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Green's currents for families of hermitian vector bundles characterizing certain vanishing loci

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Abstract In this article we generalize our integral characterization of the Andreotti-Mayer locus given in an earlier paper to a more general situation. Given a family $\rho : X \longrightarrow Y$ of smooth, projective, n-dimensional, complex varieties and a hermitian vector bundle $\overline{V}$ of rank $n - p + 1$ on X together with a holomorphic section σ of the restriction of V to a p-codimensional cycle $D \subseteq X$, we are able to determine an explicit Green's current g for the vanishing locus Z of σ for certain values of p. The integral representation for the push-forward $\rho_* g$ then gives a characterization of $\rho_* Z$. By specializing the given data suitably, we recover, apart from known results, an integral representation of the discriminant characterizing the locus of singular hypersurfaces of degree d in n-dimensional, projective space.

Keywords Green's currents · Hermitian vector bundles · Abelian varieties · Hypersurfaces

Mathematics Subject Classification (2000) 14G40 · 11G10 · 14J70

1 Introduction

1.1

In this paper we unify and generalize the results obtained in [8–10] for star-products of Green's functions and Green's currents on families $\rho : X \longrightarrow Y$ of smooth, projective vari-

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eties over a quasi-projective base variety Y, and their push-forwards by ρ to Y. The main emphasis of these investigations was to describe the Green's currents and their push-forwards in question as explicit as possible in terms of the initial data.

In the paper [8], J. Jorgenson and the first author have shown that the integral of the star-product of two Green's functions (associated to a suitable line bundle) over the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z}\tau \oplus \mathbb{Z})$ ($\tau \in \mathbb{H}$, the upper half-plane) equals $-\log \|\Delta(\tau)\|^2$, where $\Delta(\tau)$ is Dedekind's delta function, the unique (up to scale) cusp form of weight 12 with respect to $\mathrm{SL}_2(\mathbb{Z})$. Since, by the very definition, $\Delta(\tau) = E_4(\tau)^3 - 27E_6(\tau)^2$ with the two (normalized) Eisenstein series $E_4(\tau)$, $E_6(\tau)$ of weight 4, 6, respectively, is the discriminant of the cubic equation defining the elliptic curve E_τ in $\mathbb{P}^2$, at the end of [9], the question was raised, if it is possible in general to provide such an integral representation for (minus the logarithm of the norm of) the discriminant of a family of hypersurfaces in $\mathbb{P}^n$. Generalizing the constructions developed in [10], we are able to answer this question positively at the end of this paper. The details are given in Sect. 6 and summarized in Theorem 6.1.

1.2

We let $\rho : X \longrightarrow Y$ denote a family of smooth, projective, n-dimensional, complex varieties over a quasi-projective base variety Y. We let $D \subseteq X$ be a p-codimensional cycle on X such that each fiber of D over $y \in Y$ has dimension $(n-p)$ and is generically smooth. Furthermore, we let $\overline{V} := (V, \|\cdot\|_V)$ be a hermitian vector bundle of rank $(n-p+1)$ on X, $\overline{W} := \overline{V}|_D$ its restriction to D, and σ a holomorphic section of W with vanishing locus Z, which meets the zero section of W properly. By dimension reasons, the push-forward $\rho_* Z$ is a divisor in Y. Under some additional assumptions on $\overline{V}$, we are able to determine an explicit Green's current g for Z on X and its push-forward by ρ to Y in the cases $p=0$, resp. $p=1$ (see Propositions 4.1, resp. 4.3). The integral representation obtained for $\rho_* g$ characterizes the divisor $\rho_* Z$.

The main application of our formalism is a solution of a problem posed in [9], namely to give an integral representation of (minus the logarithm of the norm of) the discriminant characterizing the locus of singular hypersurfaces of degree d in n-dimensional, projective space. This is the content of Theorem 6.1.

We note that in Sects. 3 and 4 differential forms and Green's currents on singular varieties have to be used. The theoretical background for this is provided by the work of T. Bloom and M. Herrera [2], Sect. 3. We would like to point out that our constructions of differential forms and Green's currents on the singular varieties under consideration will be explicit. We also emphasize that in all our investigations secondary Bott-Chern forms play an important role. In particular, we make essential use of some of the main results of D. Mourougane (see [11]).

1.3

The paper is organized as follows. In Sect. 2, we give the main notations together with a basic construction resolving the singularities of D, which is later on used in the text. In Sect. 3, we put together the main ingredients, which allow us to construct the Green's current g for Z in X. In Sect. 4, we specialize to the cases $p=0$, resp. $p=1$. By means of some additional assumptions, we are able to prove Propositions 4.1, resp. 4.3. In Sect. 5, we show how the main results obtained in [8–10] can be derived from the main results of Sect. 4.

In Sect. 6, we finally construct an integral representation characterizing the discriminant locus of singular hypersurfaces of degree d in n-dimensional, projective space.

2 Preliminaries

2.1

In this and in the next section, we recall and generalize some basic results described in [10]. Let $\rho : X \longrightarrow Y$ be as in the introduction; we denote the fiber $\rho^{-1}(y)$ over $y \in Y$ by X_y.

Let $D \subseteq X$ be a p-codimensional cycle on X such that each fiber $D_y = D \cap X_y$ ($y \in Y$) has dimension $(n-p)$ and is generically smooth. Let g_D be a Green's current of log-type for D, i.e., g_D is a smooth current on $X \setminus D$, g_D has logarithmic growth along D, and it satisfies

$$\mathrm{dd^c} g_D + \delta_D = \omega_D,$$

where ω_D is a smooth current on the whole of X; the existence of g_D is justified by [5], Theorem 1.3.5.

Let V be a vector bundle of rank $(n-p+1)$ on X. Denote the restriction of V to D by W. Equip V with a smooth hermitian metric $\|\cdot\|_V$ and write $\overline{V} = (V, \|\cdot\|_V)$. Denote the restriction of $\|\cdot\|_V$ to W by $\|\cdot\|_W$ and put $\overline{W} = (W, \|\cdot\|_W)$.

We assume that the vector bundle W has a holomorphic section σ, which meets the zero section of W properly; we denote the cycle attached to the l.c.i. subscheme defined by the vanishing of σ by Z. Furthermore, we assume that Z contains the singular locus of D.

2.2

We write $\mathcal{I}$ for the sheaf of ideals in $\mathcal{O}_D$ given by the functions vanishing on Z, and $\mathcal{W}$ for the sheaf of holomorphic sections of W. The section σ gives rise to an epimorphism

$$\sigma^* : \mathcal{W}^\vee \longrightarrow \mathcal{I};$$

here $\mathcal{W}^\vee$ is the dual of $\mathcal{W}$. This induces an epimorphism of graded algebras of $\mathcal{O}_D$-modules

$$\bigoplus_{k\geq 0} \mathrm{Sym}^k(\mathcal{W}^\vee) \longrightarrow \bigoplus_{k\geq 0} \mathcal{I}^k,$$

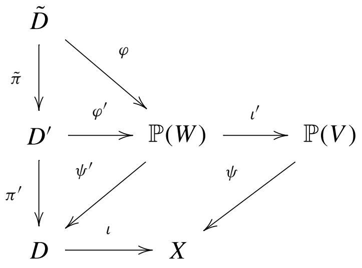
and consequently gives rise to an immersion

$$\varphi' : D' = \mathrm{Proj}\left(\bigoplus_{k\geq 0} \mathcal{I}^k\right) \longrightarrow \mathrm{Proj}\left(\bigoplus_{k\geq 0} \mathrm{Sym}^k(\mathcal{W}^\vee)\right) = \mathbb{P}(W)$$

satisfying

$$\varphi'^* \mathcal{O}_{\mathbb{P}(W)}(1) = \mathcal{O}_{D'}(1).$$

Letting $\tilde{\pi} : \tilde{D} \longrightarrow D'$ be a desingularization of D' à la Hironaka, we obtain the following commutative diagram



Here ψ, ψ' and ι, ι' are the obvious maps; φ is the morphism induced by φ' (cf. [4]).

