# Elliptic Eisenstein series for $\mathrm{PSL}_2(\mathbb{Z})$

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*To the memory of Serge Lang*

**Abstract**

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind acting by fractional linear transformations on the upper half-plane $\mathbb{H}$, and let $\Gamma\backslash\mathbb{H}$ be the associated finite volume hyperbolic Riemann surface. Associated to any cusp of $\Gamma\backslash\mathbb{H}$, there is the classically studied non-holomorphic (parabolic) Eisenstein series. In [11], Kudla and Millson studied non-holomorphic (hyperbolic) Eisenstein series associated to any closed geodesic on $\Gamma\backslash\mathbb{H}$. Finally, in [9], Jorgenson and the first named author introduced so-called elliptic Eisenstein series associated to any elliptic fixed point of $\Gamma\backslash\mathbb{H}$. In the present article, we study elliptic Eisenstein series for the full modular group $\mathrm{PSL}_2(\mathbb{Z})$. We explicitly compute the Fourier expansion of the elliptic Eisenstein series and derive from this its meromorphic continuation.

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## 1 Introduction

**1.1.** The theory of Eisenstein series plays a prominent role in the theory of automorphic functions and automorphic forms. Classically, in the theory of holomorphic modular forms, the Eisenstein series of weight $2k$ ($k \in \mathbb{N}$, $k \geq 2$) for the full modular group $\mathrm{PSL}_2(\mathbb{Z})$ are defined by

$$\mathcal{E}_{2k}(z) := \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1}} \frac{1}{(cz+d)^{2k}} \qquad (z = x+iy \in \mathbb{C},\ y>0).$$

The arithmetic significance of these series is reflected by the fact that their Fourier coefficients are given by certain divisor sums.

More generally, in the theory of automorphic functions for Fuchsian subgroups $\Gamma$ of the first kind of $\mathrm{PSL}_2(\mathbb{R})$, Eisenstein series are defined by

$$\mathcal{E}(z,s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s \qquad (s \in \mathbb{C},\ \mathrm{Re}(s) > 1);$$

here $\Gamma_\infty$ denotes the stabilizer of the cusp $i\infty$ in the group $\Gamma$. For given $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the Eisenstein series $\mathcal{E}(z,s)$ are $C^\infty$-functions in $x, y$. For given $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$, the series $\mathcal{E}(z,s)$ are holomorphic functions in $s$ as long as $\mathrm{Re}(s) > 1$. It can be shown that the Eisenstein series $\mathcal{E}(z,s)$ admit a meromorphic continuation to the whole $s$-plane. The significance of $\mathcal{E}(z,s)$ relies on the fact that these series are eigenfunctions of the hyperbolic Laplacian $\Delta_{\mathrm{hyp}}$ for the continuous spectrum. The classical approach to establishing the meromorphic continuation is based on the explicit knowledge of the Fourier expansion of $\mathcal{E}(z,s)$. Other approaches rely on the meromorphic continuation of the resolvent kernel of $\Delta_{\mathrm{hyp}}$ or Colin de Verdière's method given in [3].

Observing that the series $\mathcal{E}(z,s)$ are associated to the cusp $i\infty$, S. Kudla and J. Millson introduced in [11] so-called hyperbolic Eisenstein series $\mathcal{E}_{\mathrm{hyp}}(z,s)$ associated to geodesics in the upper half-plane $\mathbb{H}$, and proved a partial meromorphic continuation and a Kronecker limit-type formula for these series. Following this point of view, J. Jorgenson and the first author were led to consider so-called elliptic Eisenstein series $\mathcal{E}_{\mathrm{ell}}(z,s)$ associated to elliptic fixed points $z_0 \in \mathbb{H}$ for $\Gamma$. In fact, these series were introduced in [9] (see also the unpublished paper [8]) in order to derive optimal sup-norm bounds for cusp forms of weight 2 for the subgroup $\Gamma$. An alternative, more elementary proof for these sup-norm bounds avoiding elliptic Eisenstein series is given in [7].

