot d \cdot (n+1)! \cdot c(L_0);$$

here P has to be interpreted as an element of $\mathbb{P}^\vee$ by identifying P with the hyperplane E_P, and $c(L_0)$ is a constant depending only on the line bundle L_0.

5.6. Remark. Theorem 5.2 and Corollary 5.4 have been proven under the assumption that there exist $(n+1)$ global sections $s_1, ..., s_{n+1}$, which are *in general position at* $\tau_0 \in \mathfrak{H}_n$. We now introduce the following more relaxed condition: For a fixed set of global sections $s_1, ..., s_{n+1} \in \Gamma\left(\overline{A}_{n,D,\Theta}, \overline{\mathcal{M}}(m_1, m_2)\right)$, put

$$\begin{aligned} \mathcal{U}_{n+1} &:= \mathcal{S}'(s_1, ..., s_{n+1}), \\ \mathcal{U}_n &:= \{\tau \in \mathfrak{H}_n \mid \exists\, 1 \leq i_1 = i_1(\tau) < ... < i_n = i_n(\tau) \leq n+1 \\ &\qquad \text{such that } \operatorname{div}(s_{i_1,\tau}), ..., \operatorname{div}(s_{i_n,\tau}) \text{ intersect properly}\}. \end{aligned}$$

We then call the $(n+1)$ global sections $s_1, ..., s_{n+1}$ *in general position*, if

(ã) $\operatorname{codim}(\mathfrak{H}_n \setminus \mathcal{U}_{n+1}) \geq 1$,
(b̃) $\operatorname{codim}(\mathfrak{H}_n \setminus \mathcal{U}_n) \geq 2$.

We show that the statement of Corollary 5.4 holds true assuming this weaker condition. For this consider the difference

$$\int_{A_\tau(\mathbb{C})} g_{i_1(\tau_1),\tau}(z) * ... * g_{i_{n+1}(\tau_1),\tau}(z) - \sum_{1 \leq i_1 < ... < i_n \leq n+1} \int_{A_\tau(\mathbb{C})} g_{i_{n+1},\tau}(z) \wedge \delta_{D_{i_1,...,i_n;\tau}} \wedge \Omega_\nu \tag{10}$$

for τ in some transverse disk $\mathcal{D}$ around a fixed $\tau_1 \in \mathcal{U}_n \setminus \mathcal{U}_{n+1}$; here $\nu = \dim D_{i_1,...,i_n;\tau}$. Note that the set $\mathcal{U}_n \setminus \mathcal{U}_{n+1}$ consists of irreducible components and that the integer $i_{n+1}(\tau)$ is constant on each of these components. Now, rewrite (10) as

$$\int_{A_\tau(\mathbb{C})} g_{i_1(\tau_1),\tau}(z) * \ldots * g_{i_{n+1}(\tau_1),\tau}(z) - \sum_{P\in D_{i_1(\tau_1),\ldots,i_n(\tau_1);\tau}} g_{i_{n+1}(\tau_1),\tau}(P) -$$

$$\sum_{i_{n+1}\neq i_{n+1}(\tau_1)} \int_{A_\tau(\mathbb{C})} g_{i_{n+1},\tau}(z) \wedge \delta_{D_{i_1,\ldots,i_n;\tau}} \wedge \Omega_\nu,$$

where the sum over $i_{n+1} \neq i_{n+1}(\tau_1)$ is a shorthand for the sum over all n-tuples $1 \leq i_1 < \ldots < i_n \leq n+1$ different from the n-tuple $1 \leq i_1(\tau_1) < \ldots < i_n(\tau_1) \leq n+1$. Now one applies Remark 4.6 to the difference between the first integral and the first sum and the fact that the second integral is bounded from below in order to conclude as in the proof of Corollary 5.4 that the difference (10) equals the logarithm of the absolute value square of some non-zero constant on all of $\mathfrak{H}_n$. This proves the desired variant of Corollary 5.4. □

