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Sup-norm Bounds for Automorphic Forms and Eisenstein Series

Dedicated to Stephen Kudla's 60th Birthday

Jay Jorgenson* and Jürg Kramer†

Abstract

For any finite volume hyperbolic Riemann surface M of positive genus, let μ_{can} denote the $(1,1)$-form associated to the canonical metric and μ_{shyp} the $(1,1)$-form associated to the hyperbolic metric, both forms scaled such that the volume of M is one. Let d_M be the sup-norm of the quotient $\mu_{\mathrm{can}}/\mu_{\mathrm{shyp}}$. In [13] optimal bounds for d_M through covers were obtained. In the present paper, we revisit this problem and give an alternative proof for the optimal bounds established in [13]. Our approach here takes as its starting point the key relation from [14] involving as the main term an integral over time of a heat kernel on M. By suitably decomposing this integral, one obtains bounds in terms of classical parabolic Eisenstein series, the hyperbolic Eisenstein series defined in [21], and elliptic Eisentein series defined in [24].

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1 Introduction

1.1. Associated to any finite volume hyperbolic Riemann surface M of positive genus, there are a number of natural metrics to consider. The hyperbolic metric on M comes from the point of view of the uniformization theorem, which allows us to represent M as the quotient space $\Gamma\backslash\mathbb{H}$, where $\mathbb{H}$ denotes the upper half-plane and Γ is a Fuchsian subgroup of the first kind of $\mathrm{PSL}_2(\mathbb{R})$. The hyperbolic metric

*Department of Mathematics, City College of New York, Convent Avenue at 138th Street, New York, NY 10031, U.S.A., jjorgenson@mindspring.com. The first author acknowledges support from grants from the NSF and PSC-CUNY.

†Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany, kramer@math.hu-berlin.de. The second author acknowledges support from the DFG Graduate School *Berlin Mathematical School* and from the DFG Research Training Group *Arithmetic and Geometry*.

is the unique metric which is compatible with the complex structure of M and is complete with constant curvature equal to negative one. The canonical metric on M comes from the point of view of complex geometry, where one uses the Abel-Jacobi map to embed the compactification $\overline{M}$ of M into its Jacobian $\mathrm{Jac}(\overline{M})$, and then a metric on M is obtained through the pull-back of the natural flat Kähler metric on $\mathrm{Jac}(\overline{M})$, when the Jacobian is viewed as a complex torus. From the definition of the Abel-Jacobi map, one can easily show that the canonical metric equals (up to scaling) the average of the squared absolute values of an orthonormal basis of holomorphic weight two forms on M. Let us use the notation μ_{shyp} and μ_{can} to denote the $(1,1)$-forms associated to the hyperbolic and canonical metrics, respectively, where each metric has been scaled so that M has volume one.

A natural topic to investigate is to compare the hyperbolic and canonical metrics. Aside from intrinsic interest, being a question that could have been posed nearly 150 years ago, the problem possesses additional significance due to modern developments in mathematics. Specifically, Proposition 3.1 of [27] states a relation involving the degree of a modular parameterization of an elliptic curve and the Petersson norm of certain holomorphic modular forms. Since the canonical metric is an average value of such norms, one can relate the Petersson norm of such forms to the quotient $\mu_{\mathrm{can}}/\mu_{\mathrm{shyp}}$. The literature contains many articles which document the relation between degrees of modular parameterizations and fundamental questions in number theory, such as the *abc*-conjecture. Therefore, we can take the question of understanding bounds for holomorphic modular forms as having both classical significance as well as modern importance.

1.2. With the above discussion, let us focus on the following problem. For any finite volume hyperbolic Riemann surface M, let

$$d_M := \sup_{z \in M} \frac{\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{shyp}}(z)}\,.$$

The main result in [1] is the bound $d_{\Gamma_0(N)\backslash\mathbb{H}} = O(N^{2+\varepsilon})$; in [23] the bound was improved to $d_{\Gamma_0(N)\backslash\mathbb{H}} = O(N^{\varepsilon})$. The methods of proof in [1] and [23] employ a significant amount of arithmetic associated to the congruence subgroup $\Gamma_0(N)$. In [13], the authors use geometric considerations, namely heat kernel analysis and the maximum principle for holomorphic functions, and prove the following result. If M is a finite degree cover of M_0, then $d_M = O_{M_0}(1)$; in particular, for the modular curves, the main theorem in [13] asserts that $d_{\Gamma_0(N)\backslash\mathbb{H}} = O(1)$, which can be shown to be optimal.

1.3. In the present article, we revisit the problem of proving the optimal bound for d_M derived in [13] by studying one of the key identities from our previous work. Specifically, in [14], an identity is proved which relates μ_{can} to μ_{shyp} and an integral involving the hyperbolic heat kernel on M. The analysis in the present paper involves a detailed study of the hyperbolic heat kernel on M, sufficient to reprove the main result from [13] by examining the aforementioned relation between μ_{can} and μ_{shyp}. The hyperbolic heat kernel on M is written as a sum over the uniformizing group Γ, and this series is decomposed into terms involving parabolic, hyperbolic, and elliptic terms. In each case, the series of terms can be bounded involving special values of non-holomorphic Eisenstein series at $s = 2$, where s is the spectral parameter. By now, non-holomorphic Eisenstein series attached to parabolic subgroups are classical objects in mathematics. Kudla and Millson defined non-holomorphic Eisenstein series attached to hyperbolic subgroups; see [21]. With [21] as motivation, we defined non-holomorphic Eisenstein series attached to elliptic subgroups in [17]. The results in the unpublished article [17] have been absorbed into the present paper.

Bounds for d_M in terms of various Eisenstein series at $s = 2$ stem from the group theoretic decomposition of a sum over Γ of a function, a method that is well-known from the development of the Selberg trace formula. The series obtained by studying the parabolic and elliptic terms are bounded by finite sums of special values of Eisenstein series, in effect because there are a finite number of non-conjugate (in Γ), primitive classes of parabolic and elliptic elements in Γ. By contrast, there are an infinite number of non-conjugate (in Γ), primitive, hyperbolic elements, so the resulting bound for d_M in terms of special values of hyperbolic Eisenstein series needs to be studied in depth. Beyond the formal bound for d_M, we utilize two additional results. First, we employ the spectral expansion of scalar-valued hyperbolic Eisenstein series, which differ slightly from the form-valued hyperbolic Eisenstein series defined in [21]. The spectral expansion we need is proved in [18]. Second, we use results from [31] which studies the asymptotic behavior of periods of functions when integrated over primitive, closed geodesics of increasing length. To be honest, we require a version of the main theorem of [31] which asserts the dependence on the error term in the asymptotic formula proven in terms of the eigenvalue of the function whose period is under study; see Subsection 2.7.

As stated, the analysis of the present paper yields another proof of the main theorem of [13] using hyperbolic geometry, Eisenstein series of various types (parabolic, hyperbolic, and elliptic), asymptotics for periods of eigenfunctions of the Laplacian, as proved in [31], and the main identity from [14] relating μ_{can} to hyperbolic geometric quantities.

1.4. We find it fascinating that the special value of the Eisenstein series which appears is at the value $s = 2$ of the spectral parameter. The same special value of parabolic Eisenstein series has appeared elsewhere. In [29], the author is studying various problems in the theory of infinite energy harmonic maps, and numerous bounds involve the parabolic Eisenstein series at $s = 2$. The results from [29] are used in [30] when studying the disappearance of cusp forms through deformations, which itself is related to the outstanding problem of determining the existence of L^2-eigenfunctions on M, i.e., the Phillips-Sarnak conjecture. In [28], the authors study curvature forms of metrics on the moduli space of Riemann surfaces. If the surfaces are non-compact, the curvature forms include a term involving the parabolic Eisenstein series at $s = 2$.

1.5. This article is organized as follows. In Section 2, we establish notation and recall necessary background results from throughout the literature. In Section 3, we define the three types of non-holomorphic Eisenstein series associated to M; parabolic, hyperbolic, and elliptic. In Section 4, we extend the key identity from

[14] to the general setting of finite volume, hyperbolic Riemann surfaces which may have cusps as well as elliptic fixed points. As stated above, the identity involves the heat kernel on M, which can be written as a sum of three sub-series corresponding to the parabolic, hyperbolic, and elliptic elements in Γ. These three sub-series are studied separately in Sections 5, 6, and 7. Finally, in Section 8, we put together the main computations from Sections 5, 6, and 7 in order to establish the optimal bound for d_M.

1.6. We are very happy to dedicate the present paper to Stephen S. Kudla in honor of his 60th birthday, and to call attention to his article [21]. The study of the arithmetic and geometry of hyperbolic and elliptic Eisenstein series seems to be a fruitful area of investigation, and we first learned of hyperbolic Eisenstein series when James Cogdell suggested to the first-named author to study [21]. Many important theorems in mathematics have been obtained from non-holomorphic, parabolic Eisenstein series, and each result can be posed as a question to be studied for either hyperbolic or elliptic Eisenstein series. We refer the reader to the articles [5], [6], [19], and [24] for recent results.

We thank the editors of the present volume for the opportunity to contribute our article in honor of Stephen Kudla. We thank Anna von Pippich for many suggestions which helped with the exposition of the article, as well as D. Hejhal and S. Zelditch for informative mathematical discussions relating to this article. Also, the first-named author thanks James Cogdell for the suggestion to study [21], which begin the mathematics in this article. We anticipate that the mathematical influence of [21] will continue from here.

2 Background material

2.1. Hyperbolic metric. Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ be any Fuchsian group of the first kind acting by fractional linear transformations on the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \,|\, z = x + iy\,,\, y > 0\}$. Let M be the quotient space $\Gamma\backslash\mathbb{H}$ and g the genus of M. Denote by $\mathcal{T}$ the set of torsion points (i.e., elliptic fixed points) of M and by $\mathcal{C}$ the set of cusps of M; we put $t := |\mathcal{T}|$ and $c := |\mathcal{C}|$. If $p \in \mathcal{T}$, we let m_p denote the order of the torsion point p; we set $m_p = 1$, if p is a regular point of M. Locally, away from the torsion points, we identify M with its universal cover $\mathbb{H}$, and hence, denote the points on $M \setminus \mathcal{T}$ by the same letter as the points on $\mathbb{H}$.

We denote by $\mathrm{d}s^2_{\mathrm{hyp}}$ the line element, resp. by μ_{hyp} the volume form corresponding to the hyperbolic metric on M, which is compatible with the complex structure of M and has constant curvature equal to minus one. Locally on $M \setminus \mathcal{T}$, we have

$$\mathrm{d}s^2_{\mathrm{hyp}}(z) := \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}\,, \quad \text{resp.} \quad \mu_{\mathrm{hyp}}(z) := \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}\,.$$

We denote the hyperbolic distance between $z, w \in M$ by $\mathrm{dist}_{\mathrm{hyp}}(z, w)$ and we recall that the hyperbolic volume $\mathrm{vol}_{\mathrm{hyp}}(M)$ of M is given by the formula

$$\mathrm{vol}_{\mathrm{hyp}}(M) = 4\pi\bigg(g-1+\frac{c}{2}+\frac{1}{2}\sum_{p\in\mathcal{T}}\Big(1-\frac{1}{m_p}\Big)\bigg).$$

We define the $(1,1)$-form corresponding to the scaled hyperbolic metric, which measures the volume of M to be one, by

$$\mu_{\mathrm{shyp}} := \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(M)}\,\mu_{\mathrm{hyp}}.$$

We note that μ_{hyp} and μ_{shyp} are smooth differential forms on $M\setminus\mathcal{T}$.

In addition to the cartesian coordinates x,y, we will also make use of euclidean and hyperbolic polar coordinates. The euclidean polar coordinates $\rho=\rho(z)$, $\theta=\theta(z)$ of the point z centered at the origin are related to x,y in the way

$$x := e^{\rho}\cos(\theta)\,,\quad y := e^{\rho}\sin(\theta). \tag{1}$$

The line element $\mathrm{d}s^2_{\mathrm{hyp}}$, resp. volume form μ_{hyp} are expressed as follows in these coordinates

$$\mathrm{d}s^2_{\mathrm{hyp}}(z) = \frac{\mathrm{d}\rho^2+\mathrm{d}\theta^2}{\sin^2(\theta)}\,,\quad \text{resp.}\quad \mu_{\mathrm{hyp}}(z) = \frac{\mathrm{d}\rho\wedge\mathrm{d}\theta}{\sin^2(\theta)}\,.$$

The hyperbolic polar coordinates $\varrho=\varrho(z)$, $\vartheta=\vartheta(z)$ of the point z centered at i are given by

$$\varrho(z) := \mathrm{dist}_{\mathrm{hyp}}(i,z)\,,\quad \vartheta(z) := \angle(\widetilde{L},T_z), \tag{2}$$

where $\widetilde{L}$ denotes the positive y-axis and T_z is the tangent line to the geodesic passing through i and z at the point i. The line element $\mathrm{d}s^2_{\mathrm{hyp}}$, resp. volume form μ_{hyp} are expressed as follows in these coordinates

$$\mathrm{d}s^2_{\mathrm{hyp}}(z) = \mathrm{d}\varrho^2+\sinh^2(\varrho)\,\mathrm{d}\vartheta^2\,,\quad \text{resp.}\quad \mu_{\mathrm{hyp}}(z) = \sinh(\vartheta)\,\mathrm{d}\varrho\wedge\mathrm{d}\vartheta.$$

The hyperbolic Laplacian Δ_{hyp} is locally, on $M\setminus\mathcal{T}$, given by

$$\Delta_{\mathrm{hyp}} := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

From [22], p. 10, we have for any smooth function f on M

$$\mathrm{d}_z\mathrm{d}_z^c f = -\frac{1}{4\pi}(\Delta_{\mathrm{hyp}}f)\mu_{\mathrm{hyp}}.$$

2.2. Hyperbolic heat kernels. For $t>0$ and $\rho\geq 0$, we define

$$K_{\mathbb{H}}(t;\rho) := \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}}\int_{\rho}^{\infty}\frac{r\,e^{-r^2/t}}{\sqrt{\cosh(r)-\cosh(\rho)}}\,\mathrm{d}r.$$

The hyperbolic heat kernel on $\mathbb{H}$ is defined by

$$K_{\mathbb{H}}(t;z,w) := K_{\mathbb{H}}\big(t;\mathrm{dist}_{\mathrm{hyp}}(z,w)\big) \qquad (z,w\in\mathbb{H}).$$

The hyperbolic heat kernel on M is now defined by

$$K_{\mathrm{hyp}}(t;z,w) := \sum_{\gamma\in\Gamma} K_{\mathbb{H}}(t;z,\gamma w) \qquad (z,w\in M).$$

We note that the hyperbolic heat kernel satisfies the heat equation, i.e.,

$$\left(\Delta_{\mathrm{hyp}} + \frac{\partial}{\partial t}\right) K_{\mathrm{hyp}}(t;z,w) = 0.$$

We end this subsection by recalling the following heat kernel estimates:

$$K_{\mathrm{hyp}}(t;z,w) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(M)} = O_M\big(e^{-c_{M;z,w}t}\big) \qquad (z,w\in M;\, t\to\infty), \tag{3}$$

$$K_{\mathrm{hyp}}(t;z,w) = O_M\big(e^{-c_{M;z,w}/t}\big) \qquad (z,w\in M;\, z\neq w;\, t\to 0), \tag{4}$$

$$K_{\mathrm{hyp}}(t;z,z) - K_{\mathbb{H}}(t;0) = O_M\big(e^{-c_{M;z}/t}\big) \qquad (z\in M;\, t\to 0); \tag{5}$$

here $c_{M;z,w}$ refers to a positive constant depending on M and z, w; furthermore we let $c_{M;z} := c_{M;z,z}$. The constant $c_{M;z,w}$ can be uniformly bounded away from zero when z and w are restricted to any compact subsets of M, which is the only situation we will consider when employing the above bound. The bound (3) can be derived from [11], p. 103, Theorem 7.3, formula (7.15), applied to $K_{\mathrm{hyp}}(t;z,w)$ for fixed $w\in M$; the bounds (4) and (5) follow from [4], p. 154, formula (45).