The elliptic Eisenstein series associated to an elliptic fixed point $z_0 \in \mathbb{H}$ for the subgroup $\Gamma$ is defined by

$$\mathcal{E}_{\mathrm{ell}}(z,s) = \sum_{\gamma \in \Gamma_{z_0} \backslash \Gamma} \sinh\big(\varrho\big(\sigma_{z_0}^{-1}\gamma z\big)\big)^{-s} \qquad (z \neq z_0),$$

where $\Gamma_{z_0}$ denotes the stabilizer of $z_0$ in $\Gamma$, $\sigma_{z_0} \in \mathrm{PSL}_2(\mathbb{R})$ is a scaling matrix for $z_0$, i.e., $\sigma_{z_0}(i) = z_0$, and $\varrho(z)$ denotes the hyperbolic distance from $z$ to $i$. In the Ph.D. thesis [13] by the second named author, the meromorphic continuation of the elliptic Eisenstein series $\mathcal{E}_{\mathrm{ell}}(z,s)$ for any Fuchsian subgroup $\Gamma$ of the first kind to the whole $s$-plane is proven using a variation of Colin de Verdière's method mentioned above. Moreover, various expansions of the series $\mathcal{E}_{\mathrm{ell}}(z,s)$ are computed and a Kronecker limit type formula is established there.

In this note we study elliptic Eisenstein series in the special case $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and $z_0 = i$. Following the classical approach, the main goal of this paper is to establish the meromorphic continuation of the series $\mathcal{E}_{\mathrm{ell}}(z,s)$ by means of its Fourier expansion thereby complementing work carried out in [13] in the special case $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. In order to achieve our goal, the Fourier expansion of $\mathcal{E}_{\mathrm{ell}}(z,s)$ has to be explicitly computed and the growth of the Fourier coefficients has to be controlled.

**1.2.** The paper is organized as follows. In Section 2, we recall and summarize basic notation and definitions used in this article.

In Section 3, we recall the classical Poincaré series $P_m(z,s)$ and relate them to the more recent Poincaré-type series $V_m(z,s)$ studied in [14]. We review how the meromorphic continuation of $P_m(z,s)$ can be obtained via its spectral expansion. Via the aforementioned relation we obtain the meromorphic continuation of $V_m(z,s)$ to the whole $s$-plane.

In Section 4, we define the elliptic Eisenstein series $\mathcal{E}_{\mathrm{ell}}(z,s)$ associated to the elliptic fixed point $i$ of $\mathrm{PSL}_2(\mathbb{Z})$. We show that it is holomorphic for $\mathrm{Re}(s) > 1$ and an automorphic function for $\mathrm{PSL}_2(\mathbb{Z})$. In contrast to the parabolic situation, the elliptic Eisenstein series fails to be an eigenfunction of $\Delta_{\mathrm{hyp}}$; instead it satisfies the differential equation

$$\big(\Delta_{\mathrm{hyp}} - s(1-s)\big)\mathcal{E}_{\mathrm{ell}}(z,s) = -s^2\mathcal{E}_{\mathrm{ell}}(z,s+2).$$

In Section 5, we calculate the Fourier coefficients of $\mathcal{E}_{\mathrm{ell}}(z,s)$. In order to simplify the exposition, we restrict our study to the case $z \in \mathbb{H}$ with $\mathrm{Im}(z) > 1$.

In Section 6, we obtain the meromorphic continuation of $\mathcal{E}_{\mathrm{ell}}(z,s)$ via its Fourier expansion. The main task here is to first meromorphically continue the $m$th Fourier coefficients $a_m(y,s)$ of $\mathcal{E}_{\mathrm{ell}}(z,s)$ and then to achieve suitable bounds for $a_m(y,s)$ with respect to $m$. The main result is stated in Theorem 6.10.

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## 2 Basic notation

**2.1.** Let $\Gamma := \mathrm{PSL}_2(\mathbb{Z})$ be the modular group acting by fractional linear transformations on the upper half-plane $\mathbb{H} := \{z = x + iy \in \mathbb{C} \,|\, y > 0\}$, i.e., for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $z \in \mathbb{H}$, we have

$$\gamma z := \frac{az+b}{cz+d}\,.$$

We denote by $\mathcal{F}_\Gamma$ a fundamental domain of $\Gamma$ in $\mathbb{H}$. By $\Gamma_z := \mathrm{Stab}_\Gamma(z)$ we denote the stabilizer of $z \in \mathbb{H}$ in $\Gamma$, and we set

