DEVELOPING AN INTUITIVE CONCEPT OF LIMIT WHEN APPROACHING THE DERIVATIVE FUNCTION

André Henning and Andrea Hoffkamp

Humboldt-Universität zu Berlin

The central idea of calculus is the concept of limit. German secondary school curricula claim to introduce the concept of limit in an intuitive way refraining from a rigorous mathematical definition. However, it is unclear what can be regarded as a "good intuitive basis". After the discussion of typical student problems we present a DGS-based activity to support the introduction of the derivative function fostering a dynamic concept of limit. Aside from the step from derivation as a local phenomenon to the global view of the derivative function we also show how the activity can be used to visualize and talk about the variety of limit processes.

Keywords: conceptualization processes, DGS, interactive learning activity, concept of limit, teaching and learning of calculus

INTRODUCTION AND RATIONALE

The problems of contemporary calculus courses at secondary school mainly result from the tension between learning or teaching of routines and the development of a structural understanding of the underlying concepts (Tall 1996, p. 306).

In fact this problem exists since the introduction of calculus to German school curricula: For example Toeplitz (1928) urges to remove calculus from school if teachers are not able to bring out more than the teaching of mere routines.

The fundamental concepts of calculus (e.g. the concept of limits, derivation and integration) are mathematically advanced. Therefore teachers are always required to make didactical decisions about what to teach in a visual-intuitive way and what to teach in a mathematically rigorous way.

For example the secondary school curriculum in Berlin, Germany, explicitly demands an intuitive or propaedeutic approach to calculus concepts at the end of grade 10. This is established through introducing a special „Modul" called „Describing change with functions" (Jugend und Sport Senatsverwaltung für Bildung 2006a). In this context Hoffkamp designed and investigated several DGS-based activities to support such a qualitative approach (Hoffkamp 2009, 2011), which could lead to a sustainable intuitive basis of certain calculus concepts.

Considering the central role of the concept of limit within calculus, the described situation needs particular attention. On the one hand the Berlin secondary school curriculum for grade 11 mentions that the concept of limit can only be taught in a propaedeutic way since the necessary exact notions (series, convergence tests) are not available to the students. On the other hand, the teachers are required to introduce the derivative as limit of the difference quotient (Jugend und Sport Senatsverwaltung für Bildung 2006b).

Sfard (1991) describes the process of mathematical concept formation. Starting with operational conceptions (e.g. functions as computational processes, rational numbers as results of division of integers) structural conceptions (e.g. function as set of ordered pairs, rational numbers as pairs of integers) develop, the latter leading to the establishment of abstract objects.

Fischbein (1989) offers another perspective on the issue of intuitive approaches as well as the process of mathematical concept formation. He describes how, during the process of mathematical abstraction, mental models of the abstract concepts develop in the learners mind. According to him these intuitive models, tacitly or not, influence the way we conduct mathematical reasoning processes. These tacit models are one reason for the difficulties students are facing in the process of learning mathematics. He suggests allowing the students to consciously analyze the influence of those tacit models and in this way to allow them to avoid the development of misconceptions.

Taking the above into account means to realize the necessity of a profound intuitive basis or cognitive root (Tall 2006) for the limit concept that could lead to the development of an object view of the derivative. Cognitive roots as described by Tall (2006) are not mathematical foundations but approaches that build "on concepts which have the dual role of being familiar to the students and also provide the basis for later mathematical development" (Tall 2006). According to him they require a combination of empirical research and mathematical knowledge in order to find them. In our opinion, DGS appears to be a good means in this context, therefore we present an activity using DGS-based interactive visualizations fostering a dynamic idea of limits and leading to an object view of derivation.

In the following we will describe the conceptual change approach and the potential of DGS-based activities. We will give an example of an activity that is related to our rationale. After that we will present our research questions and the results of a video-study.

Spontaneous conceptions of limit and conceptual change

The above considerations conform to a genetic view of learning as described by Wagenschein (1992). Especially for the process of conceptualization, a theoretical perspective like the conceptual change approach is helpful to design activities like the one described in this article, and to understand the students' learning processes. The conceptual change approach itself is a genetic learning theory. A description can be found in Verschaffel & Vosniadou (2004). Conceptual change does not mean to switch from one concept to another by replacing the old concept by a better new one. It is the process of reintegration and reorganization of cognitive structures in order to develop mental conceptions and to activate the appropriate conceptions dependent on given contexts (Verschaffel & Vosniadou 2004, p. 448).