5.7. Remark. Theorem 5.2 together with Corollary 5.4 leads to the following determination of the analytic contribution to the arithmetic degree of line bundles on abelian varieties over number fields; for the definition of the arithmetic degree of hermitian vector bundles on arithmetic varieties we refer to [4] or [23]: Suppose that $s_1, \ldots, s_{n+1} \in \Gamma\left(\overline{A}_{n,D,\Theta}, \overline{\mathcal{L}}\right)$, are $(n+1)$ global sections, which are defined over $\mathbb{Q}$ and in general position; Remark 4.5 shows that this is possible, eventually after replacing $\overline{\mathcal{L}}$ by $\overline{\mathcal{M}}(m_1, m_2)$ and choosing $m_1 >> m_2 >> 0$; to simplify the exposition we assume $m_1 = 0, m_2 = 1$. Let now $\tau \in \mathcal{S}'(s_1, \ldots, s_{n+1})$ be such that the abelian variety $A = A_\tau$ is defined over some number field K and has good reduction at all the finite places of the ring of integers $\mathcal{O}_K$ of K, i.e., gives rise to an abelian scheme $\tilde{A}/\mathrm{Spec}\,\mathcal{O}_K$. As usual, let $L = L_\tau$ be the line bundle on A equipped with the smooth hermitian metric $\|\cdot\|$ having translation invariant curvature. Furthermore, denote by $\tilde{L}$ the unique extension of L to a symmetric, relatively ample line bundle on $\tilde{A}$ satisfying the theorem of the cube (cf. [17], chapitre II). Then, the analytic contribution $\deg_\infty(L, \|\cdot\|)$ to the arithmetic degree $\widehat{\deg}(L, \|\cdot\|)$ of L (or rather of $\tilde{L}$) is given by the formula

$$\deg_\infty(L, \|\cdot\|) = -\frac{1}{2} \sum_{\sigma: K \hookrightarrow \mathbb{C}} \left(\log |F(\tau^{(\sigma)})|^2 + \frac{1}{2} \cdot d \cdot (n+1)! \cdot \log\det \eta^{(\sigma)} \right);$$

here $\tau^{(\sigma)} \in \mathfrak{H}_n$ is such that $A \times_\sigma \mathbb{C} = A_{\tau^{(\sigma)}}$ and $F \in M_{\frac{1}{2}\cdot d\cdot(n+1)!}(\Gamma_{n,D,\Theta}, \chi)$, a modular form, which is explicitly determined by the formula given in Corollary 5.4. The recent preprint [14] of K. Künnemann shows that the above result can also be applied, if the abelian variety A/K in question has semi-stable reduction at all the finite places of the ring of integers $\mathcal{O}_K$.

In [13], Proposition 13.1, K. Künnemann proves a result which is analogous to ours. He shows that the analytic contribution to the arithmetic degree in question can be expressed in terms of classical Thetanullwerte provided that they do not vanish on the abelian schemes under consideration. We therefore view our application of Theorem 5.2 and Corollary 5.4 as being supplementary to the result of K. Künnemann.

6 Examples

6.1. The case $n = 1, d = 1$. This example is included here for the sake of completeness; we also refer to [1] or [10]. We point out that it is not covered by Corollary 5.4 nor the subsequent

Remark 5.6 because of dimension reasons and since 4 does not divide d. Since $D = (1)$, we are working here with the universal elliptic surface $A_{1,(1),\Theta}$ over the modular curve $\mathcal{A}_{1,(1),\Theta}$ attached to Igusa's group $\Gamma_1(1,2)$ (cf. [7], p. 178) and with the line bundle $\mathcal{L}_{00} := \mathcal{L}$ over $A_{1,(1),\Theta}$ induced by the theta function $\vartheta(\tau, z)$. The functions

$$s_{00,\tau}(z) := \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau, z)^2 , \; s_{11,\tau}(z) := \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (\tau, z)^2$$

then are global sections of the line bundle $\mathcal{L}_{00}^{\otimes 2}$ over $A_{1,(1),\Theta}$. Using the integral formula given in 1.1, we then compute (with the obvious notations)

$$\int_{A_\tau(\mathbb{C})} g_{11,\tau}(z) * g_{00,\tau}(z) = \int_{A_\tau(\mathbb{C})} \left(g_{11,\tau}(z) \wedge \omega + g_{00,\tau}(z) \wedge \delta_{\mathrm{div}(s_{11,\tau})}\right) =$$

$$-4 \int_0^1 \int_0^1 \log \left| \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau, 0) \right|^2 d\alpha d\beta - 4 \log \left| \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau, 0) \right|^2 - 4 \log \eta =$$

$$-\frac{1}{6} \log |\Delta(\tau)|^2 - 4 \log \left| \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau, 0) \right|^2 - 4 \log \eta.$$