2.3. Hyperbolic Green's functions. For $z,w\in\mathbb{H}$ and $s\in\mathbb{C}$ satisfying $\mathrm{Re}(s) > 1/2$, the free-space Green's function on $\mathbb{H}$ is defined by (see [9], p. 31, taking into account that our normalization differs from Hejhal's by a factor of -4π)

$$g_{\mathbb{H},s}(z,w) := \frac{\Gamma(s)^2}{\Gamma(2s)}\left(1-\left|\frac{z-w}{z-\bar{w}}\right|^2\right)^s F\left(s,s,2s;1-\left|\frac{z-w}{z-\bar{w}}\right|^2\right),$$

where $\Gamma(s)$ is the Gamma function and $F(a,b,c;w)$ is Gauss' hypergeometric function. The hyperbolic Green's function on $\mathbb{H}$ is now defined by

$$g_{\mathbb{H}}(z,w) := g_{\mathbb{H},1}(z,w).$$

The well-known formula for $F(1,1,2;w)$ yields the useful simple formula

$$g_{\mathbb{H}}(z,w) = -\log\left(\left|\frac{z-w}{z-\bar{w}}\right|^2\right). \tag{6}$$

The hyperbolic Green's function on $\mathbb{H}$ is related to the hyperbolic heat kernel on $\mathbb{H}$ through the formula

$$g_{\mathbb{H}}(z,w) = 4\pi\int_0^\infty K_{\mathbb{H}}(t;z,w)\,\mathrm{d}t. \tag{7}$$

For $z, w \in M$ and $s \in \mathbb{C}$ satisfying $\mathrm{Re}(s) > 1$, the free-space Green's function on M is now defined by the series (see [9], p. 31)

$$g_{\mathrm{hyp},s}(z,w) := \sum_{\gamma\in\Gamma} g_{\mathbb{H},s}(z,\gamma w).$$

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, this series is absolutely and locally uniformly convergent on $\mathbb{H} \times \mathbb{H}$ as long as (the lifts of) z and w do not lie in the same Γ-orbit (see [9], Proposition 6.2). For fixed $z, w \in M$, the series defines a holomorphic function in the half-plane $\{s \in \mathbb{C} \,|\, \mathrm{Re}(s) > 1\}$, which admits a meromorphic continuation to all $s \in \mathbb{C}$ with a pole at $s = 1$ (see [9], p. 250). The hyperbolic Green's function $g_{\mathrm{hyp}}(z,w)$ on M ($z, w \in M$; $z \neq w$) is defined as the constant term in the Laurant expansion of $g_{\mathrm{hyp},s}(z,w)$ at $s = 1$. The hyperbolic Green's function on M is related to the hyperbolic heat kernel on M through the formula

$$g_{\mathrm{hyp}}(z,w) = 4\pi \int_0^\infty \left(K_{\mathrm{hyp}}(t;z,w) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(M)} \right) \mathrm{d}t.$$

The heat kernel estimates (3) and (4) imply that the above integrand is integrable whenever $z \neq w$.

We conclude this subsection by recalling the characteristic properties of the hyperbolic Green's function on M. For $z, w \in M$, $z \neq w$, $z \notin \mathcal{T}$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the free-space Green's function on M satisfies the differential equation (see [9], p. 250)

$$\mathrm{d}_z \mathrm{d}_z^c g_{\mathrm{hyp},s}(z,w) = s(s-1) g_{\mathrm{hyp},s}(z,w) \mu_{\mathrm{hyp}}(z),$$

which gives rise to the differential equation

$$\mathrm{d}_z \mathrm{d}_z^c g_{\mathrm{hyp}}(z,w) + \delta_w(z) = \mu_{\mathrm{shyp}}(z)$$

for the hyperbolic Green's function on M; here $z, w \in M$ with $z \notin \mathcal{T}$. Next, the hyperbolic Green's function on M satisfies the normalization condition

$$\int_M g_{\mathrm{hyp}}(z,w) \mu_{\mathrm{hyp}}(z) = 0 \qquad (w \in M),$$

which follows from

$$\int_M K_{\mathrm{hyp}}(t;z,w) \mu_{\mathrm{hyp}}(z) = 1 \qquad (w \in M).$$

The third property to be mentioned states that the hyperbolic Green's function on M is bounded for $z \neq w$ and that the function $g_{\mathrm{hyp}}(z,w) + m_p \log|z-w|^2$ is bounded as z approaches the point $w = p$. The last property to be recalled states that the hyperbolic Green's function on M inverts the $\mathrm{d}_z \mathrm{d}_z^c$-operator; more precisely, if f is a bounded function on M with

$$\int_M f(z) \mu_{\mathrm{hyp}}(z) = 0,$$

then

$$\int\limits_M g_{\mathrm{hyp}}(z,w)\mathrm{d}_z\mathrm{d}_z^c f(z) = f(w) \qquad (w \in M).$$

The analysis given in [9] readily extends to prove that

$$\mathrm{d}_z\mathrm{d}_z^c g_{\mathrm{hyp},s}(z,z) = \sum_{\substack{\gamma\in\Gamma \\ \gamma\neq \mathrm{id}}} \mathrm{d}_z\mathrm{d}_z^c g_{\mathbb{H},s}(z,\gamma z)$$

for $z \in M$ and $s \in \mathbb{C}$ satisfying $\mathrm{Re}(s) > 1$. This shows in particular that the series on the right-hand side is absolutely and locally uniformly convergent in the range under consideration. Ultimately, our work will extend this equality to $s = 1$ in a conditionally converging sense (see Section 8).

2.4. Canonical metric. Assuming $g > 0$, we denote by $S_2(\Gamma)$ the g-dimensional $\mathbb{C}$-vector space of cusp forms of weight 2 with respect to Γ equipped with the Petersson inner product

$$\langle f,h\rangle := \int\limits_M f(z)\,\overline{h(z)}\,\mathrm{d}x\wedge\mathrm{d}y;$$

here $f,h \in S_2(\Gamma)$. Choosing an orthonormal basis $\{f_1,\ldots,f_g\}$ of $S_2(\Gamma)$, we define the $(1,1)$-form μ_{can} corresponding to the canonical metric of M by

$$\mu_{\mathrm{can}}(z) := \frac{1}{g}\cdot\frac{i}{2}\sum_{j=1}^{g}|f_j(z)|^2\mathrm{d}z\wedge\mathrm{d}\bar{z}.$$

Elementary linear algebra shows that μ_{can} is independent of the choice of the chosen orthonormal basis of $S_2(\Gamma)$. Further computations using the classical Riemann-Roch theorem show that μ_{can} is non-vanishing, and hence defines a metric on $M \setminus \mathcal{T}$.

2.5. Canonical Green's functions. The canonical Green's function $g_{\mathrm{can}}(z,w)$ on M ($z,w \in M$; $z \neq w$) is a smooth function on $(M\times M)\setminus\Delta(M)$ with a logarithmic singularity along the diagonal $\Delta(M)$. It is uniquely characterized by the two properties

$$\mathrm{d}_z\mathrm{d}_z^c g_{\mathrm{can}}(z,w) + \delta_w = \mu_{\mathrm{can}}(z) \qquad (z,w\in M; z\notin\mathcal{T}), \tag{8}$$

$$\int\limits_M g_{\mathrm{can}}(z,w)\mu_{\mathrm{can}}(z) = 0 \qquad (w\in M). \tag{9}$$

The existence and uniqueness of $g_{\mathrm{can}}(z,w)$ is obtained as follows: Consider the compactification $\overline{M}$ of M by adding the cusps. After reuniformization one obtains a compact Riemann surface M' of genus g with the same underlying complex structure as $\overline{M}$. The existence and uniqueness of the canonical Green's function on M' satisfying the analogues of properties (8) and (9) is now well-known. Transferring the resulting function back to $\overline{M}$ and restricting it to M, yields the desired function $g_{\mathrm{can}}(z,w)$.

Canonical Green's functions on quotient spaces of genus zero having elliptic fixed points can be explicitly written down using affine coordinates; see [22], p. 26, in the compact and smooth setting. For the sake of brevity, we omit the general formulas from the present discussion.

2.6. The prime geodesic theorem. Let $\mathcal{H}(\Gamma)$ denote a complete set of representatives of the Γ-conjugacy classes of the maximal hyperbolic subgroups of Γ. If $H \in \mathcal{H}(\Gamma)$, then $H = \langle \gamma_H \rangle$ with generator γ_H uniquely determined up to inversion. The real number ℓ_H denotes the length of the closed geodesic L_H on M determined by γ_H.

The prime geodesic counting function $\pi(u)$ is defined for $u \in \mathbb{R}_{>1}$ by the formula

$$\pi(u) := \sum_{\substack{H \in \mathcal{H}(\Gamma) \\ e^{\ell_H} < u}} 1.$$

Equivalently, the quantity $\pi(u)$ counts the number of non-conjugate, primitive, hyperbolic elements of Γ such that the corresponding geodesics have length less than $\log(u)$. From [8], p. 46, Proposition 2.11, we recall that the prime geodesic counting function can be "trivially" bounded as $\pi(u) = O(u)$.

For any eigenvalue λ_j of the hyperbolic Laplacian Δ_{hyp} satisfying $0 \leq \lambda_j < 1/4$, we set

$$s_j := \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_j}\,;$$

we note that $1/2 < s_j \leq 1$. In addition, we define for any $0 < \varepsilon \leq 1/4$ analogously

$$s_\varepsilon := \frac{1}{2} + \sqrt{\frac{1}{4} - \varepsilon}\,.$$

Recalling the logarithmic integral

$$\mathrm{li}\big(u^{s_j}\big) := \int_2^{u^{s_j}} \frac{\mathrm{d}\xi}{\log(\xi)}\ ,$$

the prime geodesic theorem states

$$\pi(u) = \sum_{0 \leq \lambda_j < 1/4} \mathrm{li}\big(u^{s_j}\big) + O\left(u^{3/4}\log(u)^{-1/2}\right)$$

for $u > 2$, where the implied constant depends solely on M (see [10], p. 474).

2.7. Periods of eigenfunctions. Let ψ be any smooth, bounded function on M and let $\mathcal{C}$ denote any bounded, continuous path on M. The period of ψ along $\mathcal{C}$ is defined to be the integral

$$\int_{\mathcal{C}} \psi(z)\, \mathrm{d}s_{\mathrm{hyp}}(z)\,.$$

A generalization of the prime geodesic theorem can be considered by studying the counting function

$$\pi(u;\psi) := \sum_{\substack{H\in\mathcal{H}(\Gamma)\\ e^{\ell_H}<u}} \int_{L_H} \psi(z)\, \mathrm{d}s_{\mathrm{hyp}}(z)\,.$$

The asymptotic behavior of $\pi(u;\psi)$ as u tends to infinity is studied in detail in [31]. Specifically, the following statement is proven. Let ψ be either a non-constant, L^2-normalized eigenfunction of the hyperbolic Laplacian with eigenvalue $\lambda = s(1-s)$, or the parabolic Eisenstein series with spectral parameter $s = 1/2 + ir$, hence $\lambda = 1/4 + r^2$ (see Subsection 3.1 for more details). Then, there exists $\varepsilon = \varepsilon(\Gamma) > 0$ such that the asymptotic formula

$$\pi(u;\psi) = O_{\lambda}\big(u^{1-\varepsilon}\big) \tag{10}$$

holds as $u \to \infty$.

In fact, a stronger statement than (10) is given on p. 5 of [31]: Specifically, an asymptotic expansion of the counting function $\pi(u;\psi)$ is proven out to $O_{\lambda}(u^{19/20})$, with lead terms expressed in terms of certain integrals involving ψ and eigenfunctions as well as constants which depend on small eigenvalues. Elementary considerations, as discussed on p. 85 of [31], bound these constants in the form $O(r^k e^{\pi r/2})$, and the main results from [25] bound the integrals involving ψ and the eigenfunctions associated to the small eigenvalues by $O(e^{-\pi r/2})$. Combining these results, the finite series which appears in the expansion of (10) on p. 5 of [31] has a bound of the form $O(\lambda^k)$ for some k.

With the above discussion, we can assume in the sequel that the dependence of the estimate (10) with respect to λ can be bounded in the form $O(\lambda^k)$ for some k. These considerations, which were provided to us by S. Zelditch, constitute the crucial step in providing all details in proving this statement. We will leave a complete proof analysis of the method of proof in [31] which utilizes the bounds from [25] to the interested reader.