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}.$$

As usual, we put $e(z) := \exp(2\pi i z)$ and denote by $\zeta(s)$ the Riemann zeta function.
In the rectangular coordinates $x, y$, the hyperbolic line element $ds^2_{\mathrm{hyp}}$, the hyperbolic volume element $\mu_{\mathrm{hyp}}$, and the hyperbolic Laplacian $\Delta_{\mathrm{hyp}}$ on $\mathbb{H}$ are given by

$$ds^2_{\mathrm{hyp}} = \frac{dx^2 + dy^2}{y^2}\,, \quad \mu_{\mathrm{hyp}} = \frac{dx\,dy}{y^2}\,, \quad \Delta_{\mathrm{hyp}} = -y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big).$$

We recall that the hyperbolic volume $\mathrm{vol}_{\mathrm{hyp}}(\mathcal{F}_\Gamma)$ of $\mathcal{F}_\Gamma$ is given by

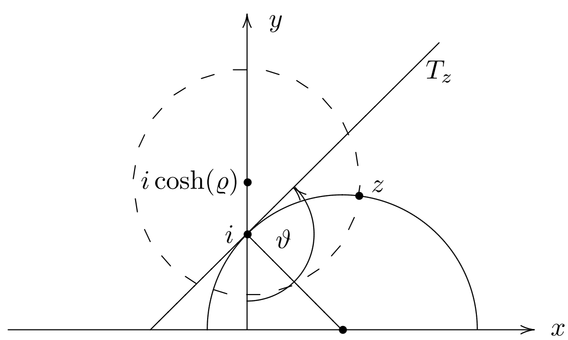
$$\mathrm{vol}_{\mathrm{hyp}}(\mathcal{F}_\Gamma) = \int\limits_{\mathcal{F}_\Gamma} \mu_{\mathrm{hyp}}(z) = \frac{\pi}{3}.$$

By $d_{\mathbb{H}}(z, w)$ we denote the hyperbolic distance from $z \in \mathbb{H}$ to $w \in \mathbb{H}$.

**2.2. Hyperbolic polar coordinates.** For $z = x + iy \in \mathbb{H}$, we define the hyperbolic polar coordinates $\varrho = \varrho(z), \vartheta = \vartheta(z)$ centered at $i \in \mathbb{H}$ by

$$\varrho(z) := d_{\mathbb{H}}(i, z)\,, \quad \vartheta(z) := \measuredangle(L, T_z),$$

where $L$ denotes the positive $y$-axis and $T_z$ is the tangent at the unique geodesic passing through $i$ and $z$ at the point $i$.




The relation between the $x, y$-coordinates and the $\varrho, \vartheta$-coordinates is expressed through the formulas

$$x = \frac{\sinh(\varrho)\sin(\vartheta)}{\cosh(\varrho) + \sinh(\varrho)\cos(\vartheta)}\,, \quad y = \frac{1}{\cosh(\varrho) + \sinh(\varrho)\cos(\vartheta)}\,. \tag{1}$$

Using the above formulas, the hyperbolic line element and the hyperbolic Laplacian in terms of the hyperbolic polar coordinates take the form

$$ds^2_{\mathrm{hyp}} = \sinh^2(\varrho)d\vartheta^2 + d\varrho^2\,, \quad \Delta_{\mathrm{hyp}} = -\frac{\partial^2}{\partial\varrho^2} - \frac{1}{\tanh(\varrho)}\frac{\partial}{\partial\varrho} - \frac{1}{\sinh^2(\varrho)}\frac{\partial^2}{\partial\vartheta^2}\,.$$

From the well-known formula for the hyperbolic distance (see [2], p. 131)

$$\cosh\big(d_{\mathbb{H}}(z, w)\big) = 1 + \frac{|z - w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)},$$

we obtain for $z = x + iy \in \mathbb{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

$$\cosh\big(\varrho(\gamma z)\big) = \cosh\big(d_{\mathbb{H}}(z, \gamma^{-1} i)\big) = \frac{1}{2y}\Big(2y + (a^2 + c^2)\big|z - \gamma^{-1} i\big|^2\Big).$$