2.3

On $\mathbb{P}(V)$ we have the tautological short exact sequence of vector bundles

$$\mathcal{E}' :\ 0 \longrightarrow S'^{\vee} \longrightarrow \psi^* V \longrightarrow Q' \longrightarrow 0,$$

where S', resp. Q' is the tautological line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$, resp. the tautological quotient vector bundle of rank $(n-p)$. By means of the short exact sequence $\mathcal{E}'$, the hermitian metric on $\overline{V}$ induces hermitian metrics $\|\cdot\|_{S'}$ on S', resp. $\|\cdot\|_{Q'}$ on Q'; we set $\overline{S}' := (S', \|\cdot\|_{S'})$, resp. $\overline{Q}' := (Q', \|\cdot\|_{Q'})$, and denote by $\overline{\mathcal{E}}'$ the short exact sequence $\mathcal{E}'$ equipped with the metrics described above. We note that $\|\cdot\|_{S'}$ is nothing but the Fubini-Study metric on S'. From the Whitney formula, we deduce

$$\mathrm{c}_{n-p+1}(\psi^* \overline{V}) - \mathrm{c}_1(\overline{S}'^{\vee}) \wedge \mathrm{c}_{n-p}(\overline{Q}') = -\mathrm{dd}^{\mathrm{c}} \eta', \tag{1}$$

where, by definition, the $(n-p, n-p)$-form η' is given by the secondary Bott-Chern form

$$\eta' = \widetilde{\mathrm{c}}_{n-p+1}(\overline{\mathcal{E}}')$$

associated to the $(n-p+1)$-st Chern form $\mathrm{c}_{n-p+1}(\cdot)$; it is unique up to $\mathrm{im}\,\partial + \mathrm{im}\,\overline{\partial}$ (see [6, 7]). With

$$\overline{S} := (\iota' \circ \varphi)^* \overline{S}', \qquad \overline{Q} := (\iota' \circ \varphi)^* \overline{Q}',$$

we obtain the short exact sequence of vector bundles on $\tilde{D}$

$$\mathcal{E} :\ 0 \longrightarrow S^{\vee} \longrightarrow \pi^* W \longrightarrow Q \longrightarrow 0;$$

here $\pi = \pi' \circ \tilde{\pi}$.

2.4

For our purposes it is more convenient to work with the twisted sequence

$$\mathcal{E}'\otimes S':\ 0\longrightarrow \mathcal{O}_{\mathbb{P}(V)}\longrightarrow \psi^*V\otimes S'\longrightarrow T_{\mathbb{P}(V)/X}\longrightarrow 0,$$

where $T_{\mathbb{P}(V)/X}$ is the relative tangent bundle of $\mathbb{P}(V)$ over X. In this case, we deduce from the Whitney formula

$$\mathrm{c}_{n-p+1}(\psi^*\overline{V}\otimes\overline{S}')=\sum_{j=0}^{n-p+1}\mathrm{c}_{n-p-j+1}(\psi^*\overline{V})\wedge\mathrm{c}_1(\overline{S}')^j=-\mathrm{dd^c}\zeta', \tag{2}$$

where the $(n-p,n-p)$-form ζ' is given by the secondary Bott-Chern form

$$\zeta'=\widetilde{\mathrm{c}}_{n-p+1}(\overline{\mathcal{E}}'\otimes\overline{S}')=\sum_{j=0}^{n-p}\tilde{\mathrm{c}}_{j+1}(\overline{\mathcal{E}}')\wedge\mathrm{c}_1(\overline{S}')^{n-p-j}.$$

3 Technical results

In the subsequent Lemma 3.1 we will explicitly construct a Green's current and its associated differential form on the possibly singular variety D. As pointed out in the introduction, the theoretical framework for this is provided by the work of T. Bloom and M. Herrera in [2], Sect. 3.

Lemma 3.1 *With the notations of Sect. 2, let g_Z denote an Euler-Green's current for the hermitian vector bundle $\overline{W}$ on the (possibly singular) variety D associated to the global section σ. Then, g_Z satisfies the* dd$^\mathrm{c}$*-equation*

$$\mathrm{dd^c}g_Z+\delta_Z=\mathrm{c}_{n-p+1}(\overline{V})|_D, \tag{3}$$

where $\mathrm{c}_{n-p+1}(\overline{V})|_D$ is the restriction of the $(n-p+1)$-st Chern form of $\overline{V}$ to D. On $D^0:=D\setminus Z$, we have

$$g_Z|_{D^0}=-\log\|\sigma\|_W^2\wedge\sum_{j=0}^{n-p}\zeta_1^j\wedge\mathrm{c}_{n-p-j}(\overline{V})|_{D^0}-\zeta_2, \tag{4}$$

where ζ_1 is a $(1,1)$-form and ζ_2 a $(n-p,n-p)$-form on D^0, which will be made explicit in course of the proof of Lemma 3.1.

Proof We denote by Z', resp. $\tilde{Z}$ the exceptional divisor in D', resp. its desingularization in $\tilde{D}$. The Green's current associated to the canonical section $\tilde{\sigma}=\pi^*\sigma$ of $S^\vee=\mathcal{O}_{\tilde{D}}(\tilde{Z})$ therefore satisfies the dd$^\mathrm{c}$-equation

$$\mathrm{dd^c}\left(-\log\|\tilde{\sigma}\|^2_{S^\vee}\right)+\delta_{\tilde{Z}}=\mathrm{c}_1(\overline{S}^\vee)=-\mathrm{c}_1(\overline{S}). \tag{5}$$

Multiplying (5) with the $(n-p, n-p)$-form α given by the formula

$$\begin{aligned}\alpha &:= (\iota' \circ \varphi)^* \left(\sum_{j=0}^{n-p} \mathrm{c}_1(\overline{S}')^j \wedge \mathrm{c}_{n-p-j}(\psi^* \overline{V}) \right) \\ &= \varphi^* \left(\sum_{j=0}^{n-p} \mathrm{c}_1(\iota'^* \overline{S}')^j \wedge \mathrm{c}_{n-p-j}(\psi'^* \overline{W}) \right) \\ &= \sum_{j=0}^{n-p} \mathrm{c}_1(\overline{S})^j \wedge \mathrm{c}_{n-p-j}(\pi^* \overline{W}),\end{aligned}$$

we derive from (2)

$$\mathrm{dd^c}\left(-\log \|\tilde{\sigma}\|^2_{S^\vee} \wedge \alpha\right) + \delta_{\tilde{Z}} \wedge \alpha = -\mathrm{c}_1(\overline{S}) \wedge \alpha = \mathrm{c}_{n-p+1}(\pi^* \overline{W}) + \mathrm{dd^c}\eta,$$

where, by the functoriality of secondary Bott-Chern forms, η is given by

$$\eta = \widetilde{\mathrm{c}}_{n-p+1}(\overline{\mathcal{E}} \otimes \overline{S}) = (\iota' \circ \varphi)^* \zeta'.$$