3 Eisenstein series

3.1. Parabolic Eisenstein series. Let $\mathcal{P}(\Gamma)$ denote a complete set of representatives of the Γ-conjugacy classes of the maximal parabolic subgroups of Γ; we note that the set $\mathcal{P}(\Gamma)$ is finite. If $P \in \mathcal{P}(\Gamma)$, then $P = \langle\gamma_P\rangle$ with generator γ_P uniquely determined up to inversion. There exists $\sigma_P \in \mathrm{PSL}_2(\mathbb{R})$ such that

$$\sigma_P^{-1}\gamma_P\,\sigma_P = \begin{pmatrix}1&1\\0&1\end{pmatrix} =: \gamma_\infty \quad\Longleftrightarrow\quad \gamma_P = \sigma_P\gamma_\infty\sigma_P^{-1}. \tag{11}$$

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the *parabolic Eisenstein series* $\mathcal{E}_{\mathrm{par},P}(z,s)$ associated to P is defined by the series

$$\mathcal{E}_{\mathrm{par},P}(z,s) := \sum_{\eta \in P\backslash\Gamma} \mathrm{Im}\big(\sigma_P^{-1}\eta z\big)^s. \tag{12}$$

The parabolic Eisenstein series (12) is absolutely and locally uniformly convergent for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$; it is invariant under the action of Γ and satisfies the differential equation

$$\big(\Delta_{\mathrm{hyp}} - s(1-s)\big)\mathcal{E}_{\mathrm{par},P}(z,s) = 0.$$

For proofs of theses facts and further details, we refer to [9] or [20].

3.2. Hyperbolic Eisenstein series. Let $\mathcal{H}(\Gamma)$ denote a complete set of representatives of the Γ-conjugacy classes of the maximal hyperbolic subgroups of Γ. If $H \in \mathcal{H}(\Gamma)$, then $H = \langle\gamma_H\rangle$ with generator γ_H uniquely determined up to inversion. There exists $\sigma_H \in \mathrm{PSL}_2(\mathbb{R})$ such that

$$\sigma_H^{-1}\gamma_H\,\sigma_H = \begin{pmatrix} e^{\ell_H/2} & 0 \\ 0 & e^{-\ell_H/2} \end{pmatrix} =: \gamma_{0,H} \quad\Longleftrightarrow\quad \gamma_H = \sigma_H\gamma_{0,H}\sigma_H^{-1}; \tag{13}$$

the real number ℓ_H denotes the length of the closed geodesic L_H on M determined by γ_H.

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the *hyperbolic Eisenstein series* $\mathcal{E}_{\mathrm{hyp},H}(z,s)$ associated to H is defined by the series

$$\mathcal{E}_{\mathrm{hyp},H}(z,s) := \sum_{\eta \in H\backslash\Gamma} \sin\big(\theta(\sigma_H^{-1}\eta z)\big)^s \tag{14}$$

using the polar coordinates (1). The hyperbolic Eisenstein series (14) is absolutely and locally uniformly convergent for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$; it is invariant under the action of Γ and satisfies the differential equation

$$\big(\Delta_{\mathrm{hyp}} - s(1-s)\big)\mathcal{E}_{\mathrm{hyp},H}(z,s) = s^2\,\mathcal{E}_{\mathrm{hyp},H}(z,s+2). \tag{15}$$

Denoting by $\widetilde{L}_H$ the lift of L_H to the universal cover $\mathbb{H}$, we have the relation

$$\sin\big(\theta(\sigma_H^{-1}z)\big)\cdot\cosh\big(\mathrm{dist}_{\mathrm{hyp}}(z,\widetilde{L}_H)\big) = 1\,,$$

and the hyperbolic Eisenstein series (14) can be rewritten as

$$\mathcal{E}_{\mathrm{hyp},H}(z,s) = \sum_{\eta \in H\backslash\Gamma} \cosh\big(\mathrm{dist}_{\mathrm{hyp}}(\eta z,\widetilde{L}_H)\big)^{-s}.$$

For proofs of theses facts and further details, we refer to [21].

In the sequel we will have to make use of the spectral expansion of the hyperbolic Eisenstein series which was established in [18]. Theorem 4.1 therein states

$$\mathcal{E}_{\mathrm{hyp},H}(z,s) = \sum_{j=0}^{\infty} a_{j,H}(s)\,\psi_j(z) + \frac{1}{4\pi}\sum_{P\in\mathcal{P}(\Gamma)}\int_{-\infty}^{\infty} a_{1/2+ir,H,P}(s)\,\mathcal{E}_{\mathrm{par},P}(z,1/2+ir)\,\mathrm{d}r\,. \tag{16}$$

The coefficient $a_{j,H}(s)$ is given by the formula

$$a_{j,H}(s) = \sqrt{\pi}\,\frac{\Gamma((s-1/2+ir_j)/2)\Gamma((s-1/2-ir_j)/2)}{\Gamma(s/2)^2}\int_{L_H}\psi_j(z)\,\mathrm{d}s_{\mathrm{hyp}}(z);$$

here we have written the eigenvalue λ_j of the eigenfunction ψ_j in the form $\lambda_j = 1/4 + r_j^2$. An analogous formula holds for the coefficient $a_{1/2+ir,H,P}(s)$; it is given at the end of the proof of Theorem 4.1 in [18].

3.3. Elliptic Eisenstein series. Let $\mathcal{E}(\Gamma)$ denote a complete set of representatives of the Γ-conjugacy classes of the maximal elliptic subgroups of Γ; we note that the set $\mathcal{E}(\Gamma)$ is finite. If $E \in \mathcal{E}(\Gamma)$, then $E = \langle\gamma_E\rangle$ with generator γ_E uniquely determined up to inversion. There exists $\sigma_E \in \mathrm{PSL}_2(\mathbb{R})$ such that

$$\sigma_E^{-1}\gamma_E\,\sigma_E = \begin{pmatrix}\cos\big(\pi/m_E\big) & \sin\big(\pi/m_E\big)\\ -\sin\big(\pi/m_E\big) & \cos\big(\pi/m_E\big)\end{pmatrix} =: \gamma_{i,E} \quad\Longleftrightarrow\quad \gamma_E = \sigma_E\gamma_{i,E}\sigma_E^{-1}; \tag{17}$$

the natural number m_E denotes the order of the torsion point t_E on M determined by the maximal elliptic subgroup E, which also equals the order $|E|$ of the finite group E.

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the *elliptic Eisenstein series* $\mathcal{E}_{\mathrm{ell},E}(z,s)$ associated to E is defined by the series

$$\mathcal{E}_{\mathrm{ell},E}(z,s) := \sum_{\eta\in E\backslash\Gamma} \sinh\big(\varrho(\sigma_E^{-1}\eta z)\big)^{-s} \tag{18}$$

using the hyperbolic polar coordinates (2). The elliptic Eisenstein series (18) is absolutely and locally uniformly convergent for $z \in \mathbb{H}$ with $z \neq \eta\sigma_E i$ for any $\eta \in \Gamma$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$; it is invariant under the action of Γ and satisfies the differential equation

$$\big(\Delta_{\mathrm{hyp}} - s(1-s)\big)\mathcal{E}_{\mathrm{ell},E}(z,s) = -s^2\,\mathcal{E}_{\mathrm{ell},E}(z,s+2).$$

For proofs of theses facts and further details, we refer to [24] or [19].

4 A fundamental identity

In this section, we discuss a fundamental identity relating the hyperbolic and the canonical metric. This relation has already been presented in special contexts (see [15] and [16]).

4.1. Proposition. *With the above notations, the following relation holds true for* $z \in M \setminus \mathcal{T}$

$$g\,\mu_{\mathrm{can}}(z) = \left(\frac{1}{4\pi} + \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(M)}\right)\mu_{\mathrm{hyp}}(z) + \frac{1}{2}\left(\int_0^\infty \Delta_{\mathrm{hyp}}\, K_{\mathrm{hyp}}(t;z,z)\mathrm{d}t\right)\mu_{\mathrm{hyp}}(z). \tag{19}$$

Proof. In case Γ is cocompact and torisonfree, the proof is given in [15]. If Γ is cocompact with torison, the proof given in [15] can easily be adapted as long as $z \in M \setminus \mathcal{T}$. If Γ is no longer cocompact, but cofinite, the proof given in [16] applies (see also [3]). $\square$

4.2. Remark. Formally, we have

$$\int_0^\infty \Delta_{\mathrm{hyp}}\, K_{\mathrm{hyp}}(t;z,z)\,\mathrm{d}t = \int_0^\infty \Delta_{\mathrm{hyp}} \sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}} K_{\mathbb{H}}(t;z,\gamma z)\,\mathrm{d}t$$
$$= \sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}} \Delta_{\mathrm{hyp}} \int_0^\infty K_{\mathbb{H}}(t;z,\gamma z)\,\mathrm{d}t = \frac{1}{4\pi}\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z)\,.$$

It will turn out that this formal computation can be justified if the latter sum is taken in a conditionally convergent sense, i.e., by individually summing over the parabolic, the elliptic, and (suitably) the hyperbolic elements of Γ; the details will be specified in the subsequent Sections 5, 6, 7, and 8.

4.3. Lemma. *For any* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{R})$*, we have*

$$\Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) = -4y^2\left(\frac{1}{(cz\bar{z}+dz-a\bar{z}-b)^2} + \frac{1}{(cz\bar{z}+d\bar{z}-az-b)^2}\right).$$

Proof. Using the explicit formula (6) for $g_{\mathbb{H}}(z,w)$, the proof is a straightforward computation, namely we have

$$\Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) = -4y^2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\left(-\log\left|\frac{z-\gamma z}{z-\gamma\bar{z}}\right|^2\right)$$
$$= 4y^2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\big(\log(z-\gamma z) + \log(\bar{z}-\gamma\bar{z}) - \log(z-\gamma\bar{z}) - \log(\bar{z}-\gamma z)\big)$$
$$= 4y^2\frac{\partial}{\partial z}\Big(\frac{1}{z-\gamma\bar{z}}\cdot\frac{\partial\gamma\bar{z}}{\partial\bar{z}} - \frac{1}{\bar{z}-\gamma z}\Big)$$
$$= -4y^2\bigg(\frac{1}{(z-\gamma\bar{z})^2}\cdot\frac{\partial\gamma\bar{z}}{\partial\bar{z}} + \frac{1}{(\bar{z}-\gamma z)^2}\cdot\frac{\partial\gamma z}{\partial z}\bigg).$$

Employing

$$\frac{\partial\gamma z}{\partial z} = \frac{1}{(cz+d)^2}\,,$$

the claim follows. $\square$

5 Estimates in the parabolic case

5.1. Lemma. (i) *For a parabolic element* $\gamma = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{R})$*, we have*

$$\Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \gamma z) = -8y^2 \frac{u^2 - 4y^2}{(u^2 + 4y^2)^2}\,.$$

(ii) *Let* $P = \langle \gamma_P \rangle$ *be a parabolic subgroup of* $\mathrm{PSL}_2(\mathbb{R})$ *generated by* $\gamma_P = \left(\begin{smallmatrix} 1 & u_P \\ 0 & 1 \end{smallmatrix}\right)$*. Then, we have*

$$\sum_{\gamma \in P \setminus \{\mathrm{id}\}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \gamma z) = 2 \left(\frac{2\pi/u_P \cdot y}{\sinh(2\pi/u_P \cdot y)} \right)^2 - 2\,.$$

Proof. The first assertion follows immediately from Lemma 4.3. As for the second claim, we recall the identity

$$\sum_{n=-\infty}^{\infty} \frac{n^2 - w^2}{(n^2 + w^2)^2} = -\left(\frac{\pi}{\sinh(\pi w)} \right)^2,$$

which can be proven using elementary arguments from complex analysis (see, e.g., [7], p. 36, formula 1.421.5). □

5.2. Lemma. *The series*

$$\sum_{\substack{\gamma \in \Gamma \setminus \{\mathrm{id}\} \\ \gamma \text{ parabolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \gamma z)$$

is absolutely and locally uniformly convergent for $z \in \mathbb{H}$.

Proof. Using the finite set $\mathcal{P}(\Gamma)$ of representatives of Γ-conjugacy classes of maximal parabolic subgroups of Γ (see subsection 3.1), we have the disjoint union decompositions

$$\big\{\gamma \in \Gamma \setminus \{\mathrm{id}\} \,|\, \gamma \text{ parabolic}\big\} = \bigcup_{P \in \mathcal{P}(\Gamma)} \bigcup_{Q \text{ conj. to } P} \big(Q \setminus \{\mathrm{id}\}\big)$$

$$= \bigcup_{P \in \mathcal{P}(\Gamma)} \bigcup_{\eta \in P \backslash \Gamma} \big(\eta^{-1} P \eta \setminus \{\mathrm{id}\}\big) = \bigcup_{P = \langle \gamma_P \rangle \in \mathcal{P}(\Gamma)} \bigcup_{\eta \in P \backslash \Gamma} \bigcup_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \{\eta^{-1} \gamma_P^n\, \eta\}\,.$$

This gives

$$\sum_{\substack{\gamma \in \Gamma \setminus \{\mathrm{id}\} \\ \gamma \text{ parabolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \gamma z) = \sum_{P = \langle \gamma_P \rangle \in \mathcal{P}(\Gamma)} \sum_{\eta \in P \backslash \Gamma} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \eta^{-1} \gamma_P^n\, \eta z).$$

For $P = \langle \gamma_P \rangle \in \mathcal{P}(\Gamma)$, we choose σ_P and γ_∞ as in (11), which leads to

$$\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\text{parabolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) = \sum_{P=\langle\gamma_P\rangle\in\mathcal{P}(\Gamma)} \sum_{\eta\in P\setminus\Gamma} \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}\big(z,\eta^{-1}\sigma_P\gamma_\infty^n\sigma_P^{-1}\eta z\big)$$
$$= \sum_{P=\langle\gamma_P\rangle\in\mathcal{P}(\Gamma)} \sum_{\eta\in P\setminus\Gamma} \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}\big(\sigma_P^{-1}\eta z,\gamma_\infty^n\sigma_P^{-1}\eta z\big).$$

Setting $y_{P,\eta} := \mathrm{Im}(\sigma_P^{-1}\eta z)$, we estimate using Lemma 5.1 (i) with $u = n \in \mathbb{Z}$

$$\Big|\Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}\big(\sigma_P^{-1}\eta z,\gamma_\infty^n\sigma_P^{-1}\eta z\big)\Big| = 8y_{P,\eta}^2\cdot\bigg|\frac{n^2-4y_{P,\eta}^2}{(n^2+4y_{P,\eta}^2)^2}\bigg| \le \frac{8y_{P,\eta}^2}{n^2+4y_{P,\eta}^2}\,.$$

This gives

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \Big|\Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(\sigma_P^{-1}\eta z,\gamma_\infty^n\sigma_P^{-1}\eta z)\Big| \le \sum_{n=1}^{\infty}\frac{16y_{P,\eta}^2}{n^2+4y_{P,\eta}^2}\,.$$

To ease notation, set $w := 2y_{P,\eta}$ for the moment. Observing the elementary estimate

$$\sum_{n=1}^{\infty}\frac{w^2}{n^2+w^2} \le \frac{w^2}{1+w^2} + \int_1^{\infty}\frac{w^2}{t^2+w^2}\,\mathrm{d}t$$
$$\le w^2 + \int_1^{\infty}\frac{w^2}{t^2+w^2}\,\mathrm{d}t = w^2 + w\bigg(\frac{\pi}{2}-\arctan\Big(\frac{1}{w}\Big)\bigg)$$
$$\le w^2 + w\cdot w = 2w^2,$$

we get

$$\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\text{parabolic}}} \big|\Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z)\big| \le \sum_{P\in\mathcal{P}(\Gamma)}\sum_{\eta\in P\setminus\Gamma} 32\, y_{P,\eta}^2$$
$$= 32\sum_{P\in\mathcal{P}(\Gamma)}\sum_{\eta\in P\setminus\Gamma}\mathrm{Im}\big(\sigma_P^{-1}\eta z\big)^2 = 32\sum_{P\in\mathcal{P}(\Gamma)}\mathcal{E}_{\mathrm{par},P}(z,2),$$

with the parabolic Eisenstein series $\mathcal{E}_{\mathrm{par},P}(z,s)$ associated to P evaluated at $s = 2$. The absolute and locally uniform convergence of the series in question now follows. □

5.3. Proposition. *Let Γ be a subgroup of finite index in the Fuchsian subgroup Γ_0 of the first kind. Then, we have*

$$\sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\,\text{parabolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) \le \sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\text{parabolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) \le 0\,.$$

Proof. We denote by $\mathcal{M}_{\mathrm{par}}(\Gamma)$ the set of all maximal parabolic subgroups of Γ. We observe that there is a bijection $\varphi : \mathcal{M}_{\mathrm{par}}(\Gamma) \xrightarrow{\simeq} \mathcal{M}_{\mathrm{par}}(\Gamma_0)$, which is given as follows: For $P \in \mathcal{M}_{\mathrm{par}}(\Gamma)$, there exists a maximal parabolic subgroup $P_0 \subset \Gamma_0$ containing P, and we set $\varphi(P) := P_0$; the inverse map is given by $\varphi^{-1}(P_0) := P_0 \cap \Gamma$.