Analogously, we could work with the line bundle $\mathcal{L}_{01}$, resp. $\mathcal{L}_{10}$, instead of $\mathcal{L}$ over the universal elliptic surface attached to the congruence subgroup $\Gamma^0(2)$, resp. $\Gamma_0(2)$, induced by the theta functions

$$\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (\tau, z), \text{ resp. } \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (\tau, z).$$

Observing now that

$$s_{01,\tau}(z) := \vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (\tau, z)^2, \text{ resp. } s_{10,\tau}(z) := \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (\tau, z)^2,$$

together with $s_{11,\tau}(z)$ are global sections of the line bundle $\mathcal{L}_{01}^{\otimes 2}$, resp. $\mathcal{L}_{10}^{\otimes 2}$, we derive as before

$$\int_{A_\tau(\mathbb{C})} g_{11,\tau}(z) * g_{01,\tau}(z) = -\frac{1}{6} \log |\Delta(\tau)|^2 - 4 \log \left| \vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (\tau, 0) \right|^2 - 4 \log \eta,$$

respectively

$$\int_{A_\tau(\mathbb{C})} g_{11,\tau}(z) * g_{10,\tau}(z) = -\frac{1}{6} \log |\Delta(\tau)|^2 - 4 \log \left| \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (\tau, 0) \right|^2 - 4 \log \eta.$$

Adding up, we obtain

$$\int_{A_\tau(\mathbb{C})} g_{11,\tau}(z) * g_{00,\tau}(z) + \int_{A_\tau(\mathbb{C})} g_{11,\tau}(z) * g_{01,\tau}(z) + \int_{A_\tau(\mathbb{C})} g_{11,\tau}(z) * g_{10,\tau}(z) = -\log |\Delta(\tau)|^2 - 12 \log \eta - 8 \log 2, \tag{11}$$

since

$$\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau, 0) \cdot \vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (\tau, 0) \cdot \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (\tau, 0) = 2 \cdot \Delta(\tau)^{1/8}.$$

6.2. Remark. We complement example 6.1 by showing how the above computations can be used to derive an explicit formula for the arithmetic degree of hermitian line bundles on semi-stable elliptic curves: To illustrate the method, we consider for simplicity an elliptic curve $A/\mathbb{Q}$ having multiplicative reduction for one odd prime p, good reduction for the remaining odd primes and ordinary reduction for the prime 2; furthermore, we assume that the 2-torsion of A is *rational*. We denote by $\Delta_A^{\min}$ the minimal discriminant of A; by assumption, we have $\Delta_A^{\min} = p^{n_p}$ with a certain positive integer n_p, which in addition will be assumed to be *even*. An example of such an elliptic curve $A/\mathbb{Q}$ is given by the Tate equation $Y^2+XY+Y = X^3-X^2-6X-4$ with minimal discriminant $\Delta_A^{\min} = 17^2$. We denote by $\tilde{A}/\mathbb{Z}$ the Néron model associated to $A/\mathbb{Q}$. By our assumptions, only the fibre $\tilde{A}_p/\mathbb{F}_p$ of $\tilde{A}/\mathbb{Z}$ over the closed point $(p) \in \operatorname{Spec}\mathbb{Z}$ is reducible; it is given as an n_p-gon whose edges are denoted by C_ν $(\nu = 0, ..., n_p - 1)$, C_0 intersecting the image of $\operatorname{Spec}\mathbb{Z}$ by the zero-section. According to [17], chapitre II, the n_p-th tensor-power of the symmetric, ample line bundle $L^{\otimes 2} = L_\tau^{\otimes 2}$ on $A = A_\tau$ has a unique extension to a symmetric, relatively ample line bundle $\tilde{M}$ on $\tilde{A}$ satisfying the theorem of the cube. Allowing rational multiplicities, the divisor of the global section s_{11} of $L^{\otimes 2}$ is seen to extend to the following rational divisor on $\tilde{A}$

$$\widetilde{\operatorname{div}}(s_{11}) = \overline{\operatorname{div}}(s_{11}) + \sum_{\nu=0}^{n_p-1} m_\nu \cdot C_\nu,$$