A standard cohomological argument shows

$$\pi_*\left(\delta_{\tilde{Z}} \wedge \alpha\right) = \delta_Z.$$

Putting

$$g_Z := \pi_*\left(-\log \|\pi^* \sigma\|^2_{S^\vee} \wedge \alpha - \eta\right),$$

we get after a short calculation

$$\mathrm{dd^c} g_Z + \delta_Z = \mathrm{c}_{n-p+1}(\overline{V})|_D.$$

Since $\pi : \tilde{D} \setminus \tilde{Z} \longrightarrow D^0$ is an isomorphism, the definition of g shows the explicit formula

$$g_Z|_{D^0} = -\log \|\sigma\|^2_W \wedge \sum_{j=0}^{n-p} \zeta_1^j \wedge \mathrm{c}_{n-p-j}(\overline{V})|_{D^0} - \zeta_2$$

with the $(1,1)$-form $\zeta_1 := \pi_*(\mathrm{c}_1(\overline{S})|_{\tilde{D}\setminus\tilde{Z}})$, and the $(n-p, n-p)$-form $\zeta_2 := \pi_*(\eta)$ on D^0. This proves the lemma. □

Theorem 3.2 *With the notations of Sect. 2 and g_Z from Lemma 3.1, define the following current of type $(n+1, n+1)$ on X*

$$g := g_D \wedge \mathrm{c}_{n-p+1}(\overline{V}) + g_Z \wedge \delta_D. \tag{6}$$

Then, g is a Green's current on X satisfying

$$\mathrm{dd^c} g + \delta_Z = \omega_D \wedge \mathrm{c}_{n-p+1}(\overline{V}); \tag{7}$$

here Z is considered as a subscheme of X.

Proof Using Lemma 3.1, a straightforward computation yields

$$\begin{aligned}
\mathrm{dd^c} g &= \mathrm{dd^c}(g_D)\wedge \mathrm{c}_{n-p+1}(\overline{V}) + \mathrm{dd^c}(g_Z)\wedge\delta_D\\
&= (\omega_D-\delta_D)\wedge \mathrm{c}_{n-p+1}(\overline{V}) + \big(\mathrm{c}_{n-p+1}(\overline{V})|_D-\delta_Z\big)\wedge\delta_D\\
&= \omega_D\wedge \mathrm{c}_{n-p+1}(\overline{V}) - \mathrm{c}_{n-p+1}(\overline{V})|_D + \mathrm{c}_{n-p+1}(\overline{V})|_D - \delta_Z\wedge\delta_D\\
&= \omega_D\wedge \mathrm{c}_{n-p+1}(\overline{V}) - \delta_Z,
\end{aligned}$$

as claimed. □

Let now $\rho : X \longrightarrow Y$ be as in Sect. 2.1. Then, we have the following

Corollary 3.3 *With the notations of Sect.* 2 *and* g *from Theorem* 3.2*, consider the current* $\rho_* g$ *of type* $(0,0)$ *on* Y*. Then,* $\rho_* g$ *is a Green's function on* Y *satisfying*

$$\mathrm{dd^c}(\rho_* g) + \delta_{\rho_* Z} = \mathrm{c}_1(\overline{M}), \tag{8}$$

where the hermitian line bundle $\overline{M} = (M, \|\cdot\|_M)$ *is given by* $M = \mathcal{O}_Y(\rho_* Z)$ *equipped with a suitable hermitian metric* $\|\cdot\|_M$.

Remark 3.4 The determination of the class of M in terms of the given data on X amounts to an application of the Hirzebruch-Riemann-Roch Theorem. Furthermore, the determination of the hermitian metric $\|\cdot\|_M$ in terms of the given metric $\|\cdot\|_V$ and the Kählerian structure under consideration amounts to an application of the arithmetic Riemann-Roch Theorem. In the next section we specialize the above formalism to distinguished cases which we will be able to handle quite explicitely.

4 Explicit formulas

In this section we are going to make explicit the results obtained in the previous section in the case when $D = X$, i.e., $p = 0$, and when D is a divisor in X, i.e., $p = 1$. In order to get results, which are as explicit as possible, we will have to impose some additional assumptions. Again, we point out that when working with differential forms and Green's currents on singular varieties in this section, this has to be understood in the framework of the article [2].

4.1 The case of codimension 0

If $D = X$, i.e., $p = 0$, the Green's current g in formula (6) of Theorem 3.2 satisfies $g = g_Z$ and, hence, formulas (3), resp. (4) of Lemma 3.1 give

$$\mathrm{dd^c} g + \delta_Z = \mathrm{c}_{n+1}(\overline{V}), \tag{9}$$

resp. (recall that $X^0 = X \setminus Z$)

$$g|_{X^0} = -\log\|\sigma\|_V^2 \wedge \sum_{j=0}^{n} \zeta_1^j \wedge \mathrm{c}_{n-j}(\overline{V})|_{X^0} - \zeta_2, \tag{10}$$

where ζ_1 and ζ_2 are described in the proof of Lemma 3.1. Applying ρ_* to (9), we find the $\mathrm{dd^c}$-equation

$$\mathrm{dd^c}(\rho_* g) + \delta_{\rho_* Z} = \rho_* \left(\mathrm{c}_{n+1}(\overline{V})\right). \tag{11}$$

In the subsequent proposition we will make explicit formula (11).

Proposition 4.1 *In addition to the hypotheses made in Sect.* 2, *we assume* $D = X$ *and* $\overline{V} = \rho^*\overline{E} \otimes \overline{L}$, *where* $\overline{E} = (E, \|\cdot\|_E)$ *is a hermitian vector bundle of rank* $(n+1)$ *on* Y, *and* $\overline{L} = (L, \|\cdot\|_L)$ *is a hermitian line bundle on* X. *We write* $\overline{L}_y$ *for the restriction of* $\overline{L}$ *to the fiber* X_y *over* $y \in Y$. *Then, for* $y \in Y \setminus \rho_* Z$, *the push-forward* $\rho_* g$ *of* g *given by formula* (6) *takes the form*

$$(\rho_* g)(y) = -\int_{X_y} \log \|\sigma\|_V^2 \wedge \sum_{j=0}^{n} \binom{n+1}{j+1} (\zeta_1|_{X_y})^j \wedge \mathrm{c}_1(\overline{L}_y)^{n-j} - \int_{X_y} (\zeta_2|_{X_y}). \tag{12}$$

Furthermore, the right-hand side of (11) *can be made explicit as*

$$\rho_* \left(\mathrm{c}_{n+1}(\overline{V})\right) = \rho_* \left(\mathrm{c}_1(\overline{L})^{n+1}\right) + \mathrm{c}_1 \left(\det(\overline{E})^{\otimes \deg(L)}\right). \tag{13}$$

The quantity $\rho_*(\mathrm{c}_1(\overline{L})^{n+1})$ will be studied in Sect. 4.3. The integral of the secondary Bott-Chern form ζ_2 will be discussed in Sect. 4.4.