For the process of conceptualization it is important to know about the students' spontaneous conceptions and to build on them. In fact the students' spontaneous conceptions can be considered a learning opportunity and a starting point for further development by dealing with them explicitly (Prediger 2004).

Therefore we tried to find out about the students' spontaneous conceptions of limits. We asked students at the end of grade 10 and 11 to write a letter to an imaginary „clueless" friend explaining the mathematical notion of limit. We present two excerpts from letters here. One student wrote:

> The limit is the outermost value of a number range. For example if one says „all numbers from one to five" then one and five are limits.

Another student wrote:

> Considering the graph of a function over a large interval one can observe that some functions come closer and closer to a certain value. The function tends only to this value, without drifting away again. However, the function does not reach or exceed it.

The first excerpt shows that the student considers limits as bounds of intervals and formulates a static conception of limit. The second excerpt reflects the student's experiences with limits in connection with asymptotic behaviour. Although the student formulates a dynamic conception of limit, his conception is not elaborated since limit processes seem to be always „monotonous" and limits „cannot be reached or exceeded".

These observations are confirmed by the work of Cornu (1991) who described typical spontaneous conceptions of limits. In fact all limiting processes like the concepts of continuity, differentiation or integration contain similar cognitive problems: To overcome or integrate the spontaneous conceptions in the learner's individual concept.

THE USE OF DGS-BASED ACTIVITIES

As already mentioned we think that DGS is a good means to establish an intuitive basis of the concept of limit. With respect to the mentioned spontaneous conceptions and the conceptual change approach, DGS-based activities could not only help to develop a dynamic view of limit and limit processes, but also help to develop a more elaborated conception of limit by visualizing the variety of limit processes. Therefore we combine interactive visualizations with special tasks stimulating verbalization and exploration processes. Especially the role of verbalization as a mediator between the representations and the students' mental concepts when working with interactive visualizations was pointed out in the work of Hoffkamp (2011) and based on Janvier's work (1978).

Which role of the computer do we focus on? At first, we benefit from the various possibilities of visualizing mathematical concepts. Therefore, we make use of the possibility to visualize a holistic representation in contrast to a linear order of mathematical content (see also Sfard 1991). Moreover we add interaction to enable learners to explore the interactive activities without negative consequences.

While giving the opportunity for exploration we use the computer to „restrict the actions of learners and thus help them to develop appropriate mental models of representation“ (Kortenkamp 2007, p. 148). In this sense visualizations play a heuristic role and can be used before exact mathematical notions or concepts are available.

THE ACTIVITY "TOWARDS THE DERIVATIVE FUNCTION"

In the following we present a DGS-based learning activity. It introduces the derivative function as an object and its relation to the original function. This activity is meant to be exemplary. It shows how conceptualization processes in the context of the concept of limit can be supported by using special DGS-based activities. Our idea of using the difference quotient and its extension by continuity for an object-based approach to obtaining the derivative function has already been mentioned by Mueller and Forster (2003) quoting Yerushalmy and Schwartz (1999). However, they did not explicate the full didactic potential of this approach and did not use the dynamic approach towards limit that can be fostered by a DGS-based activity.

The two focal points for our activity are emphasizing the difference quotient and permitting a look on various limit processes that constitute the analytical step. The difference quotient is not only a (theory generating) precursor for the differential quotient and the derivative (the way it is often used in schools). It is the concept that has a direct relation to reality through the concept of average change (of speed etc.). Therefore it is evident for students and is thus worth taking a closer look at.

The activity is meant to be used when the derivative of a function at a certain value is already known to the students. However, the students have no elaborate limit concept so far. This is usually the case at the start of grade 11 in secondary schools. The derivative function, however, is not known to the students yet. The way from a local (derivative at a point) view on derivation towards a more global, object based view (derivative function) shall be supported, while several limit processes get examined on the way. This is in line with our idea of broadening the view on limits as well as bringing about an object view on the derivative function.

The activity can be found at

http://www2.mathematik.hu-berlin.de/~hoffkamp/Material/ableitungsfunktion.html.

Description of the activity and didactical analysis

The activity consists of three separate worksheets or tasks. It is very rich concerning mathematical concepts. We will however only describe and didactically analyze selected parts that have a direct relation to the presented results and aims of this article. Figure 1 gives a general idea of what one of the tasks looks like. The text above the applet gives an introduction to the task. The text next to the applet poses special questions and tasks that also ask students to verbalize their thoughts and observations and are meant to help the students go through the conceptual change process we intend to initiate. This also relates to Fischbein. Verbalization makes tacit models and intuitive concepts accessible to a conscious process of reflection.