where $\overline{\operatorname{div}}(s_{11})$ denotes the Zariski closure in $\tilde{A}$ of $\operatorname{div}(s_{11})$, namely the image of $\operatorname{Spec}\mathbb{Z}$ by the zero-section, and the rational multiplicities m_ν are given by

$$m_\nu = \frac{\nu(\nu - n_p)}{n_p} \quad (\nu = 0, ..., n_p - 1).$$

Since n_p is assumed to be even, precisely one of the divisors of the three global sections s_{00}, s_{01}, s_{10} of $L^{\otimes 2}$ extends to a rational divisor on $\tilde{A}$ with the same vertical part as $\widetilde{\operatorname{div}}(s_{11})$ and with horizontal part being determined by the Zariski closure in $\tilde{A}$ of the corresponding divisor on the generic fibre A of $\tilde{A}$. The divisors of the remaining two global sections of $L^{\otimes 2}$ then extend to rational divisors on $\tilde{A}$ with horizontal part being given by the Zariski closure of the corresponding divisor on the generic fibre A of $\tilde{A}$ and with vertical part $\sum_{\nu=0}^{n_p-1} m'_\nu \cdot C_\nu$, where the rational multiplicities m'_ν equal the multiplicities m_ν up to a cyclic permutation of the indices determined by the property $m'_{n_p/2} = 0$. The intersection number of $\widetilde{\operatorname{div}}(s_{11})$ with the sum $\widetilde{\operatorname{div}}(s_{00}) + \widetilde{\operatorname{div}}(s_{01}) + \widetilde{\operatorname{div}}(s_{10})$ is then easily computed to

$$\widetilde{\operatorname{div}}(s_{11}) \cdot \Big(\widetilde{\operatorname{div}}(s_{00}) + \widetilde{\operatorname{div}}(s_{01}) + \widetilde{\operatorname{div}}(s_{10})\Big) = -2n_p + \frac{1}{n_p}.$$

Combining the analytical formula (11) together with the above geometric considerations, we finally arrive at

$$\widehat{\deg}(L, \|\cdot\|) = \frac{1}{4n_p^2} \cdot \widehat{\deg}(\tilde{M}, \|\cdot\|) =$$

$$-\frac{1}{12}\log|\Delta(\tau)| - \frac{1}{2}\log\eta + \left(-\frac{n_p}{6} + \frac{1}{12n_p}\right)\log p + \frac{2}{3}\log 2;$$

concerning the determination of the last summand, we used the fact that the intersection of $\widetilde{\operatorname{div}}(s_{11})$ with the second rational divisor, which passes through the origin in the fibre $\tilde{A}_2/\mathbb{F}_2$, is transversal.

6.3. The case $n = 2, d = 1$. We point out that this example will not follow directly from Corollary 5.4 but rather from Remark 5.6; the fact that d is not divisible by 4 does not lead to serious problems.

Let $\mathcal{H}$ denote the 1-codimensional subset of $\mathfrak{H}_2$ consisting of those $\tau \in \mathfrak{H}_2$, which are diagonalizable with respect to the action of $\mathrm{Sp}_2(\mathbb{Z})$. The entries of $\tau \in \mathcal{H}$ are the moduli of abelian surfaces, which are Jacobians of singular curves of genus 2, while the entries of $\tau \notin \mathcal{H}$ are the moduli of abelian surfaces, which are Jacobians of smooth, projective curves of genus 2 (which are necessarily hyperelliptic).

First, let τ vary in $\mathfrak{H}_2$ away from $\mathcal{H}$ and denote as usual by A_τ the corresponding abelian surface equipped with the line bundle L_τ associated to the divisor of the theta function $\vartheta(\tau, z)$. For each one of the six odd 2-torsion points $P \in A_\tau$ consider the global section

$$s_{P,\tau}(z) := \vartheta(\tau, z - P)^2 \in \Gamma(A_\tau, L_\tau^{\otimes 2});$$