A straightforward computation yields

$$\cosh\bigl(\varrho(\gamma z)\bigr) = \frac{1}{2y}\bigl((a^2+c^2)(x^2+y^2)+2(ab+cd)x+(b^2+d^2)\bigr). \tag{2}$$

**2.3. Hypergeometric functions.** For $a,b,c\in\mathbb{C}$, $c\neq -n$ ($n\in\mathbb{N}$), and $w\in\mathbb{C}$, we denote Gauss's hypergeometric function by $F(a,b;c;w)$. For $w\in\mathbb{C}$ with $|w|<1$ it is defined by the series

$$F(a,b;c;w) := \sum_{k=0}^{\infty}\frac{(a)_k\cdot(b)_k}{(c)_k\cdot k!}\cdot w^k,$$

where $(\lambda)_k := \Gamma(\lambda+k)/\Gamma(\lambda)$ ($\lambda\in\mathbb{C}$, $k\in\mathbb{N}$) is the Pochhammer symbol; for $k\in\mathbb{N}$ with $k>0$, we note the alternative formula $(\lambda)_k=\prod_{j=0}^{k-1}(\lambda+j)$. For $\mathrm{Re}(c)>\mathrm{Re}(b)>0$, the hypergeometric function $F(a,b;c;w)$ has the integral representation (see [1], formula 15.3.1)

$$F(a,b;c;w) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tw)^{-a}\,dt. \tag{3}$$

**2.4. Parabolic Eisenstein series.** For $z\in\mathbb{H}$ and $s\in\mathbb{C}$, the parabolic Eisenstein series $\mathcal{E}_{\mathrm{par}}$ is given by

$$\mathcal{E}_{\mathrm{par}}(z,s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\mathrm{Im}(\gamma z)^s.$$

The parabolic Eisenstein series is known to be holomorphic for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$ with Fourier expansion given by

$$\mathcal{E}_{\mathrm{par}}(z,s) = y^s+\varphi(s)y^{1-s}+\sum_{n\neq 0}\varphi(n,s)y^{1/2}K_{s-1/2}(2\pi|n|y)e(nx), \tag{4}$$

where $K_{s-1/2}(\cdot)$ is the modified Bessel function of the second kind,

$$\varphi(s) = \frac{\sqrt{\pi}\,\Gamma(s-1/2)}{\Gamma(s)}\cdot\frac{\zeta(2s-1)}{\zeta(2s)} = \frac{\Lambda(2s-1)}{\Lambda(2s)},$$

and

$$\varphi(n,s) = \frac{2\pi^s|n|^{s-1/2}}{\Gamma(s)\zeta(2s)}\sum_{d|n}d^{-2s+1} = \frac{2}{\Lambda(2s)}\sum_{ab=|n|}\Bigl(\frac{a}{b}\Bigr)^{s-1/2};$$

here we set $\Lambda(s):=\pi^{-s/2}\Gamma(s/2)\zeta(s)$. The Fourier expansion (4) provides the meromorphic continuation of $\mathcal{E}_{\mathrm{par}}(z,s)$ to the whole $s$-plane with a simple pole at $s=1$ with residue $\mathrm{res}_{s=1}\,\mathcal{E}_{\mathrm{par}}(z,s) = 1/\mathrm{vol}_{\mathrm{hyp}}(\mathcal{F}_\Gamma)=3/\pi$, and other poles contributed by the non-trivial zeros of $\zeta(2s)$ in the strip $0<\mathrm{Re}(s)<1/2$. From the functional equation $\Lambda(s)=\Lambda(1-s)$, we get $\varphi(s)\varphi(1-s)=1$, and hence the relation

$$\varphi(s)\varphi(n,1-s) = \frac{2\,\Lambda(2s-1)}{\Lambda(2s)\Lambda(-2s+2)}\sum_{ab=|n|}\Bigl(\frac{b}{a}\Bigr)^{s-1/2} = \frac{2}{\Lambda(2s)}\sum_{ab=|n|}\Bigl(\frac{a}{b}\Bigr)^{s-1/2} = \varphi(n,s), \tag{5}$$

which, using (4), proves the functional equation

$$\mathcal{E}_{\mathrm{par}}(z,s) = \varphi(s)\,\mathcal{E}_{\mathrm{par}}(z,1-s). \tag{6}$$

## 3 Poincaré series

In this section we recall results for two types of Poincaré series which are mostly known to the experts. However, for the lack of complete reference, some proofs have to be elaborated.