We note that the set of parabolic elements of Γ different from the identity can be written as the disjoint union of the sets $P \setminus \{\mathrm{id}\}$ with P running through $\mathcal{M}_{\mathrm{par}}(\Gamma)$. Choosing for $P = \langle \gamma_P \rangle \in \mathcal{M}_{\mathrm{par}}(\Gamma)$, the quantities σ_P and γ_∞ as in (11), letting $w_P := \sigma_P^{-1} z$, and applying Lemma 5.1 (ii) with $u_P = 1$, we compute

$$\begin{aligned}\sum_{\substack{\gamma \in \Gamma \setminus \{\mathrm{id}\} \\ \gamma \text{ parabolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \gamma z) &= \sum_{P \in \mathcal{M}_{\mathrm{par}}(\Gamma)} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}\big(z, \sigma_P \gamma_\infty^n \sigma_P^{-1} z\big) \\ &= \sum_{P \in \mathcal{M}_{\mathrm{par}}(\Gamma)} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}\big(w_P, \gamma_\infty^n w_P\big) \\ &= 2 \sum_{P \in \mathcal{M}_{\mathrm{par}}(\Gamma)} \left(\left(\frac{2\pi \operatorname{Im}(w_P)}{\sinh(2\pi \operatorname{Im}(w_P))} \right)^2 - 1 \right). \end{aligned} \tag{20}$$

Elementary calculus shows that the function

$$h(x) := \left(\frac{x}{\sinh(x)} \right)^2 - 1$$

is negative, bigger than -1, and strictly monotone decreasing for $x > 0$. Using the negativity of the function $h(x)$, we derive the claimed upper bound immediately from formula (20).

We are left to prove the claimed lower bound. Replacing Γ by Γ_0, we obtain as above

$$\sum_{\substack{\gamma \in \Gamma_0 \setminus \{\mathrm{id}\} \\ \gamma \text{ parabolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \gamma z) = 2 \sum_{P_0 \in \mathcal{M}_{\mathrm{par}}(\Gamma_0)} \left(\left(\frac{2\pi \operatorname{Im}(w_{P_0})}{\sinh(2\pi \operatorname{Im}(w_{P_0}))} \right)^2 - 1 \right); \tag{21}$$

here $w_{P_0} := \sigma_{P_0}^{-1} z$ with the scaling matrix σ_{P_0} for a generator γ_{P_0} of $P_0 \in \mathcal{M}_{\mathrm{par}}(\Gamma_0)$. Letting $p := [P_0 : P]$, the relation between σ_{P_0} and σ_P is given by the formula

$$\sigma_{P_0} = \sigma_P \begin{pmatrix} 1/\sqrt{p} & 0 \\ 0 & \sqrt{p} \end{pmatrix}.$$

This together with the bijection between $\mathcal{M}_{\mathrm{par}}(\Gamma_0)$ and $\mathcal{M}_{\mathrm{par}}(\Gamma)$ allows us to rewrite (21) in the form

$$\sum_{\substack{\gamma \in \Gamma_0 \setminus \{\mathrm{id}\} \\ \gamma \text{ parabolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \gamma z) = 2 \sum_{P \in \mathcal{M}_{\mathrm{par}}(\Gamma)} \left(\left(\frac{2\pi p \operatorname{Im}(w_P)}{\sinh(2\pi p \operatorname{Im}(w_P))} \right)^2 - 1 \right).$$

The claimed lower bound now immediately follows from the fact that the function $h(x)$ is monotone decreasing. $\square$

5.4. Proposition. *Let* Γ *be a subgroup of finite index in* Γ_0. *Then, we have*

$$\sup_{z\in\mathbb{H}}\Bigg|\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\text{parabolic}}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z)\Bigg| = O_{\Gamma_0}(1).$$

Proof. By Proposition 5.3, it suffices to prove

$$\sup_{z\in\mathbb{H}}\Bigg(\sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\,\text{parabolic}}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z)\Bigg) = O_{\Gamma_0}(1).$$

By the Γ_0-invariance of the series under consideration, it suffices to bound it on $M_0 := \Gamma_0\backslash\mathbb{H}$. On the compact set given by M_0 minus the union of sufficiently small neigborhoods around the cusps of M_0, the series in question can be uniformly bounded. It remains to find uniform bounds in the neighborhoods of the (finitely many) cusps of M_0. For this we proceed as follows.

Recall that $\mathcal{P}(\Gamma_0)$ denotes a complete set of representatives of the Γ_0-conjugacy classes of the maximal parabolic subgroups of Γ_0. Then, arguing as in the proof of Lemma 5.2, we have

$$\big\{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\,|\,\gamma\,\text{parabolic}\big\} = \bigcup_{P\in\mathcal{P}(\Gamma_0)}\;\bigcup_{Q\,\text{conj. to}\,P}\big(Q\setminus\{\mathrm{id}\}\big)$$
$$= \bigcup_{P\in\mathcal{P}(\Gamma_0)}\;\bigcup_{\eta\in P\backslash\Gamma_0}\big(\eta^{-1}P\eta\setminus\{\mathrm{id}\}\big) = \bigcup_{P=\langle\gamma_P\rangle\in\mathcal{P}(\Gamma_0)}\;\bigcup_{\eta\in P\backslash\Gamma_0}\;\bigcup_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\{\eta^{-1}\gamma_P^n\,\eta\},$$

which leads to

$$\sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\,\text{parabolic}}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z) = \sum_{P=\langle\gamma_P\rangle\in\mathcal{P}(\Gamma_0)}\;\sum_{\eta\in P\backslash\Gamma_0}\;\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}\big(z,\eta^{-1}\sigma_P\gamma_\infty^n\sigma_P^{-1}\eta z\big)$$
$$= \sum_{P=\langle\gamma_P\rangle\in\mathcal{P}(\Gamma_0)}\;\sum_{\eta\in P\backslash\Gamma_0}\;\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}\big(\sigma_P^{-1}\eta z,\gamma_\infty^n\sigma_P^{-1}\eta z\big).$$

Using Lemma 5.1 (ii) with $u_P = 1$, we get

$$\sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\,\text{parabolic}}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z) = 2\sum_{P=\langle\gamma_P\rangle\in\mathcal{P}(\Gamma_0)}\;\sum_{\eta\in P\backslash\Gamma_0}\left(\left(\frac{2\pi\,\mathrm{Im}(\sigma_P^{-1}\eta z)}{\sinh(2\pi\,\mathrm{Im}(\sigma_P^{-1}\eta z))}\right)^2 - 1\right).$$

We are now able to show that the series in question is uniformly bounded as z tends to a cusp of M_0. By suitably conjugating Γ_0 inside of $\mathrm{PSL}_2(\mathbb{R})$, if necessary, we may assume that the cusp in question is $i\infty$ with cusp width one. To simplify notation, we denote the resulting Fuchsian subgroup again by Γ_0. Let then P_0 be the stabilizer of $i\infty$ in Γ_0. We can view P_0 as an element of $\mathcal{P}(\Gamma_0)$, and we may choose $\sigma_{P_0} = \mathrm{id}$. We have

$$\sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\,\text{parabolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) = 2\sum_{\substack{P\in\mathcal{P}(\Gamma_0)\\ P\neq P_0}} \sum_{\eta\in P\setminus\Gamma_0} \left(\left(\frac{2\pi\,\mathrm{Im}(\sigma_P^{-1}\eta z)}{\sinh(2\pi\,\mathrm{Im}(\sigma_P^{-1}\eta z))}\right)^2 - 1\right)$$
$$+\, 2\sum_{\substack{\eta\in P_0\setminus\Gamma_0\\ \eta\neq\mathrm{id}}} \left(\left(\frac{2\pi\,\mathrm{Im}(\eta z)}{\sinh(2\pi\,\mathrm{Im}(\eta z))}\right)^2 - 1\right) + 2\left(\left(\frac{2\pi\,\mathrm{Im}(z)}{\sinh(2\pi\,\mathrm{Im}(z))}\right)^2 - 1\right).$$

Applying the bound

$$0 \geq \left(\frac{x}{\sinh(x)}\right)^2 - 1 \geq -x^2$$

for $x > 0$, we get

$$0 \geq \sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\,\text{parabolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) \geq -8\pi^2 \sum_{\substack{P\in\mathcal{P}(\Gamma_0)\\ P\neq P_0}} \sum_{\eta\in P\setminus\Gamma_0} \mathrm{Im}(\sigma_P^{-1}\eta z)^2$$
$$-\, 8\pi^2 \sum_{\substack{\eta\in P_0\setminus\Gamma_0\\ \eta\neq\mathrm{id}}} \mathrm{Im}(\eta z)^2 + 2\left(\left(\frac{2\pi\,\mathrm{Im}(z)}{\sinh(2\pi\,\mathrm{Im}(z))}\right)^2 - 1\right).$$

With the parabolic Eisenstein series $\mathcal{E}_{\mathrm{par},P}(z,s)$, we get

$$0 \geq \sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\,\text{parabolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) \geq -8\pi^2 \sum_{\substack{P\in\mathcal{P}(\Gamma_0)\\ P\neq P_0}} \mathcal{E}_{\mathrm{par},P}(z,2)$$
$$-\, 8\pi^2\big(\mathcal{E}_{\mathrm{par},P_0}(z,2) - \mathrm{Im}(z)^2\big) + 2\left(\left(\frac{2\pi\,\mathrm{Im}(z)}{\sinh(2\pi\,\mathrm{Im}(z))}\right)^2 - 1\right).$$

If $\mathrm{Im}(z)$ tends to ∞, all terms involving the parabolic Eisenstein series are uniformly bounded (see [20], p. 12, Theorem 2.1.2). The last term is trivially bounded by -2 from below. This completes the proof of the proposition. □

6 Estimates in the elliptic case

6.1. Hyperbolic polar coordinates. From Subsection 2.1, formula (2), we recall the hyperbolic polar coordinates $\varrho(z)$ and $\vartheta(z)$ for $z = x + iy \in \mathbb{H}$. These coordinates satisfy the set of equations

$$x^2 + \big(y - \cosh\big(\varrho(z)\big)\big)^2 = \sinh^2\big(\varrho(z)\big)\,,$$
$$\big(x + \tan\big(\vartheta(z)\big)\big)^2 + y^2 = \tan^2\big(\vartheta(z)\big) + 1\,,$$

which are equivalent to

$$x^2 + y^2 + 1 = 2y\cosh\big(\varrho(z)\big)\,,$$
$$x^2 + y^2 - 1 = -2x\tan\big(\vartheta(z)\big)\,.$$

6.2. Lemma. (i) *For an elliptic element* $\gamma = \left(\begin{smallmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{R})$*, we have*

$$\begin{aligned}\Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) &= -8y^2\,\frac{(x^2+y^2+1)\sin^2(\alpha) - 4y^2\cos^2(\alpha)}{\big((x^2+y^2+1)\sin^2(\alpha) + 4y^2\cos^2(\alpha)\big)^2} \\ &= -2\,\frac{\cosh^2\big(\varrho(z)\big)\sin^2(\alpha) - \cos^2(\alpha)}{\big(\cosh^2\big(\varrho(z)\big)\sin^2(\alpha) + \cos^2(\alpha)\big)^2}\,.\end{aligned}$$

(ii) *Let* $E = \langle\gamma_{i,E}\rangle$ *be an elliptic subgroup of* $\mathrm{PSL}_2(\mathbb{R})$ *of order* $m_E > 1$ *generated by the element* $\gamma_{i,E} = \left(\begin{smallmatrix} \cos(\pi/m_E) & \sin(\pi/m_E) \\ -\sin(\pi/m_E) & \cos(\pi/m_E) \end{smallmatrix}\right)$*. Then, we have*

$$\sum_{\gamma\in E\setminus\{\mathrm{id}\}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) = 2\left(\frac{4m_E^2}{\sinh^2\big(\varrho(z)\big)}\cdot\frac{\tanh^{2m_E}\big(\varrho(z)/2\big)}{\big(1-\tanh^{2m_E}\big(\varrho(z)/2\big)\big)^2} - 1\right).$$

Proof. The first assertion follows immediately from Lemma 4.3 and the use of the hyperbolic polar coordinates given in 6.1. In order to prove the second claim, we first note the formula

$$\sum_{\gamma\in E\setminus\{\mathrm{id}\}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) = -2\sum_{n=1}^{m_E-1}\frac{\cosh^2\big(\varrho(z)\big)\sin^2\big(\pi n/m_E\big) - \cos^2\big(\pi n/m_E\big)}{\big(\cosh^2\big(\varrho(z)\big)\sin^2\big(\pi n/m_E\big) + \cos^2\big(\pi n/m_E\big)\big)^2}\,.$$