Proof For $y \in Y \setminus \rho_* Z$, we derive from (10) after integrating along the fibers

$$(\rho_* g)(y) = -\int_{X_y} \log \|\sigma\|_V^2 \wedge \sum_{j=0}^{n} (\zeta_1|_{X_y})^j \wedge \mathrm{c}_{n-j}(\overline{V})|_{X_y} - \int_{X_y} (\zeta_2|_{X_y}).$$

Since the restriction of $\rho^* E$ to X_y is trivial, we get by functoriality

$$\mathrm{c}_{n-j}(\overline{V})|_{X_y} = \mathrm{c}_{n-j}\left(\overline{L}_y^{\oplus(n+1)}\right) = \binom{n+1}{n-j} \mathrm{c}_1(\overline{L}_y)^{n-j},$$

which proves formula (12). Concerning the proof of formula (13), we note

$$\begin{aligned} \rho_* \left(\mathrm{c}_{n+1}(\overline{V})\right) &= \rho_* \left(\mathrm{c}_{n+1}(\rho^*\overline{E} \otimes \overline{L})\right) = \rho_* \left(\sum_{j=0}^{n+1} \mathrm{c}_{n-j+1}(\rho^*\overline{E}) \wedge \mathrm{c}_1(\overline{L})^j\right) \\ &= \rho_* \left(\mathrm{c}_1(\overline{L})^{n+1}\right) + \mathrm{c}_1(\overline{E}) \wedge \rho_* \left(\mathrm{c}_1(\overline{L})^n\right) = \rho_* \left(\mathrm{c}_1(\overline{L})^{n+1}\right) + \mathrm{c}_1 \left(\det(\overline{E})^{\otimes \deg(L)}\right), \end{aligned}$$

where the third equality is justified by dimension reasons. □

Remark 4.2 If $\overline{V} = \overline{L}_1 \oplus \cdots \oplus \overline{L}_{n+1}$ (orthogonal direct sum), we can avoid the preceding, sophisticated construction. First, we note in this case that

$$\mathrm{c}_{n+1}(\overline{V}) = \mathrm{c}_1(\overline{L}_1) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_{n+1}).$$

Choosing the global section σ of V of the form $\sigma := (\sigma_1, \ldots, \sigma_{n+1})$, where σ_j is a global section of L_j $(j = 1, \ldots, n+1)$ and $\sigma_1, \ldots, \sigma_{n+1}$ are in general position (in the notation of [8], p. 353), we find for a Green's current g associated to $\overline{V}$ (restricted to X^0)

$$g|_{X^0} = -\sum_{j=1}^{n+1} \log \|\sigma_j\|^2_{L_j} \wedge \delta_{\mathrm{div}(\sigma_1)} \wedge \cdots \wedge \delta_{\mathrm{div}(\sigma_{j-1})} \wedge \mathrm{c}_1(\overline{L}_{j+1}) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_{n+1}) \tag{14}$$

as a replacement for formula (10). We note that the Green's current g in (14) is nothing but the star-product $g_1 * \cdots * g_{n+1}$ of the Green's functions $g_j := -\log \|\sigma_j\|^2_{L_j}$ $(j = 1, \ldots, n+1)$. If $y \in Y \setminus \rho_* Z$, we find for the push-forward $\rho_* g$ of g

$$\begin{aligned}(\rho_* g)(y) = -\sum_{j=1}^{n+1} \int_{\mathrm{div}(\sigma_1)\cdots\mathrm{div}(\sigma_{j-1})} &\log \|\sigma_j\|^2_{L_j} \wedge \delta_{\mathrm{div}(\sigma_1)} \wedge \cdots \wedge \delta_{\mathrm{div}(\sigma_{j-1})} \\ &\wedge \mathrm{c}_1(\overline{L}_{j+1}) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_{n+1})\end{aligned} \tag{15}$$

as a replacement for formula (12). If $L_j = L$ for $j = 1, \ldots, n+1$, formulas (14) and (15) simplify accordingly. We observe that in both cases the secondary Bott-Chern forms do not occur.

4.2 The case of codimension 1

If D is a divisor in X, i.e., $p = 1$, we let L_D be the line bundle given by $L_D = \mathcal{O}_X(D)$ with section s satisfying $D = \mathrm{div}(s)$, equipped with a suitable hermitian metric $\|\cdot\|_D$; we put $\overline{L}_D = (L_D, \|\cdot\|_D)$. The Green's current g in formula (6) of Theorem 3.2 is then given by

$$g = -\log \|s\|^2_D \wedge \mathrm{c}_n(\overline{V}) + g_Z \wedge \delta_D\,; \tag{16}$$

it satisfies the $\mathrm{dd^c}$-equation (see formula (7))

$$\mathrm{dd^c} g + \delta_Z = \mathrm{c}_1(\overline{L}_D) \wedge \mathrm{c}_n(\overline{V}). \tag{17}$$

By means of formula (4) of Lemma 3.1, g_Z is given by the explicit formula (recall that $D^0 = D \setminus Z$)

$$g_Z|_{D^0} = -\log \|\sigma\|^2_W \wedge \sum_{j=0}^{n-1} \zeta_1^j \wedge \mathrm{c}_{n-j-1}(\overline{V})|_{D^0} - \zeta_2, \tag{18}$$

where ζ_1 and ζ_2 are described in the proof of Lemma 3.1. Applying ρ_* to (17), we find the $\mathrm{dd^c}$-equation

$$\mathrm{dd^c}(\rho_* g) + \delta_{\rho_* Z} = \rho_*\left(\mathrm{c}_1(\overline{L}_D) \wedge \mathrm{c}_n(\overline{V})\right). \tag{19}$$

In the subsequent proposition we will make explicit formula (19).

Proposition 4.3 *In addition to the hypotheses made in Sect. 2, we assume that D is a divisor in X, $L_D = L^{\otimes m}$ for some $m \in \mathbb{N}$, and $\overline{V} = \rho^* \overline{E} \otimes \overline{L}^{\otimes \ell}$, where $\overline{E} = (E, \|\cdot\|_E)$ is a hermitian* *vector bundle of rank n on Y*, $\overline{L} = (L, \|\cdot\|_L)$, *and* $\ell \in \mathbb{N}$. *Then, for* $y \in Y \setminus \rho_* Z$, *the push-forward* $\rho_* g$ *of g in formula* (6) *takes the form*

$$(\rho_* g)(y) = \int_{D_y} \log \|\sigma\|_W^2 \wedge \sum_{j=0}^{n-1} \ell^{n-j-1} \binom{n}{j+1} (\zeta_1|_{D_y})^j \wedge \mathrm{c}_1(\overline{L}|_{D_y})^{n-j-1}$$

$$- \ell^n \int_{X_y} \log \|s\|_D^2 \wedge \mathrm{c}_1(\overline{L}_y)^n - \int_{D_y} (\zeta_2|_{D_y}), \tag{20}$$

and

$$\rho_* \left(\mathrm{c}_1(\overline{L}_D) \wedge \mathrm{c}_n(\overline{V})\right) = m\ell^n \cdot \rho_* \left(\mathrm{c}_1(\overline{L})^{n+1}\right) + m\ell^{n-1} \cdot \mathrm{c}_1 \left(\det(\overline{E})^{\otimes \deg(L)}\right). \tag{21}$$

The quantity $\rho_*(\mathrm{c}_1(\overline{L})^{n+1})$ will be studied in Sect. 4.3. The integral of the secondary Bott-Chern form ζ_2 will be discussed in Sect. 4.4.