**3.1. Definition.** For $z \in \mathbb{H}$, $s \in \mathbb{C}$, and $m \in \mathbb{Z}$, the *Poincaré series* $P_m$ is defined by

$$P_m(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \operatorname{Im}(\gamma z)^s \exp\bigl(-2\pi |m| \operatorname{Im}(\gamma z)\bigr) e\bigl(m \operatorname{Re}(\gamma z)\bigr).$$

The Poincaré series is known to be holomorphic for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, since it can be majorized by $P_0(z, \operatorname{Re}(s)) = \mathcal{E}_{\mathrm{par}}(z, \operatorname{Re}(s))$.

**3.2. Remark.** For $m \neq 0$, the Poincaré series $P_m(z,s)$ is bounded on $\mathbb{H}$ (see [10], p. 83) and hence admits a spectral expansion in terms of the eigenfunctions $\psi_j$ associated to the discrete eigenvalues $\lambda_j$ of $\Delta_{\mathrm{hyp}}$ and the parabolic Eisenstein series $\mathcal{E}_{\mathrm{par}}$, namely

$$P_m(z,s) = \sum_{j=0}^{\infty} a_{j,m}(s)\, \psi_j(z) + \frac{1}{4\pi} \int_{-\infty}^{\infty} a_{1/2+ir,m}(s)\, \mathcal{E}_{\mathrm{par}}(z, 1/2+ir)\, dr\,, \tag{7}$$

where the coefficients $a_{j,m}(s)$, resp. $a_{1/2+ir,m}(s)$, are given by

$$a_{j,m}(s) = \int_{\mathcal{F}_\Gamma} P_m(z,s) \overline{\psi}_j(z) \mu_{\mathrm{hyp}}(z), \text{ resp. } a_{1/2+ir,m}(s) = \int_{\mathcal{F}_\Gamma} P_m(z,s) \overline{\mathcal{E}}_{\mathrm{par}}(z, 1/2+ir) \mu_{\mathrm{hyp}}(z).$$

The expansion (7) is absolutely and locally uniformly convergent for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$.

As ususal, we enumerate the eigenvalues of the discrete spectrum by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$; since $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, we have $\lambda_j = 1/4 + t_j^2 = s_j(1-s_j)$, i.e., $s_j = 1/2 + it_j$ with $t_j > 0$, as long as $j > 0$. For $j = 0$, the eigenfunction is given by $\psi_0(z) = \sqrt{3/\pi}$. For $j > 0$, the eigenfunction $\psi_j$ is a cusp form and admits a Fourier expansion of the form

$$\psi_j(z) = \sum_{n \neq 0} \rho_j(n) y^{1/2} K_{s_j - 1/2}(2\pi |n| y) e(nx). \tag{8}$$

The eigenvalues of the continuous spectrum are of the form $\lambda = 1/4 + r^2 = s(1-s)$, i.e., $s = 1/2 + ir$ with $r \in \mathbb{R}$. The corresponding eigenfunctions are given by the parabolic Eisenstein series $\mathcal{E}_{\mathrm{par}}(z, 1/2+ir)$.

**3.3. Proposition.** *For $z \in \mathbb{H}$, $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, and $m \neq 0$, the Poincaré series $P_m(z,s)$ has the following explicit spectral expansion:*

$$P_m(z,s)\, 2^{2s-1} \pi^{s-1} \Gamma(s) |m|^{s-1/2} = \sum_{j=1}^{\infty} \Gamma(s-s_j)\Gamma(s+s_j-1) \overline{\rho}_j(m)\, \psi_j(z)$$
$$+ \frac{1}{4\pi} \int_{-\infty}^{\infty} \Gamma(s-1/2-ir)\Gamma(s-1/2+ir) \overline{\varphi}(m, 1/2+ir)\, \mathcal{E}_{\mathrm{par}}(z, 1/2+ir)\, dr. \tag{9}$$

*Proof.* The proof can easily be deduced from the spectral expansion given for the function $\widetilde{P}_m(z,s) = \pi^{s-1/2}\Gamma(s+1/2)^{-1}|m|^{s-1/2} P_m(z,s)$ in [12], p. 58. □