Put $X := \cosh^2\big(\varrho(z)\big)$ for the moment; note $X \geq 1$. For real α, we then consider the smooth real function

$$F(\alpha) := \frac{X\sin^2(\alpha) - \cos^2(\alpha)}{\big(X\sin^2(\alpha) + \cos^2(\alpha)\big)^2}\,.$$

Since $F(\alpha)$ is an even function with period π, it has a Fourier expansion of the form

$$F(\alpha) = \sum_{m=1}^{\infty} a_m(X)\cdot\cos(2m\alpha)\,.$$

A straightforward computation using residue calculus shows

$$a_m(X) = -\frac{4m}{X-1} \cdot q(X)^m \quad (m = 1, 2, 3, \ldots)$$

with $q(X) = 1 - 2/(1+\sqrt{X})$; note that $0 \leq q(X) < 1$ for $1 \leq X < \infty$. From this we derive

$$\sum_{n=1}^{m_E-1} \frac{\cosh^2\big(\varrho(z)\big)\sin^2\big(\pi n/m_E\big) - \cos^2\big(\pi n/m_E\big)}{\big(\cosh^2\big(\varrho(z)\big)\sin^2\big(\pi n/m_E\big) + \cos^2\big(\pi n/m_E\big)\big)^2} = 1 + \sum_{n=0}^{m_E-1} F\bigg(\frac{\pi n}{m_E}\bigg)$$

$$= 1 + \sum_{m=1}^{\infty} a_m(X) \sum_{n=0}^{m_E-1} \cos\bigg(\frac{2\pi m}{m_E} n\bigg) = 1 + m_E \sum_{m=1}^{\infty} a_{mm_E}(X)$$

$$= 1 - \frac{4m_E^2}{X-1} \sum_{m=1}^{\infty} m \cdot q(X)^{mm_E} = 1 - \frac{4m_E^2}{X-1} \cdot \frac{q(X)^{m_E}}{\big(1 - q(X)^{m_E}\big)^2}\,.$$

By observing

$$q(X) = \tanh^2\big(\varrho(z)/2\big),$$

we complete the proof of the lemma. □

6.3. Lemma. *The series*

$$\sum_{\substack{\gamma \in \Gamma \setminus \{\mathrm{id}\} \\ \gamma\,\mathrm{elliptic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \gamma z)$$

is absolutely and locally uniformly convergent for $z \in \mathbb{H}$ away from the elliptic fixed points modulo Γ.

Proof. Using the finite set $\mathcal{E}(\Gamma)$ of representatives of Γ-conjugacy classes of maximal elliptic subgroups of Γ (see Subsection 3.3), we have the disjoint union decompositions

$$\big\{\gamma \in \Gamma \setminus \{\mathrm{id}\} \,|\, \gamma\,\mathrm{elliptic}\big\} = \bigcup_{E \in \mathcal{E}(\Gamma)} \bigcup_{F\,\mathrm{conj.\,to}\,E} \big(F \setminus \{\mathrm{id}\}\big)$$

$$= \bigcup_{E \in \mathcal{E}(\Gamma)} \bigcup_{\eta \in E \setminus \Gamma} \big(\eta^{-1} E \eta \setminus \{\mathrm{id}\}\big) = \bigcup_{E = \langle \gamma_E \rangle \in \mathcal{E}(\Gamma)} \bigcup_{\eta \in E \setminus \Gamma} \bigcup_{n=1}^{m_E-1} \{\eta^{-1}\gamma_E^n\,\eta\}\,,$$

where $m_E = |E|$. This gives

$$\sum_{\substack{\gamma \in \Gamma \setminus \{\mathrm{id}\} \\ \gamma\,\mathrm{elliptic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \gamma z) = \sum_{E = \langle \gamma_E \rangle \in \mathcal{E}(\Gamma)} \sum_{\eta \in E \setminus \Gamma} \sum_{n=1}^{m_E-1} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \eta^{-1}\gamma_E^n\,\eta z).$$

For $E=\langle\gamma_E\rangle\in\mathcal{E}(\Gamma)$, we choose σ_E and $\gamma_{i,E}$ as in (17), which leads to

$$\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\text{elliptic}}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z)=\sum_{E=\langle\gamma_E\rangle\in\mathcal{E}(\Gamma)}\sum_{\eta\in E\setminus\Gamma}\sum_{n=1}^{m_E-1}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}\big(z,\eta^{-1}\sigma_E\gamma_{i,E}^{n}\sigma_E^{-1}\eta z\big)$$
$$=\sum_{E=\langle\gamma_E\rangle\in\mathcal{E}(\Gamma)}\sum_{\eta\in E\setminus\Gamma}\sum_{n=1}^{m_E-1}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}\big(\sigma_E^{-1}\eta z,\gamma_{i,E}^{n}\sigma_E^{-1}\eta z\big).$$

Setting

$$M_\Gamma:=\max_{E\in\mathcal{E}(\Gamma)}\big(m_E\big),$$

we estimate using Lemma 6.2 (i) with $\alpha=n\pi/m_E$

$$\begin{aligned}&\big|\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}\big(\sigma_E^{-1}\eta z,\gamma_{i,E}^{n}\sigma_E^{-1}\eta z\big)\big|\\
&=2\left|\frac{\cosh^2\big(\varrho(\sigma_E^{-1}\eta z)\big)\sin^2\big(n\pi/m_E\big)-\cos^2\big(n\pi/m_E\big)}{\big(\cosh^2\big(\varrho(\sigma_E^{-1}\eta z)\big)\sin^2\big(n\pi/m_E\big)+\cos^2\big(n\pi/m_E\big)\big)^2}\right|\\
&\leq\frac{2}{\cosh^2\big(\varrho(\sigma_E^{-1}\eta z)\big)\sin^2\big(n\pi/m_E\big)+\cos^2\big(n\pi/m_E\big)}\\
&\leq\frac{2}{\sin^2\big(n\pi/M_\Gamma\big)}\cdot\frac{1}{\sinh^2\big(\varrho(\sigma_E^{-1}\eta z)\big)}\,.\end{aligned}$$

From this we derive the estimate

$$\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\text{elliptic}}}\big|\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z)\big|\leq\sum_{E\in\mathcal{E}(\Gamma)}\sum_{\eta\in E\setminus\Gamma}\sum_{n=1}^{m_E-1}\frac{2}{\sin^2\big(n\pi/M_\Gamma\big)}\cdot\frac{1}{\sinh^2\big(\varrho(\sigma_E^{-1}\eta z)\big)}$$
$$\leq\frac{2M_\Gamma}{\sin^2\big(\pi/M_\Gamma\big)}\sum_{E\in\mathcal{E}(\Gamma)}\sum_{\eta\in E\setminus\Gamma}\frac{1}{\sinh^2\big(\varrho(\sigma_E^{-1}\eta z)\big)}=\frac{2M_\Gamma}{\sin^2\big(\pi/M_\Gamma\big)}\sum_{E\in\mathcal{E}(\Gamma)}\mathcal{E}_{\mathrm{ell},E}(z,2),$$

with the elliptic Eisenstein series $\mathcal{E}_{\mathrm{ell},E}(z,s)$ associated to E evaluated at $s=2$. The absolute and locally uniform convergence of the series in question away from the elliptic fixed points modulo Γ now follows. □

6.4. Proposition. *Let Γ be a subgroup of finite index in the Fuchsian subgroup Γ_0 of the first kind. Then, we have*

$$\sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\,\text{elliptic}}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z)\leq\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\text{elliptic}}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z)\;\leq\;0\,.$$

Proof. We denote by $\mathcal{M}_{\mathrm{ell}}(\Gamma)$ the set of all maximal elliptic subgroups of Γ. We observe that there is an injection $\varphi : \mathcal{M}_{\mathrm{ell}}(\Gamma) \hookrightarrow \mathcal{M}_{\mathrm{ell}}(\Gamma_0)$, which is given as follows: For $E \in \mathcal{M}_{\mathrm{ell}}(\Gamma)$, there exists a maximal elliptic subgroup $E_0 \subset \Gamma_0$ containing E, and we set $\varphi(E) := E_0$; note that this map need not be surjective since the intersection $E_0 \cap \Gamma$ might be trivial.

We note that the set of elliptic elements of Γ different from the identity can be written as the disjoint union of the sets $E \setminus \{\mathrm{id}\}$ with E running through $\mathcal{M}_{\mathrm{ell}}(\Gamma)$. Choosing for $E = \langle \gamma_E \rangle \in \mathcal{M}_{\mathrm{ell}}(\Gamma)$, the quantities σ_E and $\gamma_{i,E}$ as in (17), letting $w_E := \sigma_E^{-1} z$, and applying Lemma 6.2 (ii), we compute

$$\sum_{\substack{\gamma \in \Gamma \setminus \{\mathrm{id}\} \\ \gamma \text{ elliptic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \gamma z) = \sum_{E \in \mathcal{M}_{\mathrm{ell}}(\Gamma)} \sum_{n=1}^{m_E - 1} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}\left(z, \sigma_E \gamma_{i,E}^n \sigma_E^{-1} z\right)$$

$$= \sum_{E \in \mathcal{M}_{\mathrm{ell}}(\Gamma)} \sum_{n=1}^{m_E - 1} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}\left(w_E, \gamma_{i,E}^n w_E\right)$$

$$= 2 \sum_{E \in \mathcal{M}_{\mathrm{ell}}(\Gamma)} \left(\frac{4 m_E^2}{\sinh^2\left(\varrho(w_E)\right)} \cdot \frac{\tanh^{2m_E}\left(\varrho(w_E)/2\right)}{\left(1 - \tanh^{2m_E}\left(\varrho(w_E)/2\right)\right)^2} - 1 \right). \tag{22}$$

Setting $Y := \tanh^2(\varrho/2)$ and observing $0 \leq Y < 1$, we consider the function

$$h_m(Y) := \frac{4m^2}{\sinh^2(\varrho)} \cdot \frac{Y^m}{(1 - Y^m)^2} - 1$$

for $m \geq 2$. From the elementary estimate

$$\frac{1 - Y^m}{1 - Y^{m+1}} \leq \frac{m}{m+1} \cdot \frac{1}{\sqrt{Y}},$$

which is valid for $0 \leq Y < 1$, we derive

$$h_{m+1}(Y) \leq h_m(Y).$$

Since

$$h_2(Y) = \frac{16}{\sinh^2(\varrho)} \cdot \frac{Y^2}{(1 - Y^2)^2} - 1 = -\frac{1}{\cosh^2(\varrho)} \leq 0,$$

we conclude $h_m(Y) \leq 0$ for all $m \geq 2$ and $0 \leq Y < 1$. Using the negativity of the function $h_m(Y)$, we derive the claimed upper bound immediately from formula (22).

We are left to prove the claimed lower bound. Replacing Γ by Γ_0, we obtain as above

$$\sum_{\substack{\gamma \in \Gamma_0 \setminus \{\mathrm{id}\} \\ \gamma \text{ elliptic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \gamma z)$$

$$= 2 \sum_{E_0 \in \mathcal{M}_{\mathrm{ell}}(\Gamma_0)} \left(\frac{4 m_{E_0}^2}{\sinh^2\left(\varrho(w_{E_0})\right)} \cdot \frac{\tanh^{2m_{E_0}}\left(\varrho(w_{E_0})/2\right)}{\left(1 - \tanh^{2m_{E_0}}\left(\varrho(w_{E_0})/2\right)\right)^2} - 1 \right); \tag{23}$$

here $w_{E_0} := \sigma_{E_0}^{-1} z$ with the scaling matrix σ_{E_0} for a generator γ_{E_0} of $E_0 \in \mathcal{M}_{\mathrm{ell}}(\Gamma_0)$. Letting $e := [E_0 : E]$, we note that we may choose $\gamma_E = \gamma_{E_0}^e$, which permits us to take $\sigma_{E_0} = \sigma_E$. This and the relation $m_{E_0} = em_E$ together with the injection from $\mathcal{M}_{\mathrm{ell}}(\Gamma)$ into $\mathcal{M}_{\mathrm{ell}}(\Gamma_0)$ allows us to estimate (23) in the form

$$\sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\,\mathrm{elliptic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z)$$

$$\leq 2 \sum_{E\in\mathcal{M}_{\mathrm{ell}}(\Gamma)} \left(\frac{4(em_E)^2}{\sinh^2\big(\varrho(w_E)\big)} \cdot \frac{\tanh^{2em_E}\big(\varrho(w_E)/2\big)}{\big(1-\tanh^{2em_E}\big(\varrho(w_E)/2\big)\big)^2} - 1 \right).$$

The claimed lower bound now immediately follows from the fact that the function $h_m(Y)$ is monotone decreasing with respect to m. □

6.5. Proposition. *Let Γ be a subgroup of finite index in Γ_0. Then, we have*

$$\sup_{z\in\mathbb{H}} \left| \sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\mathrm{elliptic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) \right| = O_{\Gamma_0}(1).$$

Proof. By Proposition 6.4, it suffices to prove

$$\sup_{z\in\mathbb{H}} \left(\sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\,\mathrm{elliptic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) \right) = O_{\Gamma_0}(1).$$

By the Γ_0-invariance of the series under consideration, it suffices to bound it on $M_0 := \Gamma_0\backslash\mathbb{H}$. On the compact set given by M_0 minus the union of sufficiently small neigborhoods around the cusps and the elliptic fixed points of M_0, the series in question can be uniformly bounded. It remains to find uniform bounds in the neighborhoods of the (finitely many) cusps and elliptic fixed points of M_0. For this we proceed as follows.