its divisor $\Theta_{P,\tau}$ contains the origin of A_τ. In fact, it can be shown by using the Abel-Jacobi map for curves of genus 2 and the Jacobi inversion formula that $\Theta_{P,\tau}$ contains precisely six 2-torsion points (each with multiplicity 2). For each one of the remaining ten 2-torsion points $Q \notin \Theta_{P,\tau}$ consider the global section $s_{Q,\tau}(z) := s_{P,\tau}(z - Q) \in \Gamma(A_\tau, L_\tau^{\otimes 2})$; its divisor $\Theta_{Q,\tau}$ contains Q. Since $\Theta_{P,\tau}$ and $\Theta_{Q,\tau}$ are irreducible and since $Q \in \Theta_{Q,\tau}$, but $Q \notin \Theta_{P,\tau}$, the intersection $\Theta_{P,\tau} \cap \Theta_{Q,\tau}$ is proper, hence consists of two points (each with multiplicity 4), which are 2-torsion points (use again the Abel-Jacobi map and the Jacobi inversion formula). By extrapolating from the diagonal case, it can finally be shown that there are four choices for a 2-torsion point $R \notin \Theta_{P,\tau} \cup \Theta_{Q,\tau}$ such that the divisor $\Theta_{R,\tau}$ of the global section $s_{R,\tau}(z) := s_{P,\tau}(z - R) \in \Gamma(A_\tau, L_\tau^{\otimes 2})$ has proper, i.e., empty intersection with $\Theta_{P,\tau} \cap \Theta_{Q,\tau}$. With the notations of Remark 5.6 we conclude $\mathcal{U}_3 \supseteq \mathfrak{H}_2 \setminus \mathcal{H}$.

Let now τ degenerate to a point in $\mathcal{D}$, where $\mathcal{D}$ is an irreducible component of $\mathcal{H}$; by translating $\mathcal{D}$ by a suitable element of $\mathrm{Sp}_2(\mathbb{Z})$, we may assume without loss of generality that we deal with the component, where τ is diagonal. We note that each one of the divisors $\Theta_{P,\tau}, \Theta_{Q,\tau}, \Theta_{R,\tau}$ constructed above then contains a seventh 2-torsion point; this point is just the singular point on these divisors. By a direct inspection, it can then be shown that either the three divisors $\Theta_{P,\tau}, \Theta_{Q,\tau}, \Theta_{R,\tau}$ have proper, i.e., empty intersection (as in the case $\tau \notin \mathcal{H}$) or at least two of them intersect properly (note that, if two of the divisors do not intersect properly, they have precisely one component in common). In terms of Remark 5.6 this means that $\mathcal{U}_2 = \mathfrak{H}_2$.

For a given triple P, Q, R of 2-torsion points on A_τ as above, let $\Gamma_{\{P,Q,R\}}$ denote the subgroup of $\mathrm{Sp}_2(\mathbb{Z})$, consisting of those isomorphisms of A_τ, which fix P, Q, R as a set. Since the sections s_P, s_Q, s_R constructed above are in general position, we derive from Remark 5.6 that there is an explicitly given Siegel modular form $F_{\{P,Q,R\}}$ of weight 24 with respect to $\Gamma_{\{P,Q,R\}}$ such that the equality

$$\int_{A_\tau(\mathbb{C})} g_{P,\tau}(z) * g_{Q,\tau}(z) * g_{R,\tau}(z) = -\log |F_{\{P,Q,R\}}(\tau)|^2 - 24 \log \det \eta \qquad (12)$$

holds. Summing now both sides of (12) over the various choices for P, Q, R, one obtains two sums, which can be subdivided into ten partial sums according to the ten different choices for Q. Since $\vartheta(\tau, 0) \neq 0$, one possible choice for Q is $Q = P$; the corresponding partial sum on the right hand side leads to the Siegel modular form

$$G(\tau) := \prod_{P,R} F_{\{P,P,R\}}(\tau)$$

of weight 576 with respect to the subgroup of $\mathrm{Sp}_2(\mathbb{Z})$, which permutes the six odd 2-torsion points and stabilizes the origin of A_τ, hence is nothing but Igusa's group $\Gamma_2(1,2)$ (cf. [6], p. 405). The remaining nine partial sums lead to Siegel modular forms of the same weight, but with respect to the nine different conjugate subgroups of $\Gamma_2(1,2)$ inside of $\mathrm{Sp}_2(\mathbb{Z})$ (note that $(\mathrm{Sp}_2(\mathbb{Z}) : \Gamma_2(1,2)) = 10$). Adding up, we obtain a Siegel modular form of weight 5760 with respect to the full modular group $\mathrm{Sp}_2(\mathbb{Z})$, which vanishes by construction along $\mathcal{H}$. Since $\mathrm{Sp}_2(\mathbb{Z})\backslash\mathcal{H}$ is irreducible and since Igusa's $\chi_{10} \in M_{10}(\mathrm{Sp}_2(\mathbb{Z}))$ (given by the product of the squares of the ten even theta functions, cf. [6], p. 404) vanishes also precisely along $\mathcal{H}$, we conclude that