**3.4. Proposition.** *For $z \in \mathbb{H}$ and $m \neq 0$, the Poincaré series $P_m(z,s)$ admits a meromorphic continuation to the whole $s$-plane with simple poles at $s = s_j - N$ and $s = -s_j - N + 1$ ($N \in \mathbb{N}$) with residues*

$$\operatorname{res}_{s=s_j-N} P_m(z,s) = \frac{(-1)^N 2^{-2s_j+2N+1} \pi^{-s_j+N+1} \Gamma(2s_j - N - 1)}{N!\, \Gamma(s_j - N) |m|^{s_j - N - 1/2}} \sum_{s_\ell = s_j} \overline{\rho}_\ell(m)\, \psi_\ell(z)$$

*and*

$$\operatorname{res}_{s=-s_j-N+1} P_m(z,s) = \frac{(-1)^N 2^{2s_j+2N-1}\pi^{s_j+N}\Gamma(-2s_j-N+1)}{N!\,\Gamma(-s_j-N+1)|m|^{-s_j-N+1/2}} \sum_{s_\ell=s_j} \overline{\rho}_\ell(m)\,\psi_\ell(z),$$

*respectively.*

*Proof.* Because of the lack of reference for the claimed residues, we have to discuss the proof briefly. In order to obtain the desired meromorphic continuation we will follow closely [12] and [10], and base our argument on the spectral expansion (9).

We start by discussing the meromorphic continuation of the discrete part

$$D(s) := \sum_{j=1}^{\infty} \Gamma(s-s_j)\Gamma(s+s_j-1)\overline{\rho}_j(m)\,\psi_j(z)$$

of the spectral expansion (9). The argument given in [10], p. 87, shows that $D(s)$ has a meromorphic continuation to the whole $s$-plane with simple poles at $s = s_j - N$ and $s = -s_j - N + 1$ ($N \in \mathbb{N}$) arising from the $\Gamma$-factors. For later purposes, we note the bound (see [10], p. 87, adapted to the present situation)

$$\big|D(s)\big| \ll y^{-3/2}, \tag{10}$$

where the implied constant depends only on $s$ (not a pole), but is independent of $z$ and $m$. The dependence of the implied constant on $s$ is uniform as long as $s$ is contained in a compact set not containing $s_j - N$ or $-s_j - N + 1$ for some $N \in \mathbb{N}$. For the residues we compute

$$\operatorname{res}_{s=s_j-N} D(s) = \frac{(-1)^N}{N!}\Gamma(2s_j-N-1) \sum_{s_\ell=s_j} \overline{\rho}_\ell(m)\,\psi_\ell(z)$$

and

$$\operatorname{res}_{s=-s_j-N+1} D(s) = \frac{(-1)^N}{N!}\Gamma(-2s_j-N+1) \sum_{s_\ell=s_j} \overline{\rho}_\ell(m)\,\psi_\ell(z),$$

respectively.

We now turn to the meromorphic continuation of the continuous part

$$Q(s) := \frac{1}{4\pi}\int_{-\infty}^{\infty} \Gamma(s-1/2-ir)\Gamma(s-1/2+ir)\overline{\varphi}(m,1/2+ir)\,\mathcal{E}_{\mathrm{par}}(z,1/2+ir)\,dr \tag{11}$$

of the spectral expansion (9). By substituting $t := 1/2 + ir$, the integral (11) can be rewritten as

$$Q(s) = \frac{1}{4\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \Gamma(s-t)\Gamma(s-1+t)\varphi(m,1-t)\,\mathcal{E}_{\mathrm{par}}(z,t)\,dt. \tag{12}$$

By construction, the integral (12) exists for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and represents a holomorphic function in this range. The argument given in [10], p. 89, shows that $Q(s)$ extends to a holomorphic function for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \neq -N+1/2$, and for $s = -N+1/2$, where $N \in \mathbb{N}$. In order to extend $Q(s)$ to the whole $s$-plane, we rewrite the integral (12) by means of a different path of integration (see [12], p. 51) using the residue theorem as follows. Let $s_0 \in \mathbb{C}$ with $\mathrm{Re}(s_0) = -N+1/2$ for some $N \in \mathbb{N}$ and $\mathrm{Im}(s_0) > 0$, and let $C(s_0)$ denote the integration path, which runs on the vertical line with $\mathrm{Re}(t) = 1/2$ from $-\infty$ to $\infty$ as before, but passes on the left-hand side around $-s_0 - N + 1$ and on the right-hand side around $s_0 + N$ in such a way that the only poles of the integrand being encircled by this new integration path are located at $t = -s_0 - N + 1$ and $t = s_0 + N$. For $s$ with