Recall that $\mathcal{E}(\Gamma_0)$ denotes a complete set of representatives of the Γ_0-conjugacy classes of the maximal elliptic subgroups of Γ_0. Then, arguing as in the proof of Lemma 6.3, we have

$$\big\{\gamma \in \Gamma_0 \setminus \{\mathrm{id}\} \,|\, \gamma\,\mathrm{elliptic}\big\} = \bigcup_{E\in\mathcal{E}(\Gamma_0)} \bigcup_{F\,\mathrm{conj.\,to}\,E} \big(F\setminus\{\mathrm{id}\}\big)$$

$$= \bigcup_{E\in\mathcal{E}(\Gamma_0)} \bigcup_{\eta\in E\backslash\Gamma_0} \big(\eta^{-1}E\eta\setminus\{\mathrm{id}\}\big) = \bigcup_{E=\langle\gamma_E\rangle\in\mathcal{E}(\Gamma_0)} \bigcup_{\eta\in E\backslash\Gamma_0} \bigcup_{n=1}^{m_E-1} \big\{\eta^{-1}\gamma_E^n\,\eta\big\},$$

which leads to

$$\sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\,\mathrm{elliptic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) = \sum_{E=\langle\gamma_E\rangle\in\mathcal{E}(\Gamma_0)} \sum_{\eta\in E\backslash\Gamma_0} \sum_{n=1}^{m_E-1} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\eta^{-1}\sigma_E\gamma_{i,E}^n\sigma_E^{-1}\eta z)$$

$$= \sum_{E=\langle\gamma_E\rangle\in\mathcal{E}(\Gamma_0)} \sum_{\eta\in E\backslash\Gamma_0} \sum_{n=1}^{m_E-1} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(\sigma_E^{-1}\eta z,\gamma_{i,E}^n\sigma_E^{-1}\eta z).$$

Using Lemma 6.2 (ii), we get

$$\sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\,\text{elliptic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z)$$
$$= 2\sum_{E=\langle\gamma_E\rangle\in\mathcal{E}(\Gamma_0)}\ \sum_{\eta\in E\setminus\Gamma_0}\left(\frac{4m_E^2}{\sinh^2\big(\varrho(\sigma_E^{-1}\eta z)\big)}\cdot\frac{\tanh^{2m_E}\big(\varrho(\sigma_E^{-1}\eta z)/2\big)}{\big(1-\tanh^{2m_E}\big(\varrho(\sigma_E^{-1}\eta z)/2\big)\big)^2}-1\right).$$

We are now able to show that the series in question is uniformly bounded as z tends to an elliptic fixed point of M_0. By suitably conjugating Γ_0 inside of $\mathrm{PSL}_2(\mathbb{R})$, if necessary, we may assume that the elliptic fixed point in question is i and has order m_{E_0}. To simplify notation, we denote the resulting Fuchsian subgroup again by Γ_0. Let then E_0 be the stabilizer of i in Γ_0. We can view E_0 as an element of $\mathcal{E}(\Gamma_0)$, and we may choose $\sigma_{E_0}=\mathrm{id}$. We have

$$\sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\,\text{elliptic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z)$$
$$= 2\sum_{\substack{E\in\mathcal{E}(\Gamma_0)\\ E\neq E_0}}\ \sum_{\eta\in E\setminus\Gamma_0}\left(\frac{4m_E^2}{\sinh^2\big(\varrho(\sigma_E^{-1}\eta z)\big)}\cdot\frac{\tanh^{2m_E}\big(\varrho(\sigma_E^{-1}\eta z)/2\big)}{\big(1-\tanh^{2m_E}\big(\varrho(\sigma_E^{-1}\eta z)/2\big)\big)^2}-1\right)$$
$$+ 2\sum_{\substack{\eta\in E_0\setminus\Gamma_0\\ \eta\neq\mathrm{id}}}\left(\frac{4m_{E_0}^2}{\sinh^2\big(\varrho(\eta z)\big)}\cdot\frac{\tanh^{2m_{E_0}}\big(\varrho(\eta z)/2\big)}{\big(1-\tanh^{2m_{E_0}}\big(\varrho(\eta z)/2\big)\big)^2}-1\right)$$
$$+ 2\left(\frac{4m_{E_0}^2}{\sinh^2\big(\varrho(z)\big)}\cdot\frac{\tanh^{2m_{E_0}}\big(\varrho(z)/2\big)}{\big(1-\tanh^{2m_{E_0}}\big(\varrho(z)/2\big)\big)^2}-1\right).$$

We now observe that there is a positive constant $C(m)$ such that the estimate

$$\frac{4m^2}{\sinh^2(\varrho)}\cdot\frac{\tanh^{2m}(\varrho/2)}{(1-\tanh^{2m}(\varrho/2))^2}-1\geq-\frac{C(m)}{\sinh^2(\varrho)}$$

holds true; for this we note that the left-hand side can be written as a rational function of $\cosh(\varrho)$ of degree -2. Applying this bound, we get

$$0\geq\sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\,\text{elliptic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z)\geq-\sum_{\substack{E\in\mathcal{E}(\Gamma_0)\\ E\neq E_0}}2\,C(m_E)\sum_{\eta\in E\setminus\Gamma_0}\frac{1}{\sinh^2\big(\varrho(\sigma_E^{-1}\eta z)\big)}$$
$$-2\,C(m_{E_0})\sum_{\substack{\eta\in E_0\setminus\Gamma_0\\ \eta\neq\mathrm{id}}}\frac{1}{\sinh^2\big(\varrho(\eta z)\big)}+2\left(\frac{4m_{E_0}^2}{\sinh^2\big(\varrho(z)\big)}\cdot\frac{\tanh^{2m_{E_0}}\big(\varrho(z)/2\big)}{\big(1-\tanh^{2m_{E_0}}\big(\varrho(z)/2\big)\big)^2}-1\right).$$

With the elliptic Eisenstein series $\mathcal{E}_{\mathrm{ell},E}(z,s)$, we get

$$
\begin{aligned}
0 &\geq \sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\,\mathrm{elliptic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) \\
&\geq -\sum_{\substack{E\in\mathcal{E}(\Gamma_0)\\ E\neq E_0}} 2\,C(m_E)\,\mathcal{E}_{\mathrm{ell},E}(z,2) - 2\,C(m_{E_0})\left(\mathcal{E}_{\mathrm{ell},E_0}(z,2) - \frac{1}{\sinh^2\big(\varrho(z)\big)}\right) \\
&\quad +2\left(\frac{4m_{E_0}^2}{\sinh^2\big(\varrho(z)\big)}\cdot\frac{\tanh^{2m_{E_0}}\big(\varrho(z)/2\big)}{\big(1-\tanh^{2m_{E_0}}\big(\varrho(z)/2\big)\big)^2} - 1\right).
\end{aligned}
$$

If z tends to i, all terms involving the elliptic Eisenstein series are uniformly bounded. The last term is trivially bounded by -2 from below.

It remains to note that the right-hand side of the above inequality can be uniformly bounded when z tends to a cusp of M_0. This completes the proof of the proposition. □

7 Estimates in the hyperbolic case

7.1. Lemma. (i) *For a hyperbolic element* $\gamma = \left(\begin{smallmatrix} e^{\ell/2} & 0 \\ 0 & e^{-\ell/2}\end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{R})$*, we have using the polar coordinates* (1)*, the Fourier series expansion*

$$
\begin{aligned}
\Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) &= -4\sin^2\big(\theta(z)\big)\frac{\cosh(\ell)\cos(2\theta(z))-1}{\big(\cosh(\ell)-\cos(2\theta(z))\big)^2} \\
&= -8\sin^2\big(\theta(z)\big)\sum_{m=1}^{\infty} m e^{-m|\ell|}\cos\big(2m\theta(z)\big).
\end{aligned}
$$

(ii) *Let* $H = \langle\gamma_{0,H}\rangle$ *be a hyperbolic subgroup of* $\mathrm{PSL}_2(\mathbb{R})$ *generated by* $\gamma_{0,H} = \left(\begin{smallmatrix} e^{\ell_H/2} & 0 \\ 0 & e^{-\ell_H/2}\end{smallmatrix}\right)$. *Then, we have the Fourier series expansion*

$$
\sum_{\gamma\in H\setminus\{\mathrm{id}\}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) = -16\sin^2\big(\theta(z)\big)\sum_{m=1}^{\infty}\frac{m e^{-m\ell_H}}{1-e^{-m\ell_H}}\cos\big(2m\theta(z)\big).
$$

Proof. (i) The first claimed equality follows immediately from Lemma 4.3. In order to derive the claimed Fourier series expansion, we compute the power series expansion of the latter expression in $X := e^{-|\ell|}$. Observing that

$$
\frac{\cosh(\ell)\cos(2\theta(z))-1}{\big(\cosh(\ell)-\cos(2\theta(z))\big)^2} = 2X\frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{-\cos(2\theta(z))X+1}{X^2-2\cos(2\theta(z))X+1}\right)
$$

and

$$\frac{1}{X^2-2\cos\big(2\theta(z)\big)X+1}=\sum_{m=0}^{\infty}\frac{\sin\big(2(m+1)\theta(z)\big)}{\sin(2\theta(z))}X^m\,,$$

we obtain the Fourier series expansion

$$\frac{\cosh(\ell)\cos(2\theta(z))-1}{\big(\cosh(\ell)-\cos(2\theta(z))\big)^2}=2\sum_{m=1}^{\infty}me^{-m|\ell|}\cos\big(2m\theta(z)\big)\,,$$

from which the claim follows.

(ii) Using part (i), we find

$$\begin{aligned}\sum_{\gamma\in H\setminus\{\mathrm{id}\}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z)&=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma_H^n z)\\ &=-8\sin^2\big(\theta(z)\big)\sum_{n=1}^{\infty}\frac{\cosh(n\ell_H)\cos(2\theta(z))-1}{\big(\cosh(n\ell_H)-\cos(2\theta(z))\big)^2}\\ &=-16\sin^2\big(\theta(z)\big)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}me^{-mn\ell_H}\cos\big(2m\theta(z)\big)\\ &=-16\sin^2\big(\theta(z)\big)\sum_{m=1}^{\infty}\frac{me^{-m\ell_H}}{1-e^{-m\ell_H}}\cos\big(2m\theta(z)\big)\,,\end{aligned}$$

which proves the claimed formula. □

7.2. Proposition. *For $u\in\mathbb{R}_{>1}$, let*

$$S(\Gamma;u,z):=\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\ \mathrm{hyperbolic}\\ e^{\ell_\gamma}<u}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z)\,; \tag{24}$$

here ℓ_γ denotes the length of the closed geodesic determined by γ. Then, the series $S(\Gamma;u,z)$ is absolutely convergent for any $u\in\mathbb{R}_{>1}$, and the convergence is locally uniform in $z\in\mathbb{H}$. Furthermore, the limit

$$S(\Gamma;z):=\lim_{u\to\infty}S(\Gamma;u,z) \tag{25}$$

exists and it is locally uniform in $z\in\mathbb{H}$. In the sequel, we will simply write

$$S(\Gamma;z)=\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\ \mathrm{hyperbolic}}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z)\,.$$

Proof. Using the set $\mathcal{H}(\Gamma)$ of representatives of Γ-conjugacy classes of maximal hyperbolic subgroups of Γ (see Subsection 3.2), we have the disjoint union decompositions

$$\big\{\gamma \in \Gamma \setminus \{\mathrm{id}\} \,|\, \gamma \,\text{hyperbolic}\big\} = \bigcup_{H\in\mathcal{H}(\Gamma)} \bigcup_{J\,\text{conj. to}\,H} \big(J \setminus \{\mathrm{id}\}\big)$$

$$= \bigcup_{H\in\mathcal{H}(\Gamma)} \bigcup_{\eta\in H\backslash\Gamma} \big(\eta^{-1}H\eta \setminus \{\mathrm{id}\}\big) = \bigcup_{H=\langle\gamma_H\rangle\in\mathcal{H}(\Gamma)} \bigcup_{\eta\in H\backslash\Gamma} \bigcup_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \{\eta^{-1}\gamma_H^n\,\eta\}\,.$$

This gives

$$\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\text{hyperbolic}\\ e^{\ell_\gamma}<u}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) = \sum_{H=\langle\gamma_H\rangle\in\mathcal{H}(\Gamma)} \sum_{\eta\in H\backslash\Gamma} \sum_{\substack{n=-\infty\\ n\neq 0\\ e^{n\ell_H}<u}}^{\infty} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\eta^{-1}\gamma_H^n\,\eta z).$$

For $H=\langle\gamma_H\rangle\in\mathcal{H}(\Gamma)$, we choose σ_H and $\gamma_{0,H}$ as in (13), which leads to

$$\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\text{hyperbolic}\\ e^{\ell_\gamma}<u}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) = \sum_{H\in\mathcal{H}(\Gamma)} \sum_{\eta\in H\backslash\Gamma} \sum_{\substack{n=-\infty\\ n\neq 0\\ e^{n\ell_H}<u}}^{\infty} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}\big(z,\eta^{-1}\sigma_H\gamma_{0,H}^n\sigma_H^{-1}\eta z\big)$$

$$= \sum_{H\in\mathcal{H}(\Gamma)} \sum_{\eta\in H\backslash\Gamma} \sum_{\substack{n=-\infty\\ n\neq 0\\ e^{n\ell_H}<u}}^{\infty} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}\big(\sigma_H^{-1}\eta z,\gamma_{0,H}^n\sigma_H^{-1}\eta z\big).$$

Using Lemma 7.1 (i) with $\ell = n\ell_H$ ($n\in\mathbb{Z}$), we find (recalling (24))

$$S(\Gamma;u,z) = S_1(\Gamma;u,z) + S_2(\Gamma;u,z)\,,$$

where

$$S_1(\Gamma;u,z) := -\sum_{\substack{H\in\mathcal{H}(\Gamma)\\ e^{\ell_H}<u}} \sum_{\eta\in H\backslash\Gamma} 16\sin^2\big(\theta(\sigma_H^{-1}\eta z)\big)\cos\big(2\theta(\sigma_H^{-1}\eta z)\big)e^{-\ell_H}\,,$$

$$S_2(\Gamma;u,z) := -\sum_{H\in\mathcal{H}(\Gamma)} \sum_{\eta\in H\backslash\Gamma} \sum_{\substack{n,m=1\\ n\cdot m\neq 1\\ e^{n\ell_H}<u}}^{\infty} 16\sin^2\big(\theta(\sigma_H^{-1}\eta z)\big)\cos\big(2m\theta(\sigma_H^{-1}\eta z)\big)me^{-mn\ell_H}.$$

In a first step we will show the absolute convergence of the series $S_2(\Gamma;u,z)$ and the existence of the limit $\lim_{u\to\infty} S_2(\Gamma;u,z)$. For this we let $S_2^*(\Gamma;u,z)$ denote the series defined by summing the absolute values of the summands of the series