Proof Formula (20) is proven along the same lines as formula (12). For the second formula, we compute

$$\rho_* \left(\mathrm{c}_1(\overline{L}_D) \wedge \mathrm{c}_n(\overline{V})\right) = \rho_* \left(\mathrm{c}_1(\overline{L}^{\otimes m}) \wedge \mathrm{c}_n(\rho^* \overline{E} \otimes \overline{L}^{\otimes \ell})\right)$$

$$= m\ell^n \cdot \rho_* \left(\mathrm{c}_1(\overline{L})^{n+1}\right) + m\ell^{n-1} \cdot \mathrm{c}_1 \left(\det(\overline{E})^{\otimes \deg(L)}\right),$$

because by dimension reasons only $\mathrm{c}_0(\overline{E})$ and $\mathrm{c}_1(\overline{E})$ have to be taken into account. □

Remark 4.4 If $\overline{V} = \overline{L}_1 \oplus \cdots \oplus \overline{L}_n$ (orthogonal direct sum), we are reduced to the corresponding discussion in Remark 4.2 by considering the hermitian vector bundle $\overline{V}' = \overline{L}_D \oplus \overline{V}$ on X.

4.3 Computation of $\rho_*(\mathrm{c}_1(\overline{L})^{n+1})$

We will now compute $\rho_*(\mathrm{c}_1(\overline{L})^{n+1})$ in two ways based on two different assumptions.

For the first computation, we assume that the relative tangent bundle $T_{X/Y}$ is of the form

$$T_{X/Y} = \rho^* F,$$

where F is a vector bundle on Y. Using the Hirzebruch-Riemann-Roch Theorem in degree one, we then compute on the level of cohomology classes

$$\mathrm{c}_1(\rho_* L) = \left(\rho_*(\mathrm{ch}(L) \cdot \mathrm{Td}(T_{X/Y}))\right)^{(1)}$$

$$= \frac{\rho_*(\mathrm{c}_1(L)^{n+1})}{(n+1)!} + \frac{\rho_*(\mathrm{c}_1(L)^n)}{n!} \cdot \frac{\mathrm{c}_1(F)}{2}.$$

This shows

$$\rho_* \left(\mathrm{c}_1(L)^{n+1}\right) = (n+1)! \cdot \mathrm{c}_1(\det(\rho_* L)) - \frac{n+1}{2} \cdot \deg(L) \cdot \mathrm{c}_1(F),$$

from which we derive

$$\rho_* \left(\mathrm{c}_1(L)^{n+1}\right) = \mathrm{c}_1 \left(\det(\rho_* L)^{\otimes (n+1)!} \otimes F^{\vee \otimes (n+1) \cdot \deg(L)/2}\right).$$

Recalling (11) and (13), we therefore find that the line bundle $M = \mathcal{O}_Y(\rho_* Z)$ of Corollary 3.3 is isomorphic to the line bundle

$$\det(\rho_* L)^{\otimes(n+1)!} \otimes F^{\vee\otimes(n+1)\cdot\deg(L)/2} \otimes \det(E)^{\otimes\deg(L)} \otimes M_{\mathrm{flat}},$$

where M_{flat} is a flat line bundle on Y. In order to compute $\rho_*(\mathrm{c}_1(\overline{L})^{n+1})$, i.e., to also take into account the hermitian metric under consideration, one has to construct the corresponding Quillen metric or to make use of the arithmetic Riemann-Roch Theorem. In this way one is able to determine the hermitian metric $\|\cdot\|_M$ of M.

For the second computation, we assume that there exists a hermitian vector bundle $\overline{F} = (F, \|\cdot\|)$ of rank $(n+1)$ on Y such that the vector bundle $A := \rho^* F \otimes L^{\otimes\ell}$ has a nowhere vanishing section, and B is the quotient of A by $\mathcal{O}_X$. This gives rise to a short exact sequence

$$\mathcal{F}:\ 0 \longrightarrow \mathcal{O}_X \longrightarrow A \longrightarrow B \longrightarrow 0,$$

from which we derive

$$\mathrm{c}_{n+1}(\rho^*\overline{F} \otimes \overline{L}^{\otimes\ell}) = -\mathrm{dd}^{\mathrm{c}}\left(\widetilde{\mathrm{c}}_{n+1}(\overline{\mathcal{F}})\right).$$

Applying ρ_* to the above equation, we get

$$\ell^{n+1}\cdot\rho_*\left(\mathrm{c}_1(\overline{L})^{n+1}\right) + \ell^n\cdot\rho_*\left(\mathrm{c}_1(\overline{L})^n\right)\wedge\mathrm{c}_1(\overline{F}) = -\mathrm{dd}^{\mathrm{c}}\left(\rho_*(\widetilde{\mathrm{c}}_{n+1}(\overline{\mathcal{F}}))\right).$$

Thus, we have

$$\rho_*\left(\mathrm{c}_1(\overline{L})^{n+1}\right) = -\frac{\deg(L)}{\ell}\cdot\mathrm{c}_1(\overline{F}) - \frac{1}{\ell^{n+1}}\cdot\mathrm{dd}^{\mathrm{c}}\left(\rho_*(\widetilde{\mathrm{c}}_{n+1}(\overline{\mathcal{F}}))\right).$$

4.4 Secondary Bott-Chern class computation

The section σ induces embeddings

$$\sigma_y : D_y \longrightarrow \mathbb{P}\left(\rho^* E|_{D_y} \otimes L^{\otimes\ell}|_{D_y}\right)$$

for $y \in Y \setminus \rho_* Z$. Since the vector bundle $\rho^* E|_{D_y}$ is trivial, we have an isomorphism

$$\beta_y : \mathbb{P}\left(\rho^* E|_{D_y} \otimes L^{\otimes\ell}|_{D_y}\right) \cong D_y \times \mathbb{P}^{n-p}.$$

Defining $\alpha_y = \mathrm{pr}_2 \circ \beta_y \circ \sigma_y$ (where pr_2 denotes the projection from $D_y \times \mathbb{P}^{n-p}$ to $\mathbb{P}^{n-p}$), and proceeding as in [10], we get in case that the maps α_y are generically finite

$$\int_{D_y} (\zeta_2|_{D_y}) = -\deg(\alpha_y)\cdot\left(\sum_{j=1}^{n-p}\sum_{k=1}^{j}\frac{1}{k}\right).$$

In the other cases, the integral $\int_{D_y}(\zeta_2|_{D_y})$ vanishes.

5 Abelian varieties

5.1 Notations

In this section we compute Green's currents and their push-forwards for hermitian vector bundles related to moduli spaces of Abelian varieties using the techniques developed in the previous section and relate them to the results obtained in [8] and [10].

We denote by $\mathcal{A}_{n,\Delta}(\Delta)_0$ the moduli space of n-dimensional, polarized Abelian varieties of type $\Delta := (d_1, \ldots, d_n)$ with level Δ-theta structure; here $d_1|d_2|\cdots|d_n$; we set $d := d_1 d_2 \cdots d_n$. Let $A_{n,\Delta}(\Delta)_0$ be the universal Abelian variety, $\overline{\varpi}$ the Hodge bundle on $\mathcal{A}_{n,\Delta}(\Delta)_0$ equipped with the Petersson metric, and $\overline{\mathcal{L}}_\Delta$ the relatively ample line bundle on $A_{n,\Delta}(\Delta)_0$ associated to Δ equipped with the standard translation invariant metric.