$\mathrm{Re}(s) > -N+1/2$ being sufficiently close to $s_0$ such that $-s-N+1$ and $s+N$ are still encircled by the path $C(s_0)$, we set

$$\widetilde{Q}(s) = \frac{1}{4\pi i} \int\limits_{C(s_0)} \Gamma(s-t)\Gamma(s-1+t)\varphi(m,1-t)\,\mathcal{E}_{\mathrm{par}}(z,t)\,dt,$$

which is well defined by construction. Using the residue theorem and recalling (6) and (5), we then compute

$$\begin{aligned} Q(s) =& \frac{1}{4\pi i} \int\limits_{C(s_0)} \Gamma(s-t)\Gamma(s-1+t)\varphi(m,1-t)\,\mathcal{E}_{\mathrm{par}}(z,t)\,dt \\ &- \frac{(-1)^N}{2\,N!}\Gamma(2s+N-1)\varphi(m,-s-N+1)\,\mathcal{E}_{\mathrm{par}}(z,s+N) \\ &+ \frac{(-1)^N}{2\,N!}\Gamma(2s+N-1)\varphi(m,s+N)\,\mathcal{E}_{\mathrm{par}}(z,-s-N+1) = \widetilde{Q}(s). \end{aligned}$$

For $s$ with $\mathrm{Re}(s) < -N+1/2$ being sufficiently close to $s_0$, we define $\widetilde{Q}(s)$ as above and verify again $Q(s) = \widetilde{Q}(s)$, now using Cauchy's theorem. By the choice of the integration path $C(s_0)$ it turns out that the integral

$$\widetilde{Q}(s_0) = \frac{1}{4\pi i} \int\limits_{C(s_0)} \Gamma(s_0-t)\Gamma(s_0-1+t)\varphi(m,1-t)\,\mathcal{E}_{\mathrm{par}}(z,t)\,dt$$

is also well defined. Proceeding in an analoguous way for $s_0 \in \mathbb{C}$ with $\mathrm{Re}(s_0) = -N+1/2$ for some $N \in \mathbb{N}$, but $\mathrm{Im}(s_0) < 0$, we obtain the analytic continuation of $Q(s)$ to the whole $s$-plane.

All in all, these considerations show that $P_m(z,s)$ admits a meromorphic continuation to the whole $s$-plane with simple poles at $s = s_j - N$ and $s = -s_j - N + 1$ ($N \in \mathbb{N}$). The stated formulas for the residues are easily obtained from the residue computations for $D(s)$, taking into account that the factor $2^{-2s+1}\pi^{-s+1}\Gamma(s)^{-1}|m|^{-s+1/2}$ does not contribute further poles.

Before finishing the proof, we recall for later purposes that for $s \in \mathbb{C}$, we have the bound (see [10], p. 90)

$$\big|Q(s)\big| \ll y^{1/2}, \tag{13}$$

where the implied constant depends only on $s$ (not a pole), but is independent of $z$ and $m$. □

**3.5. Definition.** For $z \in \mathbb{H}$, $s \in \mathbb{C}$, and $m \in \mathbb{Z}$, the *Poincaré series* $V_m$ is defined by

$$V_m(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s e\big(m\,\mathrm{Re}(\gamma z)\big). \tag{14}$$

The Poincaré series is known to be holomorphic for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, since it can be majorized by $V_0(z, \mathrm{Re}(s)) = \mathcal{E}_{\mathrm{par}}(z, \mathrm{Re}(s))$.