$$\sum_{P,Q,R} \int_{A_\tau(\mathbb{C})} g_{P,\tau}(z) * g_{Q,\tau}(z) * g_{R,\tau}(z) = -576 \log |c \cdot \chi_{10}(\tau)|^2 - 5760 \log \det \eta,$$

where c is a suitable non-zero constant. We finish this example by noting that it is closely related to the explicit computations given in [1], in particular in the Appendix there.

6.4. The case $n = 3, d = 1$. As the previous example, this example will not follow directly from Corollary 5.4 but rather from Remark 5.6; again, the fact that d is not divisible by 4 does not lead to serious problems.

Let $\mathcal{H}$ denote the subset of $\mathfrak{H}_3$ consisting of those $\tau \in \mathfrak{H}_3$, which are the moduli of abelian threefolds, which are Jacobians of smooth, projective, hyperelliptic curves of genus 3. The closure $\overline{\mathcal{H}}$ of $\mathcal{H}$ is a 1-codimensional subset of $\mathfrak{H}_3$. The complement $\partial\overline{\mathcal{H}} = \overline{\mathcal{H}} \setminus \mathcal{H}$ consists of those $\tau \in \mathfrak{H}_3$, which can be put into diagonal block form by means of the action of $\mathrm{Sp}_3(\mathbb{Z})$.

First, let τ vary in $\mathfrak{H}_3$ away from $\overline{\mathcal{H}}$ and denote as usual by A_τ the corresponding abelian threefold equipped with the line bundle L_τ associated to the divisor of the theta function $\vartheta(\tau, z)$. For each one of the 28 odd 2-torsion points $P \in A_\tau$ (cf. [20], p. 169) consider the global section

$$s_{P,\tau}(z) := \vartheta(\tau, z - P)^2 \in \Gamma(A_\tau, L_\tau^{\otimes 2});$$

its divisor $\Theta_{P,\tau}$ contains the origin of A_τ. In fact, it can be shown (cf. [21], p. 3.105) that $\Theta_{P,\tau}$ contains precisely 28 2-torsion points (each with multiplicity 2). For each one of the remaining 36 2-torsion points $Q \notin \Theta_{P,\tau}$ consider the global section $s_{Q,\tau}(z) := s_{P,\tau}(z - Q) \in \Gamma(A_\tau, L_\tau^{\otimes 2})$; its divisor $\Theta_{Q,\tau}$ contains Q. Since $\Theta_{P,\tau}$ and $\Theta_{Q,\tau}$ are irreducible and since $Q \in \Theta_{Q,\tau}$, but $Q \notin \Theta_{P,\tau}$, the intersection $\Theta_{P,\tau} \cap \Theta_{Q,\tau}$ is proper and, by [2], Théorème 10.12, irreducible. Using [2] once more, now Corollaire 10.11, one can find another 2-torsion point $R \in A_\tau$ such that the three divisors $\Theta_{P,\tau}, \Theta_{Q,\tau}, \Theta_{R,\tau}$ intersect properly; here $\Theta_{R,\tau}$ is the divisor of the global section $s_{R,\tau}(z) := s_{P,\tau}(z - R) \in \Gamma(A_\tau, L_\tau^{\otimes 2})$. We let ν_R denote the number of possible choices for R. By extrapolating from the diagonal case, it can be shown that the six points in the intersection $\Theta_{P,\tau} \cap \Theta_{Q,\tau} \cap \Theta_{R,\tau}$ are 2-torsion points (each with multiplicity 8). Furthermore, one derives from that consideration that there exists a further 2-torsion point $S \in A_\tau$ such that the divisor $\Theta_{S,\tau}$ of the global section $s_{S,\tau}(z) := s_{P,\tau}(z - S) \in \Gamma(A_\tau, L_\tau^{\otimes 2})$ has proper, i.e., empty intersection with $\Theta_{P,\tau} \cap \Theta_{Q,\tau} \cap \Theta_{R,\tau}$. We let ν_S denote the number of possible choices for S. With the notations of Remark 5.6 we conclude $\mathcal{U}_4 \supseteq \mathfrak{H}_3 \setminus \overline{\mathcal{H}}$.