The tasks are consecutive. Different predetermined functions f can be chosen in every task to make sure there is not just one but many graphs available. The restriction on predetermined functions allows a broadening of the view while at the same time focusing on certain sustainable examples (see also Kortenkamp 2007). The functions were chosen as examples of certain classes of functions - symmetric and non-symmetric, polynomial and trigonometric functions and in tasks 2 and 3 also the (at the origin) non-differentiable absolute value function.

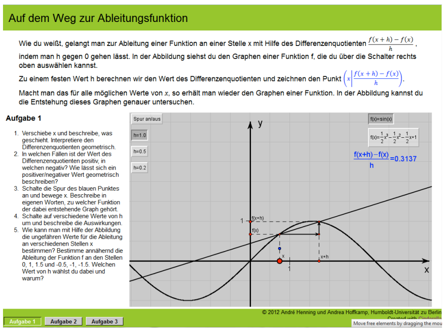

Figure 1: General overview of a worksheet.

The first task takes up the so far local ideas of the derivative at a point and differential quotient as the limit of the difference quotient. In the second task a first object view on the derivative function is reached. Also the limit process observed is changed. The third task introduces yet another variation of the limit process while the object view on the derivative function remains in focus.

The first Applet (see figure 2) offers a process view on the functions we will take an object view on in tasks 2 and 3. For three fixed values of h the term $\frac{f(x+h)-f(x)}{h}$ (the difference quotient) is evaluated at a certain x-coordinate. The x-coordinate can be changed by dragging the big red point on the x-axis. The point $\left(x\middle|\frac{f(x+h)-f(x)}{h}\right)$ is always printed in blue. The students already know the difference quotient. So far they only evaluated it for a single fixed value of x and varying values of h to gain the derivative at a point. If tracing is activated one gets a trace of the resulting points which forms the graph of the function g with $g(x)=\frac{f(x+h)-f(x)}{h}$ (for fixed h). This way we have a point wise (process) view of the construction of the graph. We change between processes by varying h. The resulting function becomes a better approximation of the derivative function for smaller values of h.

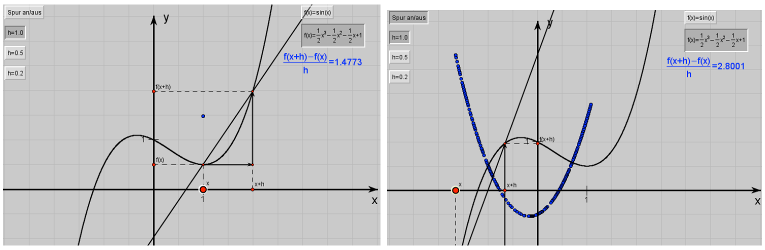

Figure 2: Applet 1 without and with tracing within the first task.

The second and third applets are pretty similar in their construction, which is why only one of the two is pictured here in figure 3. They both show graphs of functions that can be manipulated. In applet 2 there is only one function namely g with $g(x) = \frac{f(x+h) - f(x)}{h}$. In addition to that applet 3 also visualizes the function k with $k(x) = \frac{f(x+h) - f(x-h)}{2h}$.

Switching over to applet 2 we have a new situation. The function that developed as the result of a process (of changing the x value) in applet 1 now exists as a single entity. We evaluate the difference quotient for all values of x simultaneously now (note that this is not the difference quotient function as that would be $g_x(h) = \frac{f(x+h) - f(x)}{h}$ for a fixed value of x). At the same time we no longer have fixed values of h but can change those using a slider. If the value of h is changed, the graph of g moves as a whole. The students observe a family of curves that converges to a limit function. According to Sfard (1991) working with families of curves or equations with parameters is already a step towards reification. Limit becomes something more dynamic in this context as the students observe and describe the limit process that leads to the derivative function for differentiable f.

The third applet is aimed at the development of a more elaborated conception of limit. In contrast to task 2 we now take a look at not only one but two different limit processes. We show that different limit processes can lead to the same limit. The idea is to prevent or change a very restricted view on limits as could be seen in the student letters described above. An object view on the functions involved is necessary now.

One observed object is the function g, already known from task 2, the other is the above mentioned function k with $k(x) = \frac{f(x+h) - f(x-h)}{2h}$. k represents a symmetric differential quotient. It has some interesting characteristics. For example, if f is a symmetric function, k has a local extremum where f' has a local extremum for any given value of h. For g this is obviously not the case. The students are asked to describe and compare the functions g and k and to explain their observation geometrically (by using secants) and algebraically (by comparing both forms of difference quotients). One other observation is that k converges faster to f' than g. Therefore the symmetric difference quotient is better for numerical computations. For polynomial functions this can even be proved with students. The students may discover that convergence is not always the same and that different processes may converge at different speeds.