5.2

Let

$$Y := \mathcal{A}_{n,\Delta}(\Delta)_0, \qquad X := A_{n,\Delta}(\Delta)_0,$$
$$\overline{E} := \overline{\varpi}^{\otimes m_1} \otimes \mathbb{C}^{n+1}, \qquad \overline{L} := \overline{\mathcal{L}}_\Delta^{\otimes m_2},$$

i.e., $\overline{V} = \rho^* \overline{\varpi}^{\otimes m_1} \otimes \overline{\mathcal{L}}_\Delta^{\otimes m_2} \otimes \mathbb{C}^{n+1}$. By applying Proposition 4.1, or rather formula (15) of Remark 4.2, with the section $\sigma = (\sigma_1, \ldots, \sigma_{n+1})$ of V, we are able to derive the main formula in Theorem 5.2 of [8] for the integral of the star-product $g := g_1 * \cdots * g_{n+1}$ with $g_j := -\log \|\sigma_j\|_V^2$ $(j = 1, \ldots, n+1)$ along X_y $(y \in Y \setminus \rho_* Z,\ Z = \mathrm{div}(\sigma_1) \cdots \mathrm{div}(\sigma_{n+1}))$. This integral turns out to be minus the logarithm of the Petersson norm of a Siegel modular form. The weight of this modular form can be determined by the same type of computation as carried out in the first part of Sect. 4.3, namely, we have

$$\rho_* \left(\mathrm{c}_{n+1}(\overline{V}) \right) = m_2^n \cdot \left(m_1 + \frac{m_2}{2} \right) \cdot d \cdot (n+1)! \cdot \mathrm{c}_1(\overline{\varpi}). \tag{22}$$

5.3

With $\Delta := (1, \ldots, 1)$ and $\vartheta = \vartheta(\tau, z)$ the classical Riemann theta function, let

$$Y := \mathcal{A}_{n,\Delta}(\Delta)_0, \qquad X := A_{n,\Delta}(\Delta)_0, \qquad D := \Theta = \mathrm{div}(\vartheta),$$
$$\overline{E} := \rho_* \overline{\Omega}^1_{X/Y}, \qquad \overline{L} := \overline{\mathcal{L}}_\Delta,$$

i.e., $\overline{V} = \overline{\Omega}^1_{X/Y} \otimes \overline{\mathcal{L}}_\Delta$ and $\overline{W} = \overline{V}|_D$. By applying formula (20) of Proposition 4.3 with the section $\sigma := \mathrm{d}\vartheta$ of W (and $\ell = 1$), we derive the main formula of Corollary 4.5 of [10] for the integral of g along X_y $(y \in Y \setminus \rho_* Z,\ Z = \Theta_{\mathrm{sing}})$. This integral turns out to be minus the logarithm of the Petersson norm of a Siegel modular form characterizing the Andreotti-Mayer locus. The weight of this modular form appears in the relation

$$\rho_* \left(\mathrm{c}_n(\overline{W}) \right) = \frac{n+3}{2} \cdot n! \cdot \mathrm{c}_1(\overline{\varpi}),$$

which is derived in [10], Corollary 3.2.

5.4

With $n := 1$, $\Delta := 3$, let

$$Y := \mathcal{A}_{1,3}(3)_0, \qquad X := \mathbb{P}(E), \qquad D := A_{1,3}(3)_0,$$

where E is the rank 3 vector bundle over Y with fiber $E_y = H^0(X_y, \mathcal{L}_{3,y})$ ($y \in Y$). Furthermore, let

$$\overline{V} := \overline{\mathcal{O}}_{\mathbb{P}(E)}(1) \oplus \overline{\mathcal{O}}_{\mathbb{P}(E)}(1), \overline{W} := \overline{V}|_D,$$

where the hermitian metric on $\overline{\mathcal{O}}_{\mathbb{P}(E)}(1)$ is given by the Fubini-Study metric. Choosing two sections σ_1, σ_2 of $\mathcal{L}_3$, whose divisors intersect properly on D, we find as in Sect. 5.2, using Remark 4.2, the Green's current (restricted to D^0)

$$g|_{D^0} = -\log \|\sigma_1\|^2_{\mathcal{L}_3} \wedge \mathrm{c}_1(\overline{\mathcal{L}}_3) - \log \|\sigma_2\|^2_{\mathcal{L}_3} \wedge \delta_{\mathrm{div}(\sigma_1)},$$

whose push-forward to Y equals minus the logarithm of the Petersson norm of a Siegel modular form. Denoting by abuse of notation the projection from D to Y also by ρ, we derive from formula (22) (with $m_1 = 0$, $m_2 = 1$)

$$\rho_* \big(\mathrm{c}_2(\overline{W})\big) = \rho_* \big(\mathrm{c}_2(\overline{\mathcal{L}}_3 \oplus \overline{\mathcal{L}}_3)\big) = 3 \cdot \mathrm{c}_1(\overline{\omega}), \tag{23}$$

observing that $\overline{W} = \overline{\mathcal{L}}_3 \oplus \overline{\mathcal{L}}_3$.

On the other hand, using the formalism summarized in Sect. 4.2, we find the Green's current (see formulas (16) and (18))

$$\begin{aligned} g|_{X^0} &= -\log \|s|^2_D \wedge \mathrm{c}_2(\overline{V})|_{X^0} \\ &= -\big(\log \|\sigma\|^2_W \wedge \mathrm{c}_1(\overline{V})|_{D^0} + \log \|\sigma\|^2_W \wedge \zeta_1|_{D^0} + \zeta_2\big) \wedge \delta_{D^0}, \end{aligned}$$

where the elliptic surface $D = A_{1,3}(3)_0$ is determined by the cubic equation $s = 0$ in $X = \mathbb{P}(E)$; here ζ_1 and ζ_2 are as described in the proof of Lemma 3.1. From formula (21), we compute

$$\begin{aligned} \rho_* \big(\mathrm{c}_2(\overline{W})\big) &= \rho_* \big(\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(3)) \wedge \mathrm{c}_2(\overline{V})\big) \\ &= 3 \cdot \rho_* \big(\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(1))^3\big) = -3 \cdot \mathrm{c}_1(\overline{E}), \end{aligned}$$

where the last equality is justified by the very definition of $\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(1))$ and by dimension reasons. Comparing this with formula (23), yields

$$\mathrm{c}_1(\det(\overline{E})^\vee) = \mathrm{c}_1(\overline{\omega}).$$

6 Hypersurfaces

6.1 Notations

In this section, we solve one of the problems raised in [9]. In fact, we will study an example which is related to the discriminant of hypersurfaces. In particular, we will consider the moduli spaces of hypersurfaces having some suitable properties. We refer to [12] for details.