$S_2(\Gamma;u,z)$. The series $S_2^*(\Gamma;u,z)$ can trivially be bounded as

$$S_2^*(\Gamma;u,z) \leq 16 \sum_{H\in\mathcal{H}(\Gamma)} \sum_{\substack{n,m=1\\ n\cdot m\neq 1\\ e^{n\ell_H}<u}}^{\infty} \mathcal{E}_{\mathrm{hyp},H}(z,2)\, m e^{-mn\ell_H}$$

$$\leq 16 \sum_{\substack{H\in\mathcal{H}(\Gamma)\\ e^{\ell_H}<u}} \sum_{k=2}^{\infty} \mathcal{E}_{\mathrm{hyp},H}(z,2)\, k^2 e^{-k\ell_H} \tag{26}$$

with the hyperbolic Eisenstein series $\mathcal{E}_{\mathrm{hyp},H}(z,s)$ associated to H evaluated at $s=2$. In order to estimate $\mathcal{E}_{\mathrm{hyp},H}(z,2)$, we introduce the counting function

$$N_H(u;z) := \#\big\{\eta\in H\backslash\Gamma \,|\, \mathrm{dist}_{\mathrm{hyp}}(\eta z,\widetilde{L}_H)<u\big\}\,,$$

and hence we can write

$$\mathcal{E}_{\mathrm{hyp},H}(z,2) = \int_0^\infty \cosh(u)^{-2}\,\mathrm{d}N_H(u;z)\,.$$

The quantity $N_H(u;z)$ can be estimated as follows: By suitably conjugating Γ inside of $\mathrm{PSL}_2(\mathbb{R})$, if necessary, we may assume that

$$\gamma_H = \begin{pmatrix} e^{\ell_H/2} & 0 \\ 0 & e^{-\ell_H/2}\end{pmatrix},$$

so that a representative of the geodesic $L_H\subset M$ in $\mathbb{H}$ is given by the segment of the y-axis going from the point i to the point ie^{ℓ_H}; to simplify notation, we denote the resulting Fuchsian subgroup again by Γ. Eventually by modifying the representatives $\eta\in H\backslash\Gamma$ by elements of H, we can achieve that all the translates ηz of z lie in the semi-annulus $\mathcal{A}$ bounded by the semi-circles with centers at the origin of radii 1 and e^{ℓ_H}, respectively. We can now find an $r=r(z)>0$ such that the hyperbolic discs of radius r about the translates ηz do not intersect. This leads to the trivial bound

$$N_H(u;z) \leq \frac{2\ell_H\sinh(u)}{4\pi\sinh^2(r/2)}\,.$$

Introducing $c(z):=\pi\sinh^2(r/2)$, we arrive at the bound

$$\big|\mathcal{E}_{\mathrm{hyp},H}(z,2)\big| \leq \frac{\ell_H}{2c(z)}\int_0^\infty \cosh(u)^{-1}\mathrm{d}u \leq \frac{\ell_H}{c(z)}\,.$$

Letting ℓ_Γ denote the length of the shortest closed geodesic on M, we derive from this the estimate

$$S_2^*(\Gamma;u,z) \leq \frac{16}{c(z)} \sum_{\substack{H\in\mathcal{H}(\Gamma)\\ e^{\ell_H}<u}} \ell_H \sum_{k=2}^{\infty} k^2 e^{-k\ell_H} \leq \frac{80}{c(z)(1-e^{-\ell_\Gamma})^3} \sum_{H\in\mathcal{H}(\Gamma)} \ell_H e^{-2\ell_H}\,.$$

Using the trivial bound for the prime geodesic counting function $\pi(u) = O(u)$ (see Subsection 2.6), the latter sum can easily be bounded solely in terms of the quantities ℓ_Γ and $c(z)$, i.e., independent of u. This shows that the series $S_2^*(\Gamma; u, z)$ is convergent for any $u \in \mathbb{R}_{>1}$ and that the convergence is locally uniform in $z \in \mathbb{H}$. This implies that the series $S_2(\Gamma; u, z)$ is absolutely convergent for any $u \in \mathbb{R}_{>1}$, the convergence being locally uniform in $z \in \mathbb{H}$. Furthermore, since $S_2^*(\Gamma; u, z)$ viewed as a function of u, is monotone increasing and bounded (independently of u), the limit $\lim_{u\to\infty} S_2^*(\Gamma; u, z)$ exists and it is locally uniform in $z \in \mathbb{H}$. Assuming $u > v > 1$, we have

$$\big|S_2(\Gamma; u, z) - S_2(\Gamma; v, z)\big| \leq S_2^*(\Gamma; u, z) - S_2^*(\Gamma; v, z),$$

which can be made (locally uniform in $z \in \mathbb{H}$) arbitrarily small as u tends to infinity. This shows that also the limit $\lim_{u\to\infty} S_2(\Gamma; u, z)$ exists and that it is locally uniform in $z \in \mathbb{H}$.

In a second step we will show the absolute convergence of the series $S_1(\Gamma; u, z)$ and the existence of the limit $\lim_{u\to\infty} S_1(\Gamma; u, z)$. Using the differential equation (15), we can rewrite $S_1(\Gamma; u, z)$ as

$$S_1(\Gamma; u, z) = -\sum_{\substack{H\in\mathcal{H}(\Gamma)\\ e^{\ell_H}<u}} \sum_{\eta\in H\backslash\Gamma} 16\sin^2\big(\theta(\sigma_H^{-1}\eta z)\big)\big(1 - 2\sin^2\big(\theta(\sigma_H^{-1}\eta z)\big)\big)e^{-\ell_H}$$

$$= -16\sum_{\substack{H\in\mathcal{H}(\Gamma)\\ e^{\ell_H}<u}} \big(\mathcal{E}_{\mathrm{hyp},H}(z,2) - 2\,\mathcal{E}_{\mathrm{hyp},H}(z,4)\big)e^{-\ell_H} = 8\sum_{\substack{H\in\mathcal{H}(\Gamma)\\ e^{\ell_H}<u}} \Delta_{\mathrm{hyp}}\,\mathcal{E}_{\mathrm{hyp},H}(z,2)e^{-\ell_H},$$

which, using the spectral expansion (16), can be further rewritten as

$$\begin{aligned} S_1(\Gamma; u, z) = 8\Bigg(&\sum_{j=0}^{\infty} \lambda_j\, a_j\, \psi_j(z) \sum_{\substack{H\in\mathcal{H}(\Gamma)\\ e^{\ell_H}<u}} e^{-\ell_H} \int_{L_H} \psi_j(z)\,\mathrm{d}s_{\mathrm{hyp}}(z) \\ &+ \frac{1}{4\pi}\sum_{P\in\mathcal{P}(\Gamma)} \int_{-\infty}^{\infty} \left(\frac{1}{4} + r^2\right) a_r\, \mathcal{E}_{\mathrm{par},P}(z, 1/2 + ir)\,\mathrm{d}r \times \\ &\times \sum_{\substack{H\in\mathcal{H}(\Gamma)\\ e^{\ell_H}<u}} e^{-\ell_H} \int_{L_H} \mathcal{E}_{\mathrm{par},P}(z, 1/2 + ir)\,\mathrm{d}s_{\mathrm{hyp}}(z)\Bigg), \end{aligned} \tag{27}$$

where

$$a_j := \sqrt{\pi}\left|\Gamma\left(\frac{3}{4} + \frac{ir_j}{2}\right)\right|^2, \quad a_r := \sqrt{\pi}\left|\Gamma\left(\frac{3}{4} + \frac{ir}{2}\right)\right|^2.$$

We will now estimate the series $S_1(\Gamma; u, z)$ working from the representation (27). Applying the bound (10) (keeping in mind the discussion following (10)), gives the estimate

$$\sum_{\substack{H\in\mathcal{H}(\Gamma)\\ \ell_H<\log(u)}} \int_{L_H} \psi_j(z)\,\mathrm{d}s_{\mathrm{hyp}}(z) = O\big(\lambda_j^k\, u^{1-\varepsilon}\big),$$

which, for any $n \in \mathbb{N}$, leads to the bound

$$\sum_{\substack{H\in\mathcal{H}(\Gamma)\\ n\le\ell_H<n+1}} \int\limits_{L_H} \psi_j(z)\,\mathrm{d}s_{\mathrm{hyp}}(z) = O\big(\lambda_j^k\, e^{n(1-\varepsilon)}\big)\,.$$

From this we conclude

$$\begin{aligned}\Bigg|\sum_{\substack{H\in\mathcal{H}(\Gamma)\\ e^{\ell_H}<u}} e^{-\ell_H} \int\limits_{L_H} \psi_j(z)\,\mathrm{d}s_{\mathrm{hyp}}(z)\Bigg| &\ll \sum_{n=0}^{\lfloor\log(u)\rfloor} \Bigg|\sum_{\substack{H\in\mathcal{H}(\Gamma)\\ n\le\ell_H<n+1}} e^{-\ell_H} \int\limits_{L_H} \psi_j(z)\,\mathrm{d}s_{\mathrm{hyp}}(z)\Bigg| \\ &\ll \lambda_j^k \sum_{n=0}^{\lfloor\log(u)\rfloor} e^{-n\varepsilon} \ll \frac{\lambda_j^k}{1-e^{-\varepsilon}}\,. \qquad (28)\end{aligned}$$

Using the standard bound $\big|\psi_j(z)\big| \ll \lambda_j^{1/4}$ (see [11], p. 180) for the eigenfunctions $\psi_j(z)$, we derive

$$\sum_{j=0}^{\infty} \lambda_j\, a_j\, \big|\psi_j(z)\big| \cdot \Bigg|\sum_{\substack{H\in\mathcal{H}(\Gamma)\\ e^{\ell_H}<u}} e^{-\ell_H} \int\limits_{L_H} \psi_j(z)\,\mathrm{d}s_{\mathrm{hyp}}(z)\Bigg| \ll \frac{1}{1-e^{-\varepsilon}} \sum_{j=0}^{\infty} \lambda_j^{k+5/4}\, a_j\,; \quad (29)$$

a similar bound holds true for the corresponding contribution involving the parabolic Eisenstein series $\mathcal{E}_{\mathrm{par},P}(z,1/2+ir)$. Taking into account the exponential decay of the coefficients a_j in λ_j (resp. a_r in λ), we conclude that the series in (29) (resp. the corresponding series involving the parabolic Eisenstein series) is locally uniformly convergent for $z \in \mathbb{H}$.

Let now $S_1^*(\Gamma;u,z)$ denote the series defined by summing the absolute values of the summands of the series $S_1(\Gamma;u,z)$ given by (27), but with the factor

$$\Bigg|\sum_{\substack{H\in\mathcal{H}(\Gamma)\\ e^{\ell_H}<u}} e^{-\ell_H} \int\limits_{L_H} \psi_j(z)\,\mathrm{d}s_{\mathrm{hyp}}(z)\Bigg|$$

replaced by

$$\sum_{n=0}^{\lfloor\log(u)\rfloor} \Bigg|\sum_{\substack{H\in\mathcal{H}(\Gamma)\\ n\le\ell_H<n+1}} e^{-\ell_H} \int\limits_{L_H} \psi_j(z)\,\mathrm{d}s_{\mathrm{hyp}}(z)\Bigg|.$$

The above bounds show that the series $S_1^*(\Gamma;u,z)$ is convergent for any $u \in \mathbb{R}_{>1}$ and that the convergence is locally uniform in $z \in \mathbb{H}$. This implies that the series $S_1(\Gamma;u,z)$ is absolutely convergent for any $u \in \mathbb{R}_{>1}$, the convergence being locally uniform in $z \in \mathbb{H}$. Furthermore, since $S_1^*(\Gamma;u,z)$ viewed as a function of u, is monotone increasing and bounded (independently of u), the limit $\lim_{u\to\infty} S_1^*(\Gamma;u,z)$ exists and it is locally uniform in $z \in \mathbb{H}$. Arguing as before, we find that the limit $\lim_{u\to\infty} S_1(\Gamma;u,z)$ exists and that it is locally uniform in $z \in \mathbb{H}$.

Recalling that $S(\Gamma;u,z) = S_1(\Gamma;u,z) + S_2(\Gamma;u,z)$, completes the proof of the lemma. □

7.3. Proposition. *Let Γ be a subgroup of finite index in the Fuchsian subgroup Γ_0 of the first kind, and let $S(\Gamma;z)$ be given as in* (25)*. Then, there is a positive constant $C = C(\Gamma_0;z)$ depending on Γ_0 and locally uniformly on $z \in \mathbb{H}$ such that the inequality*

$$\big|S(\Gamma;z)\big| \leq \big|S(\Gamma_0;z)\big| + C(\Gamma_0;z)$$

holds.

Proof. We denote by $\mathcal{M}_{\mathrm{hyp}}(\Gamma)$ the set of all maximal hyperbolic subgroups of Γ. We observe that there is a bijection $\varphi : \mathcal{M}_{\mathrm{hyp}}(\Gamma) \stackrel{\simeq}{\longrightarrow} \mathcal{M}_{\mathrm{hyp}}(\Gamma_0)$, which is given as follows: For $H \in \mathcal{M}_{\mathrm{hyp}}(\Gamma)$, there exists a maximal hyperbolic subgroup $H_0 \subset \Gamma_0$ containing H, and we set $\varphi(H) := H_0$; the inverse map is given by $\varphi^{-1}(H_0) := H_0 \cap \Gamma$.