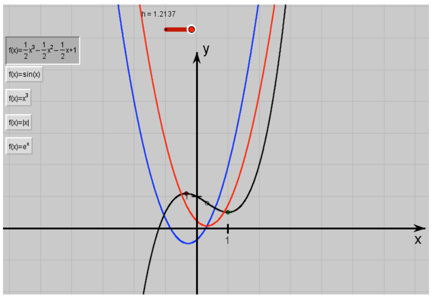

Figure 3: Applet 3 within the third task.

METHODOLOGY AND RESEARCH QUESTIONS

A first video study was conducted in August 2012. The two authors acted as researchers and teachers respectively. The activity was introduced in an 11th grade advanced course in mathematics. Students were video-taped during their work with the activity. The students' worksheets were collected for further study. Classroom interaction was noted by one of the authors while the other lead the course in classroom discussion and students' presentation of their results.

Our analysis of the collected data is lead by the following research questions:

In what way can a dynamic-visual approach as depicted in this paper support the development of a sustainable conception of limit and related mathematical concepts?

What conceptions of limit do students develop if confronted with activities as the one presented?

What spontaneous conceptions can be uncovered through analysis of students' work and how can they be used as learning opportunities?

FINDINGS

One of the tasks for the first applet was "In which cases is the value of the difference quotient positive, negative and when is it equal to zero? Explain the geometric meaning of a positive/negative value or a value equal to zero." Many students answered in the way of "Positive if the graph increases; Zero if the central point of x+h and x is similar to the local maximum; Negative if the graph decreases." Their reasoning however is only possible through the analytical step; the core achievement of school analysis. From a mathematical point of view, what we see here is a misconception: the difference quotient is interpreted as the differential quotient. The students spontaneously assumed linearity. For linear functions the slope of any secant equals that of the tangent in any point of the graph. Therefore we may infer the behaviour of linear functions from the slope of any secant. In general, this is no longer true for non-linear functions, so the slope concept of the secant needs to be evolved to that of the tangent. The students' spontaneous conception of linearity became apparent and could be productively and explicitly used directly within the classroom in order to emphasize the fundamental difference between the difference quotient and the differential quotient. The second part of the students' answer was taken up in class discussion to develop Rolle‘s theorem starting in an informal way.

If we take another look at the above excerpt, we make another interesting discovery. The task was taken up again in later class discussion. It lead to a theorem of monotonicity. What became apparent here is that mathematical logic and quantification constitute an important obstacle for students. It was difficult for them to express that the function is not increasing or decreasing for every x between x and x+h but that there is at least one x with an increasing tangent.

The third applet offered a view on different limit processes that lead to the same limit. The students were asked to verify that the usual difference quotient and the symmetric one tend to the same limit. Or rather that the limit of the two series of secant-slope-functions actually is the derivative function. It was very hard for the students to verify any of this geometrically, which sheds light on the fact that geometric representations need to be focused on more, and the geometric meaning of the limit process when approaching the differential quotient should be made more explicit in a dynamic way. However, the students were able to compute the limit for certain values of h and this way convinced themselves that the symmetric difference quotient actually leads to the same limit as the usual one. To give an example of what they did, we present the work of one of the videotaped students. He said:

> “We are not supposed to know, what function this is, but if we let h go to zero in this task, we shall only show, that it becomes roughly the same. They should move towards each other now... [uses the slider to let h go to zero]... Because it does not matter if we have plus zero or minus zero here or two times zero.”

He used the dynamic possibilities of the applet to support his assumption and realized that when he made h very small, the two graphs were almost indistinguishable. Finally he computed (in a longer way, but essentially) $\lim_{h\to 0}\frac{(1+h)^2-1^2}{h}=2=\lim_{h\to 0}\frac{(1+h)^2-(1-h)^2}{2h}$ to convince himself and the other students that his earlier observation was true. This way the students became aware of the fact that there is a variety of comparable limit processes leading to the same limit. The insight into the necessity of proving their assumption was directly evoked by the use of the dynamic possibilities provided through the applet.

CONCLUSIONS

The presented observations give an insight into the polymorphism of possibilities within the described individual cases and may raise one's awareness for the perception of specific students conceptions and the own didactical action.

We are also confident that activities like the above are valuable means in the conceptualization process and may serve as cognitive roots resp. as an intuitive basis, as can be seen in our study. They allow the development of a dynamic conception of limit that appears to be a good cognitive root. Additionally, when using the applet there can be a clearer focus on the analytical step as a core achievement of analysis.