We write $\mathcal{H}(d,n)$ for the set of all non-zero forms of degree d in $\mathbb{P}^n$. We let $\mathcal{H}^0(d,n)$ be the subset of $\mathcal{H}(d,n)$ consisting of the forms defining smooth hypersurfaces. Thus, $\mathcal{H}^0(d,n)$ is the complement of the vanishing locus of the corresponding discriminant $\Delta(d,n)$. We denote by $\mathcal{H}_s(d,n)$ the set of stable d-forms. We want to consider the GIT-quotients of $\mathcal{H}^0(d,n)$, resp. $\mathcal{H}_s(d,n)$ by $\mathrm{GL}_{n+1}(\mathbb{C})$, or better, of suitable Galois coverings $\widetilde{\mathcal{H}}^0(d,n)$, resp. $\widetilde{\mathcal{H}}_s(d,n)$ of these spaces, on which $\mathrm{GL}_{n+1}(\mathbb{C})$ acts freely. We denote by Y the GIT-quotient $\mathrm{GL}_{n+1}(\mathbb{C})\backslash\backslash\widetilde{\mathcal{H}}_s(d,n)$.

Over $\mathcal{H}_s(d,n)$ there is a universal hypersurface $\mathcal{Y}_s(d,n)\subseteq\mathcal{H}_s(d,n)\times\mathbb{P}^n$ given by

$$\mathcal{Y}_s(d,n)=\{(F,X_0,X_1,\ldots,X_n)\in\mathcal{H}_s(d,n)\times\mathbb{P}^n\mid F(X_0,X_1,\ldots,X_n)=0\}.$$

We also have a universal hypersurface $\widetilde{\mathcal{Y}}_s(d,n)\subseteq\widetilde{\mathcal{H}}_s(d,n)\times\mathbb{P}^n$. We note that $\widetilde{\mathcal{Y}}_s(d,n)$ descends to a universal hypersurface D over Y with some extra structure. Moreover, the trivial projective bundle $\widetilde{\mathcal{H}}_s(d,n)\times\mathbb{P}^n$ on $\widetilde{\mathcal{H}}_s(d,n)$ descends to a projective bundle $X:=\mathbb{P}(E)$ over Y; by construction, $D\subseteq\mathbb{P}(E)$ is defined by an equation $s=0$. The original standard metric on the trivial projective bundle on $\widetilde{\mathcal{H}}(d,n)$ induces a hermitian metric on E, and consequently on $\mathcal{O}_{\mathbb{P}(E)}(1)$, and on the relative cotangent bundle $\Omega^1_{\mathbb{P}(E)/Y}$. As usual, we write $\overline{E}$, $\overline{\mathcal{O}}_{\mathbb{P}(E)}(1)$, and $\overline{\Omega}^1_{\mathbb{P}(E)/Y}$ for the corresponding hermitian vector bundles. We set

$$\overline{V}:=\overline{\Omega}^1_{\mathbb{P}(E)/Y}\otimes\overline{\mathcal{O}}_{\mathbb{P}(E)}(d),\ \overline{W}:=\overline{V}|_D=\overline{\Omega}^1_{\mathbb{P}(E)/Y}\otimes\overline{\mathcal{O}}_D(d).$$

We recall the short exact sequence of vector bundles

$$\mathcal{G}:\ 0\longrightarrow\Omega^1_{\mathbb{P}(E)/Y}\longrightarrow\mathcal{O}_{\mathbb{P}(E)}(-1)\otimes\rho^*E^\vee\longrightarrow\mathcal{O}_{\mathbb{P}(E)}\longrightarrow 0. \tag{24}$$

Theorem 6.1 *Using the notations of Sect.* 6.1, *we have*

$$\mathrm{dd^c}(\rho_*g)+\delta_{\rho_*Z}=\rho_*\big(\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(d))\wedge\mathrm{c}_n(\overline{\Omega}^1_{\mathbb{P}(E)/Y}\otimes\overline{\mathcal{O}}_{\mathbb{P}(E)}(d))\big), \tag{25}$$

where ρ_*Z is the discriminant locus in Y. If $y\in Y\setminus\rho_*Z$, we find for the push-forward ρ_*g of g with $\xi:=\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}^n_y}(1))$

$$\begin{aligned}(\rho_*g)(y)=&-\int_{D_y}(\zeta_2|_{D_y})\\&-\int_{\mathbb{P}^n_y}\log\|s\|^2_D\wedge\sum_{k=0}^{n}\left(\binom{n+1}{k}(-\xi)^k+\mathrm{dd^c}\left(\widetilde{\mathrm{c}}_k(\overline{\mathcal{G}})\right)\right)\wedge(d\xi)^{n-k}\\&-\int_{D_y}\log\|\mathrm{d}s\|^2_W\wedge\sum_{j=0}^{n-1}(\zeta_1|_{D_y})^j\\&\wedge\sum_{k=0}^{n-j-1}\binom{n-k}{j+1}\binom{n+1}{k}(-1)^k d^{n-j-k-1}\xi^{n-j-1}|_{D_y}\\&-\int_{D_y}\log\|\mathrm{d}s\|^2_W\wedge\sum_{j=0}^{n-1}(\zeta_1|_{D_y})^j\\&\wedge\sum_{k=0}^{n-j-1}\binom{n-k}{j+1}\mathrm{dd^c}\left(\widetilde{\mathrm{c}}_k(\overline{\mathcal{G}})\right)\wedge(d\xi)^{n-j-k-1}|_{D_y}.\end{aligned} \tag{26}$$

For the right-hand side of (25), *we have*

$$\rho_*\left(\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(d))\wedge \mathrm{c}_n(\overline{\Omega}^1_{\mathbb{P}(E)/Y}\otimes\overline{\mathcal{O}}_{\mathbb{P}(E)}(d))\right)$$
$$= d(d-1)^n\cdot \mathrm{c}_1(\overline{E}^\vee)+\mathrm{dd^c}\left(\rho_*\big(\widetilde{\mathrm{c}}_{n+1}(\overline{\mathcal{G}}\otimes\overline{\mathcal{O}}_{\mathbb{P}(E)}(d))\big)\right), \tag{27}$$

where $d(d-1)^n$ *is the weight of the discriminant* $\Delta(d,n)$.

Proof We start by noting that formula (25) follows immediately from formula (19) taking into account that $\overline{L}_D=\overline{\mathcal{O}}_{\mathbb{P}(E)}(d)$.
In order to compute ρ_*g we cannot apply Proposition 4.3 directly, since $\Omega^1_{\mathbb{P}(E)/Y}$ is not the pull-back of a vector bundle on Y. We therefore modify our procedure as follows. Using the formula

$$\mathrm{c}_j(\overline{\Omega}^1_{\mathbb{P}(E)/Y}\otimes\overline{\mathcal{O}}_{\mathbb{P}(E)}(d))=\sum_{k=0}^{j}\binom{n-k}{j-k}\mathrm{c}_k(\overline{\Omega}^1_{\mathbb{P}(E)/Y})\wedge \mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(d))^{j-k}$$

for $j=0,\dots,n$, we find by means of formulas (16), (18) for the push-forward $(\rho_*g)(y)$ of g (as long as $y\in Y\setminus\rho_*Z$)

$$(\rho_*g)(y)=-\int_{\mathbb{P}^n_y}\log\|s\|^2_D\wedge\sum_{k=0}^{n}\mathrm{c}_k(\overline{\Omega}^1_{\mathbb{P}^n_y})\wedge \mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}^n_y}(d))^{n-k}$$
$$-\int_{D_y}\log\|\mathrm{d}s\|^2_W\wedge\sum_{j=0}^{n-1}(\zeta_1|_{D_y})^j$$
$$\wedge\sum_{k=0}^{n-j-1}\binom{n-k}{n-j-k-1}\mathrm{c}_k(\overline{\Omega}^1_{\mathbb{P}^n_y})\wedge \mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}^n_y}(d))^{n-j-k-1}|_{D_y}-\int_{D_y}(\zeta_2|_{D_y}).$$