Let now τ degenerate to a point in $\overline{\mathcal{H}}$. We note that each one of the divisors $\Theta_{P,\tau}, \Theta_{Q,\tau}, \Theta_{R,\tau}, \Theta_{S,\tau}$ constructed above then contains a 29th 2-torsion point (cf. [21], p. 3.105). As in the preceding example it can be shown by a direct inspection that either the four divisors $\Theta_{P,\tau}, \Theta_{Q,\tau}, \Theta_{R,\tau}, \Theta_{S,\tau}$ have proper, i.e., empty intersection (as in the case $\tau \notin \overline{\mathcal{H}}$) or at least three of them intersect properly. In terms of Remark 5.6 this means that $\mathcal{U}_3 = \mathfrak{H}_3$.

For a given quadruple P, Q, R, S of 2-torsion points on A_τ as above, let $\Gamma_{\{P,Q,R,S\}}$ denote the subgroup of $\mathrm{Sp}_3(\mathbb{Z})$, consisting of those isomorphisms of A_τ, which fix P, Q, R, S as a set. Since the sections s_P, s_Q, s_R, s_S constructed above are in general position, we derive from Remark 5.6 that there is an explicitly given Siegel modular form $F_{\{P,Q,R,S\}}$ of weight 192 with respect to $\Gamma_{\{P,Q,R,S\}}$ such that the equality

$$\int_{A_\tau(\mathbb{C})} g_{P,\tau}(z) * g_{Q,\tau}(z) * g_{R,\tau}(z) * g_{S,\tau}(z) = -\log |F_{\{P,Q,R,S\}}(\tau)|^2 - 192 \log \det \eta \qquad (13)$$

holds. Summing now both sides of (13) over the various choices for P, Q, R, S, one obtains two sums, which can be subdivided into 36 partial sums according to the 36 different choices for Q. Since $\vartheta(\tau, 0) \neq 0$, one possible choice for Q is $Q = P$; the corresponding partial sum on the right hand side leads to the Siegel modular form

$$G(\tau) := \prod_{P,R,S} F_{\{P,P,R,S\}}(\tau)$$

of weight $\nu = 28 \cdot \nu_R \cdot \nu_S \cdot 192$ with respect to the subgroup of $\mathrm{Sp}_3(\mathbb{Z})$, which permutes the 28 odd 2-torsion points and stabilizes the origin of A_τ, hence is nothing but Igusa's group $\Gamma_3(1,2)$ (cf. [24], p. 793). The remaining 35 partial sums lead to Siegel modular forms of the same weight, but with respect to the 35 different conjugate subgroups of $\Gamma_3(1,2)$ inside of $\mathrm{Sp}_3(\mathbb{Z})$ (note that $(\mathrm{Sp}_3(\mathbb{Z}) : \Gamma_3(1,2)) = 36$). Adding up, we obtain a Siegel modular form of weight $36 \cdot \nu$ with respect to the full modular group $\mathrm{Sp}_3(\mathbb{Z})$, which vanishes by construction along $\overline{\mathcal{H}}$. Since $\mathrm{Sp}_3(\mathbb{Z}) \backslash \overline{\mathcal{H}}$ is irreducible and since Igusa's $\chi_{18} \in M_{18}(\mathrm{Sp}_3(\mathbb{Z}))$ (given by the product of the 36 even theta functions, cf. [24], p. 814) vanishes also precisely along $\overline{\mathcal{H}}$, we conclude that

$$\sum_{P,Q,R,S} \int_{A_\tau(\mathbb{C})} g_{P,\tau}(z) * g_{Q,\tau}(z) * g_{R,\tau}(z) * g_{S,\tau}(z) = -2\nu \log |c \cdot \chi_{18}(\tau)|^2 - 36\nu \log \det \eta,$$

where c is a suitable non-zero constant.