We have seen that a qualitative-empirical analysis as depicted in our paper can give impulses for further didactical and mathematical analyses. The search for appropriate cognitive roots necessitates further inquiries into the nature of such roots. More work in the field of ICT-based concepts and activities, also in the way of mathematics education as a design science (Wittmann 1995), is necessary here.

ACKNOWLEDGEMENTS

This work was partly supported by Deutsche Telekom Stiftung and Humboldt-ProMINT-Kolleg. We also wish to thank Robert Bartz and Dr. Sabiene Zänker for their constructive remarks from a teacher's point of view.

REFERENCES

Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced Mathematical Thinking*. Kluwer Academic Publishers, Dordrecht, Netherlands, pp. 153-166.

Fischbein, E. (1989). Tacit Models and Mathematical Reasoning. In *For the Learning of Mathematics,* 9, 2, pp. 9-14. FLM Publishing Association, Montreal, Quebec, Canada.

Hoffkamp, A. (2009). Enhancing functional thinking using the computer for representational transfer. In *Proceedings of the Sixth Conference of European Research in Mathematical Education.* Lyon.

Hoffkamp, A. (2011). The use of interactive visualizations to foster the understanding of concepts of calculus – design principles and empirical results. *Zentralblatt für Didaktik der Mathematik,* 43, 3, pp. 359-372.

Janvier, C. (1978). *The interpretation of complex cartesian graphs representing situations.* PhD Thesis, University of Nottingham, Shell Centre for Mathematical Education.

Yerushalmy, M., & Schwartz, J. J. (1999). A procedural approach to explorations in calculus. *International Journal of Mathematical Education in Science and Technology,* 30(6), pp. 903-914.

Jugend und Sport Berlin Senatsverwaltung für Bildung (Ed.) (2006a). Rahmenlehrplan für die Sekundarstufe I, Jahrgangsstufe 7-10, Mathematik. Oktoberdruck AG, Berlin.

Jugend und Sport Berlin Senatsverwaltung für Bildung (Ed.) (2006b). Rahmenlehrplan für die gymnasiale Oberstufe, Mathematik. Oktoberdruck AG, Berlin.

Kortenkamp, U. (2007). Guidelines for Using Computers Creatively in Mathematics Education. In K. H. Ko, & D. Arganbright (Eds.), *Enhancing University Mathematics: Proceedings of the First KAIST International Symposium on Teaching*, Bd. 14. CBMS Issues in Mathematics Education. AMS, pp. 129-138.

Mueller, U.A., & Forster, P.A. (1999). Approaches to the Derivative and Integral with Technology and Their Advantages/Disadvantages. *Electronic Proceedings of the 8th Asian Technology Conference in Mathematics.* Retrieved from *http://epatcm.any2any.us/10thAnniversaryCD/EP/2003/2003C121/fullpaper.pdf*

Prediger, S. (2004). Brüche bei den Brüchen – aufgreifen oder umschiffen. *mathematik lehren*, 123, pp. 10-13.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational studies in mathematics*, 22, pp. 1-36.

Tall, D. (1996). Functions and calculus. In A. Bishop et al. (Eds.), *International Handbook of Mathematics Education.* Kluwer Academic Publishers, Dordrecht, Netherlands, pp. 289-325.

Tall, D. (2006). The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity and Proof. In D.A. Grouws (Ed.), *Handbook of Research in Mathematics Teaching and Learning*. Macmillan, New York, pp. 495-511.

Toeplitz, O. (1928). Die Spannungen zwischen den Aufgaben und Zielen der Mathematik an der Hochschule und an der höheren Schule. *Schriften des deutschen Ausschusses für den mathematischen und naturwissenschaftlichen Unterricht, Vorträge auf der 90. Versammlung deutscher Naturforscher und Ärzte in Hamburg*, 11. Folge, Heft 10.

Verschaffel, L., & Vosniadou, S. (2004). Extending the conceptual change approach to mathematics learning and teaching. In L. Verschaffel, & S. Vosniadou (Eds.), *Conceptual change in mathematics learning and teaching, special issue of Learning and Instruction*. Elsevier, pp. 445-451.

Wagenschein, M. (1992). *Verstehen lehren. Genetisch, sokratisch, exemplarisch.* Beltz, Weinheim/Basel.

Wittmann, E. C. (1995). Mathematics education as a 'design science'. *Educational studies in mathematics*, 29, pp. 355-374.