Using the short exact sequence (24), we obtain

$$\mathrm{c}_k(\overline{\Omega}^1_{\mathbb{P}(E)/Y})=\mathrm{c}_k(\overline{\mathcal{O}}_{\mathbb{P}(E)}(-1)\otimes\rho^*\overline{E}^\vee)+\mathrm{dd^c}\left(\widetilde{\mathrm{c}}_k(\overline{\mathcal{G}})\right)$$
$$=\sum_{i=0}^{k}\binom{n-i+1}{k-i}\mathrm{c}_i(\rho^*\overline{E}^\vee)\wedge \mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(-1))^{k-i}+\mathrm{dd^c}\left(\widetilde{\mathrm{c}}_k(\overline{\mathcal{G}})\right)$$

for $k=0,\dots,n$. Since the restriction of $\mathrm{c}_i(\rho^*\overline{E}^\vee)$ to the fibers is zero for $i>0$, we obtain the claimed formula (26) for $(\rho_*g)(y)$.

Tensoring the short exact sequence (24) with $\mathcal{O}_{\mathbb{P}(E)}(d)$, we compute for the right-hand side of (25)

$$\rho_*\left(\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(d))\wedge \mathrm{c}_n(\overline{\Omega}^1_{\mathbb{P}(E)/Y}\otimes\overline{\mathcal{O}}_{\mathbb{P}(E)}(d))\right)$$
$$=\rho_*\left(\mathrm{c}_{n+1}(\overline{\mathcal{O}}_{\mathbb{P}(E)}(d-1)\otimes\rho^*\overline{E}^\vee)\right)+\rho_*\left(\mathrm{dd^c}\left(\widetilde{\mathrm{c}}_{n+1}(\overline{\mathcal{G}}\otimes\overline{\mathcal{O}}_{\mathbb{P}(E)}(d))\right)\right).$$

For the first term on the right-hand side of the above equality, we obtain by dimension reasons

$$\begin{aligned}
&\rho_*\left(\mathrm{c}_{n+1}(\overline{\mathcal{O}}_{\mathbb{P}(E)}(d-1)\otimes\rho^*\overline{E}^\vee)\right)\\
&\quad=\sum_{j=0}^{n+1}\mathrm{c}_j(\overline{E}^\vee)\wedge\rho_*\left(\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(d-1))^{n-j+1}\right)\\
&\quad=\mathrm{c}_0(\overline{E}^\vee)\wedge\rho_*\left(\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(d-1))^{n+1}\right)+\mathrm{c}_1(\overline{E}^\vee)\wedge\rho_*\left(\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(d-1))^{n}\right)\\
&\quad=(d-1)^{n+1}\cdot\rho_*\left(\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(1))^{n+1}\right)+(d-1)^n\cdot\mathrm{c}_1(\overline{E}^\vee).
\end{aligned}$$

In order to determine $\rho_*(\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(1))^{n+1})$, we derive from the second computation carried out in Sect. 4.3 with the short exact sequence $\mathcal{G}^\vee$ instead of the sequence $\mathcal{F}$

$$\rho_*\left(\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(1))^{n+1}\right)=\mathrm{c}_1(\overline{E}^\vee)-\mathrm{dd^c}\left(\rho_*\left(\widetilde{\mathrm{c}}_{n+1}(\overline{\mathcal{G}}^\vee)\right)\right).$$

From Mourougane's computation in [11], Sect. 7, Theorem 3, we find

$$\rho_*\left(\widetilde{\mathrm{c}}_{n+1}(\overline{\mathcal{G}}^\vee)\right)=\sum_{j=1}^{n}\sum_{k=1}^{j}\frac{1}{k},$$

which leads to

$$\rho_*\left(\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(1))^{n+1}\right)=\mathrm{c}_1(\overline{E}^\vee).$$

After collecting all the results, we find for the right-hand side of (25)

$$\begin{aligned}
&\rho_*\left(\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(d))\wedge\mathrm{c}_n(\overline{\Omega}^1_{\mathbb{P}(E)/Y}\otimes\overline{\mathcal{O}}_{\mathbb{P}(E)}(d))\right)\\
&\quad=(d-1)^{n+1}\cdot\mathrm{c}_1(\overline{E}^\vee)+(d-1)^n\cdot\mathrm{c}_1(\overline{E}^\vee)+\rho_*\left(\mathrm{dd^c}\left(\widetilde{\mathrm{c}}_{n+1}(\overline{\mathcal{G}}\otimes\overline{\mathcal{O}}_{\mathbb{P}(E)}(d))\right)\right)\\
&\quad=d(d-1)^n\cdot\mathrm{c}_1(\overline{E}^\vee)+\mathrm{dd^c}\left(\rho_*\left(\widetilde{\mathrm{c}}_{n+1}(\overline{\mathcal{G}}\otimes\overline{\mathcal{O}}_{\mathbb{P}(E)}(d))\right)\right),
\end{aligned}$$

which is the claimed formula (27). □

Remark 6.2 In relation to Theorem 6.1, we observe that a significant example to the above construction is the case of the moduli space of marked cubic surfaces (for details, see [1]). We recall that in this case, R. Borcherds gave an automorphic form χ_{12} of weight 12 vanishing exactly along the discriminant locus (for details, see [3]). Since the GIT-compactification $\overline{Y}$ of Y satisfies $\mathrm{codim}(\overline{Y}\setminus Y)\geq 2$ in this case, we find an integral representation of minus the logarithm of the norm of Borcherds' automorphic form (see [10], Remark 3.4).

Remark 6.3 We can construct another example, which is related to Remark 4.4. For this, let X, D, Y be as in Sect. 6.1. Set $\overline{V}:=\overline{\mathcal{O}}_{\mathbb{P}(E)}(1)\otimes\mathbb{C}^n$, and let $\sigma:=(\sigma_1,\ldots,\sigma_n)$ be a global section of V as in Remark 4.2. Let s be the section of $\mathcal{O}_{\mathbb{P}(E)}(d)$, whose zero locus is D, and set $\xi := \mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}(E)}(1))$. With these notations, we obtain from Remark 4.4 the explicit Green's current

$$\begin{aligned} g|_{X^0} &= -\log\|s\|_D^2 \wedge \xi^n|_{X^0} \\ &\quad - \left(\sum_{j=1}^{n} \log\|\sigma_j\|^2_{\mathcal{O}_{\mathbb{P}(E)}(1)} \wedge \delta_{\mathrm{div}(\sigma_1)} \wedge \ldots \wedge \delta_{\mathrm{div}(\sigma_{j-1})} \wedge \xi^{n-j} \right) \wedge \delta_{D^0}. \end{aligned}$$

This Green's current satisfies

$$\mathrm{dd^c} g + \delta_Z = d \cdot \xi \wedge \mathrm{c}_n(\overline{V}) = d \cdot \xi^{n+1}.$$

From the results of Sect. 4.3, we obtain

$$\rho_*(\xi^{n+1}) = \mathrm{c}_1(\overline{E}^{\vee}),$$

hence,

$$\mathrm{dd^c}(\rho_* g_Z) + \delta_{\rho_* Z} = d \cdot \mathrm{c}_1(\overline{E}^{\vee}).$$

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