References

[1] *Bost, J.-B.; Mestre, J.-F.; Moret-Bailly, L.:* Sur le calcul explicite des "classes de Chern" des surfaces arithmétiques de genre 2. Astérisque **183** (1990), 69-105.

[2] *Debarre, O.:* Sur les variétés abéliennes dont le diviseur thêta est singulier en codimension 3. Duke Math. J. **57** (1988), 221-273.

[3] *Faltings, G.; Chai, C.-L.:* Degeneration of abelian varieties. Ergeb. Math. **22**, Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo-Hong Kong-Barcelona 1990.

[4] *Gillet, H.; Soulé, C.:* Arithmetic intersection theory. Publ. Math. I.H.E.S. **72** (1990), 93-174.

[5] *Hörmander, L.:* The analysis of linear partial differential operators I, 2nd edition. Grundlehren math. Wiss. **256**, Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo-Hong Kong 1990.

[6] *Igusa, J.:* On Siegel modular forms of genus two (II). Amer. J. Math. **86** (1964), 392-412.

[7] *Igusa, J.:* Theta functions. Grundlehren math. Wiss. **194**, Springer-Verlag, Berlin-Heidelberg-New York 1972.

[8] *Jouanolou, J.-P.:* Théorèmes de Bertini et applications. Progr. Math. **42**, Birkhäuser-Verlag, Boston-Basel-Stuttgart 1983.

[9] *Köhler, K.:* Complex analytic torsion forms for torus fibrations and moduli spaces. Preprint, Bonn 1996.

[10] *Kramer, J.:* The theory of Siegel-Jacobi forms. Habilitationsschrift, ETH-Zürich 1992.

[11] *Kramer, J.:* An arithmetic theory of Jacobi forms in higher dimensions. J. reine angew. Math. **458** (1995), 157-182.

[12] *Krantz, S.:* Function theory of several complex variables, 2nd edition. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, California 1992.

[13] *Künnemann, K.:* Arakelov Chow groups of abelian schemes, arithmetic Fourier transform, and analogues of the standard conjectures of Lefschetz type. Math. Ann. **300** (1994), 365-392.

[14] *Künnemann, K.:* Projective regular models for abelian varieties, semi-stable reduction and the height pairing. Preprint 1996.

[15] *Lang, S.:* Introduction to Arakelov theory. Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo 1988.

[16] *Lange, H.; Birkenhake, Ch.:* Complex abelian varieties. Grundlehren math. Wiss. **302**, Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo-Hong Kong-Barcelona-Budapest 1992.

[17] *Moret-Bailly, L.:* Pinceaux de variétés abéliennes. Astérisque **129** (1985).

[18] *Moret-Bailly, L.:* Sur l'équation fonctionnelle de la fonction thêta de Riemann. Compositio Math. **75** (1990), 203-217.

[19] *Mumford, D.; Fogarty, J.; Kirwan, F.:* Geometric invariant theory, 3rd enlarged edition. Ergeb. Math. **34**, Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo-Hong Kong-Barcelona-Budapest 1994.

[20] *Mumford, D.:* Tata lectures on theta I. Progr. Math. **28**, Birkhäuser-Verlag, Boston-Basel-Berlin 1994.

[21] *Mumford, D.:* Tata lectures on theta II. Progr. Math. **43**, Birkhäuser-Verlag, Boston-Basel-Berlin 1993.

[22] *Serre, J-P.:* Arithmetic Groups in Homological Group Theory, ed. C. T. C. Wall. London Math. Soc., Lecture Note Series **36** (1979), 105-136.

[23] *Soulé, C.; Abramovich, D.; Burnol, J.-F.; Kramer, J.:* Lectures on Arakelov Geometry. Cambridge University Press, Cambridge 1992.

[24] *Tsuyumine, S.:* On Siegel modular forms of degree three. Amer. J. Math. **108** (1986), 755-862; Addendum: ibid., 1001-1003.

[25] *Silverberg, A.:* Cohomology of fiber systems and Mordell-Weil groups of abelian varieties. Duke Math. J. **56** (1988), 41-46.

Jay Jorgenson
Department of Mathematics
Oklahoma State University
Stillwater, OK 74078-0613
U.S.A.
e-mail: jjorgen@littlewood.math.okstate.edu

Jürg Kramer
Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6
D-10099 Berlin
Germany
e-mail: kramer@mathematik.hu-berlin.de