We note that the set of hyperbolic elements of Γ different from the identity can be written as the disjoint union of the sets $H \setminus \{\mathrm{id}\}$ with H running through $\mathcal{M}_{\mathrm{hyp}}(\Gamma)$. Choosing for $H = \langle \gamma_H \rangle \in \mathcal{M}_{\mathrm{hyp}}(\Gamma)$, the quantities σ_H and $\gamma_{0,H}$ as in (13), letting $w_H := \sigma_H^{-1} z$, and taking into account the results of Proposition 7.2, we compute

$$\begin{aligned} S(\Gamma;z) = \sum_{\substack{\gamma \in \Gamma \setminus \{\mathrm{id}\} \\ \gamma \text{ hyperbolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \gamma z) &= \sum_{H \in \mathcal{M}_{\mathrm{hyp}}(\Gamma)} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}\big(z, \sigma_H \gamma_{0,H}^n \sigma_H^{-1} z\big) \\ &= \sum_{H \in \mathcal{M}_{\mathrm{hyp}}(\Gamma)} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}\big(w_H, \gamma_{0,H}^n w_H\big). \end{aligned} \tag{30}$$

Replacing Γ by Γ_0, we obtain as above

$$S(\Gamma_0;z) = \sum_{H_0 \in \mathcal{M}_{\mathrm{hyp}}(\Gamma_0)} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}\big(w_{H_0}, \gamma_{0,H_0}^n w_{H_0}\big);$$

here $w_{H_0} := \sigma_{H_0}^{-1} z$ with the scaling matrix σ_{H_0} for a generator γ_{H_0} of $H_0 \in \mathcal{M}_{\mathrm{hyp}}(\Gamma_0)$. Letting $h := [H_0 : H]$, we note that we may choose $\gamma_H = \gamma_{H_0}^h$, which permits us to take $\sigma_{H_0} = \sigma_H$. This together with the bijection between $\mathcal{M}_{\mathrm{hyp}}(\Gamma_0)$ and $\mathcal{M}_{\mathrm{hyp}}(\Gamma)$ allows us to rewrite (30) in the form

$$S(\Gamma;z) = \sum_{H_0 \in \mathcal{M}_{\mathrm{hyp}}(\Gamma_0)} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}\big(w_{H_0}, \gamma_{0,H_0}^{h \cdot n} w_{H_0}\big),$$

which leads to the estimate

$$\begin{aligned}\big|S(\Gamma_0;z)-S(\Gamma;z)\big| &= \Bigg|\sum_{H_0\in\mathcal{M}_{\mathrm{hyp}}(\Gamma_0)}\sum_{j=0}^{\infty}\sum_{k=1}^{h-1} 2\,\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}\big(w_{H_0},\gamma_{0,H_0}^{h\cdot j+k}w_{H_0}\big)\Bigg| \\ &\leq \Bigg|\sum_{H_0\in\mathcal{M}_{\mathrm{hyp}}(\Gamma_0)} 2\,\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}\big(w_{H_0},\gamma_{0,H_0}w_{H_0}\big)\Bigg| \\ &\quad+\Bigg|\sum_{H_0\in\mathcal{M}_{\mathrm{hyp}}(\Gamma_0)}\sum_{j=0}^{\infty}\sum_{\substack{k=1\\ h\cdot j+k>1}}^{h-1} 2\,\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}\big(w_{H_0},\gamma_{0,H_0}^{h\cdot j+k}w_{H_0}\big)\Bigg|.\end{aligned}$$

Arguing as in the proof of Proposition 7.2, we find that the last two summands in the preceding estimate are well-defined and locally uniformly bounded in $z\in\mathbb{H}$. This shows

$$\big|S(\Gamma;z)\big|-\big|S(\Gamma_0;z)\big|\leq\big|S(\Gamma;z)-S(\Gamma_0;z)\big|\leq C(\Gamma_0;z)$$

with a positive constant $C(\Gamma_0;z)$ depending on Γ_0 and locally uniformly on $z\in\mathbb{H}$. This proves the claim. □

8 The main result

Based on Lemma 5.2, Lemma 6.3, and Proposition 7.2, we make the following

8.1. Definition. For $z\in M\setminus\mathcal{T}$, we set

$$\begin{aligned}\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z) := \sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\text{parabolic}}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z)+\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\text{elliptic}}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z) \\ +\lim_{u\to\infty}\Bigg(\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\text{hyperbolic}\\ e^{\ell_\gamma}<u}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z)\Bigg).\end{aligned}$$

This provides a rigorous definition for the series in question as a conditionally convergent series as anticipated in Remark 4.2.

8.2. Lemma. *With the above notations, we have the following relation for all* $z\in M\setminus\mathcal{T}$

$$\frac{1}{4\pi}\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z)=\int_0^\infty\Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;z,z)\,\mathrm{d}t\,.$$

Proof. In addition to the quantity $S(\Gamma;u,z)$ given in (24), we introduce

$$T(\Gamma;u,z):=\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\text{hyperbolic}\\ e^{\ell_\gamma}\geq u}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z)\,.$$

As a consequence of the results established in Sections 5, 6, and 7, we have for any $u \in \mathbb{R}_{>1}$

$$\frac{1}{4\pi}\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z) = \frac{1}{4\pi}\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\ \mathrm{parabolic}}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z) + \frac{1}{4\pi}\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\ \mathrm{elliptic}}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z) + \frac{1}{4\pi}S(\Gamma;u,z) + \frac{1}{4\pi}T(\Gamma;u,z)\,. \tag{31}$$

The first three series on the right-hand side of (31) are absolutely and locally uniformly convergent for $z \in \mathbb{H}$ away from the elliptic fixed points modulo Γ, while the proof of Proposition 7.2 shows that

$$T(\Gamma;u,z) = o_z(1) \qquad (u\to\infty)\,,$$

the dependence of the implied constant being locally uniform in $z \in \mathbb{H}$. Using formula (7) together with the heat kernel estimates (3) and (4) (applied to $K_{\mathbb{H}}(t;z,w)$ instead of $K_{\mathrm{hyp}}(t;z,w)$), allows us to rewrite the right-hand side of (31) by the absolute and locally uniform convergence of the series in question as

$$\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\ \mathrm{parabolic}}}\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{\mathbb{H}}(t;z,\gamma z)\,\mathrm{d}t + \sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\ \mathrm{elliptic}}}\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{\mathbb{H}}(t;z,\gamma z)\,\mathrm{d}t + \sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\ \mathrm{hyperbolic}\\ e^{\ell_\gamma}<u}}\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{\mathbb{H}}(t;z,\gamma z)\,\mathrm{d}t + o_z(1)$$

$$= \int_0^\infty \Delta_{\mathrm{hyp}}\Bigg(\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\ \mathrm{parabolic}}} + \sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\ \mathrm{elliptic}}} + \sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\ \mathrm{hyperbolic}\\ e^{\ell_\gamma}<u}}\Bigg) K_{\mathbb{H}}(t;z,\gamma z)\,\mathrm{d}t + o_z(1)\,.$$

Letting u tend to infinity, now shows

$$\frac{1}{4\pi}\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}}\Delta_{\mathrm{hyp}}\,g_{\mathbb{H}}(z,\gamma z) = \int_0^\infty \Delta_{\mathrm{hyp}}\sum_{\gamma\in\Gamma}K_{\mathbb{H}}(t;z,\gamma z)\,\mathrm{d}t\,,$$

as claimed. □

8.3. Proposition. *Let Γ be a subgroup of finite index in the Fuchsian subgroup Γ_0 of the first kind, and let $S(\Gamma;z)$ be given as in* (25)*. Then, we have*

$$\sup_{z\in\mathbb{H}}\big|S(\Gamma;z)\big| = O_{\Gamma_0}(1).$$

Proof. By Proposition 7.3, it suffices to show that

$$S(\Gamma_0;z) = O_{\Gamma_0}(1).$$

Using the notation of (the proof of) Proposition 7.2, we have the decomposition

$$S(\Gamma_0;u,z) = S_1(\Gamma_0;u,z) + S_2(\Gamma_0;u,z)\,.$$

In order to estimate the second summand, we start by recalling the bound (26), namely

$$\big|S_2(\Gamma_0;u,z)\big| \leq 16 \sum_{\substack{H\in\mathcal{H}(\Gamma_0)\\ e^{\ell_H}<u}} \sum_{k=2}^{\infty} \mathcal{E}_{\mathrm{hyp},H}(z,2)\, k^2 e^{-k\ell_H}\,.$$

For estimating $\mathcal{E}_{\mathrm{hyp},H}(z,2)$ for $H \in \mathcal{H}(\Gamma_0)$, we decompose $M_0 := \Gamma_0\backslash\mathbb{H}$ as

$$M_0 = M_{0,c} \cup \bigcup_{P\in\mathcal{P}(\Gamma_0)} M_{0,P}\,,$$

where $M_{0,c}$ is compact and $M_{0,P}$ $(P \in \mathcal{P}(\Gamma_0))$ are sufficiently small neighborhoods about the cusps of M_0. For $z \in M_{0,c}$, we can bound $\mathcal{E}_{\mathrm{hyp},H}(z,2)$, and hence $S_2(\Gamma_0;u,z)$, as in the proof of Proposition 7.2 with the quantity $c(z)$ being bounded uniformly on $M_{0,c}$. If $z \in M_{0,P}$ and P corresponds to the cusp $i\infty$, we use the bound given in [18], Lemma 2, which, in combination with arguments analogous to the ones given in the proof of Proposition 7.2, shows that

$$\big|\mathcal{E}_{\mathrm{hyp},H}(z,2)\big| = O_{\Gamma_0}\big(\ell_H\, y^{-2}\big)\,,$$

and similarly for the other cusps. This proves

$$\lim_{u\to\infty} S_2(\Gamma_0;u,z) = O_{\Gamma_0}(1). \tag{32}$$

The fundamental identity (19) together with Definition 8.1 and Lemma 8.2 (applied to Γ_0 instead of Γ) now gives the equality

$$\frac{1}{4\pi} + \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(M_0)} + \frac{1}{8\pi}\sum_{\substack{\gamma\in\Gamma_0\\ \gamma\neq\mathrm{id}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) = \frac{g_0\,\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{hyp}}(z)}$$

for $z \in M_0 \setminus \mathcal{T}_0$; here g_0, resp. $\mathcal{T}_0$ denotes the genus, resp. the set of elliptic fixed points of M_0. More explicitly, this means

$$\frac{1}{4\pi} + \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(M_0)} + \frac{1}{8\pi}\sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\ \mathrm{parabolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) + \frac{1}{8\pi}\sum_{\substack{\gamma\in\Gamma_0\setminus\{\mathrm{id}\}\\ \gamma\ \mathrm{elliptic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z)$$
$$+ \frac{1}{8\pi}\lim_{u\to\infty} S_1(\Gamma_0;u,z) + \frac{1}{8\pi}\lim_{u\to\infty} S_2(\Gamma_0;u,z) = \frac{g_0\,\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{hyp}}(z)}\,.$$

Taking into account the estimates established in Propositions 5.4 and 6.5, and equality (32) together with the boundedness of $\mu_{\mathrm{can}}(z)/\mu_{\mathrm{hyp}}(z)$ on M_0 shows

$$\lim_{u\to\infty} S_1(\Gamma_0; u, z) = O_{\Gamma_0}(1),$$

from which the claim follows. □

8.4. Theorem. *Let Γ be a subgroup of finite index in the Fuchsian subgroup Γ_0 of the first kind. Assume that $M = \Gamma\backslash\mathbb{H}$ has positive genus g, and let $\{f_1, \ldots, f_g\}$ be an orthonormal basis of $S_2(\Gamma)$. Then, we have the bound*

$$\sup_{z\in\mathbb{H}} \left(\sum_{j=1}^{g} y^2 \big| f_j(z) \big|^2 \right) = O_{\Gamma_0}(1).$$

Proof. We start by noting that the quantity $\sum_{j=1}^{g} y^2 \big| f_j(z) \big|^2$ vanishes for $z \in \mathcal{T}$. Now, observing that

$$\sum_{j=1}^{g} y^2 \big| f_j(z) \big|^2 = \frac{g\,\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{hyp}}(z)}\,,$$

the fundamental identity (19) gives

$$\sum_{j=1}^{g} y^2 \big| f_j(z) \big|^2 = \frac{1}{4\pi} + \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(M)} + \frac{1}{2} \int_0^\infty \Delta_{\mathrm{hyp}}\, K_{\mathrm{hyp}}(t; z, z)\, \mathrm{d}t$$

for $z \in M \setminus \mathcal{T}$. Lemma 8.2 then shows

$$\sum_{j=1}^{g} y^2 \big| f_j(z) \big|^2 = \frac{1}{4\pi} + \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(M)} + \frac{1}{8\pi} \sum_{\substack{\gamma\in\Gamma \\ \gamma\neq \mathrm{id}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \gamma z)$$

for $z \in M \setminus \mathcal{T}$. Employing the well-known lower bound

$$\mathrm{vol}_{\mathrm{hyp}}(M) \geq \frac{\pi}{21}$$

from [26], p. 44, the theorem follows by applying Propositions 5.4, 6.5, and 8.3 in combination with Definition 8.1. □

8.5. Corollary. *Let Γ be a subgroup of finite index in the Fuchsian subgroup Γ_0 of the first kind. Assume that $M_0 = \Gamma_0\backslash\mathbb{H}$ has genus $g_0 \geq 2$. Then, we have the bound*

$$d_M = \sup_{z\in M} \frac{\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{shyp}}(z)} = O_{\Gamma_0}(1).$$

Proof. We have

$$\frac{\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{shyp}}(z)} = \frac{\mathrm{vol}_{\mathrm{hyp}}(M)}{g} \cdot \frac{g\,\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{hyp}}(z)} = \frac{\mathrm{vol}_{\mathrm{hyp}}(M)}{g} \sum_{j=1}^{g} y^2 \big|f_j(z)\big|^2.$$

By Theorem 8.4, it suffices to show that

$$\frac{\mathrm{vol}_{\mathrm{hyp}}(M)}{g} = O_{\Gamma_0}(1).$$

Since $g_0 \geq 2$, this follows easily from the Hurwitz formula applied to the finite covering $\pi : M \to M_0$ (more precisely one has to look at the covering of the corresponding compactifications $\overline{\pi} : \overline{M} \to \overline{M}_0$), namely from the estimate

$$2g - 2 \geq \deg(\pi)\,(2g_0 - 2), \text{ i.e., } g \geq \deg(\pi)\,(g_0 - 1),$$

which gives

$$\frac{\mathrm{vol}_{\mathrm{hyp}}(M)}{g} \leq \frac{\deg(\pi)\,\mathrm{vol}_{\mathrm{hyp}}(M_0)}{\deg(\pi)\,(g_0 - 1)} \leq \frac{2\,\mathrm{vol}_{\mathrm{hyp}}(M_0)}{g_0}.$$

This proves the corollary. □

8.6. Remark. Let Γ be a subgroup of finite index in the Fuchsian subgroup Γ_0 of the first kind. Assume that $M_0 = \Gamma_0 \backslash \mathbb{H}$ has genus $g_0 \geq 2$. Then, collecting all the estimates obtained in Sections 5, 6, 7, and above, allows us to estimate the quantity d_M as follows in terms of special values at $s = 2$ of the parabolic, elliptic, and hyperbolic Eisenstein series of Γ_0:

$$d_M \leq \frac{\mathrm{vol}_{\mathrm{hyp}}(M_0)}{4\pi g_0} \Bigg(32 \sum_{P \in \mathcal{P}(\Gamma_0)} \mathcal{E}_{\mathrm{par},P}(z,2) + \frac{2M_{\Gamma_0}}{\sin^2\big(\pi/M_{\Gamma_0}\big)} \sum_{E \in \mathcal{E}(\Gamma_0)} \mathcal{E}_{\mathrm{ell},E}(z,2)$$
$$+ 8 \sum_{H \in \mathcal{H}(\Gamma_0)} \Bigg(\big|\Delta_{\mathrm{hyp}}\, \mathcal{E}_{\mathrm{hyp},H}(z,2)\big| e^{-\ell_H} + 2 \sum_{k=2}^{\infty} \mathcal{E}_{\mathrm{hyp},H}(z,2)\, k^2 e^{-k\ell_H} \Bigg) + C(\Gamma_0; z) + 170 \Bigg).$$

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