**3.6. Lemma.** *For $z \in \mathbb{H}$, $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, and $m \neq 0$, we have the relation*

$$V_m(z,s) = \sum_{k=0}^{\infty} \frac{(2\pi|m|)^k}{k!} P_m(z,s+k).$$

*Proof.* We first check the absolute and local uniform convergence of the series in the claimed relation for fixed $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Using the estimate

$$|cz+d| \geq C\,|ci+d|,$$

where $C = C(z)$ is a positive constant depending on $z$ but which is independent of $(c,d) \in \mathbb{R}^2$, we obtain the bound

$$\begin{aligned}
\sum_{k=0}^{\infty} \left| \frac{(2\pi|m|)^k}{k!} P_m(z,s+k) \right| &\leq \sum_{k=0}^{\infty} \frac{(2\pi|m|)^k}{k!} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}(\gamma z)^{\operatorname{Re}(s)+k} \\
&= \sum_{k=0}^{\infty} \frac{(2\pi|m|)^k}{k!} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \frac{y^{\operatorname{Re}(s)}}{|cz+d|^{2\operatorname{Re}(s)}} \cdot \frac{y^k}{|cz+d|^{2k}} \\
&\leq \sum_{k=0}^{\infty} \frac{(2\pi|m|yC^{-2})^k}{k!} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \frac{y^{\operatorname{Re}(s)}}{|cz+d|^{2\operatorname{Re}(s)}} \cdot \frac{1}{(c^2+d^2)^k} \\
&\leq \exp(2\pi|m|yC^{-2}) \cdot \mathcal{E}_{\mathrm{par}}\big(z,\operatorname{Re}(s)\big).
\end{aligned}$$

This proves that the series in question converges absolutely and locally uniformly for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$.

Now the claimed relation can easily be derived by changing the order of summation; namely, we compute

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(2\pi|m|)^k}{k!} P_m(z,s+k) \\
&= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}(\gamma z)^s \exp\big(-2\pi|m|\operatorname{Im}(\gamma z)\big) e\big(m\operatorname{Re}(\gamma z)\big) \sum_{k=0}^{\infty} \frac{(2\pi|m|)^k}{k!} \operatorname{Im}(\gamma z)^k \\
&= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}(\gamma z)^s \exp\big(-2\pi|m|\operatorname{Im}(\gamma z)\big) e\big(m\operatorname{Re}(\gamma z)\big) \exp\big(2\pi|m|\operatorname{Im}(\gamma z)\big) \\
&= V_m(z,s).
\end{aligned}$$

This completes the proof of the lemma. □

**3.7. Proposition.** *For $z \in \mathbb{H}$ and $m \neq 0$, the Poincaré series $V_m(z,s)$ admits a meromorphic continuation to the whole $s$-plane with simple poles at $s = s_j - 2N$ and $s = -s_j - 2N + 1$ ($N \in \mathbb{N}$) with residues*

$$\operatorname{res}_{s=s_j-2N} V_m(z,s) = \frac{2^{2N-1}\pi^{-s_j+2N+1}\Gamma(s_j-N-1/2)}{(2N)!\,\Gamma(-N+1/2)|m|^{s_j-2N-1/2}} \sum_{s_\ell=s_j} \overline{\rho}_\ell(m)\,\psi_\ell(z) \tag{15}$$

*and*

$$\operatorname{res}_{s=-s_j-2N+1} V_m(z,s) = \frac{2^{2N-1}\pi^{s_j+2N}\Gamma(-s_j-N+1/2)}{(2N)!\,\Gamma(-N+1/2)|m|^{-s_j-2N+1/2}} \sum_{s_\ell=s_j} \overline{\rho}_\ell(m)\,\psi_\ell(z), \tag{16}$$

*respectively.*

*Proof.* We start by proving that the Poincaré series $V_m(z,s)$ has a meromorphic continuation to the half-plane

$$\mathcal{H}'_N := \{s \in \mathbb{C} \mid \operatorname{Re}(s) > -N\}$$

for any $N \in \mathbb{N}$. By Lemma 3.6, we can write

$$V_m(z,s) = \sum_{k=0}^{N} \frac{(2\pi|m|)^k}{k!} P_m(z,s+k) + \sum_{k=N+1}^{\infty} \frac{(2\pi|m|)^k}{k!} P_m(z,s+k). \tag{17}$$

We show that the series

$$\sum_{k=N+1}^{\infty} \frac{(2\pi|m|)^k}{k!} P_m(z,s+k)$$

is a holomorphic function on the half-plane $\mathcal{H}'_N$. For this we estimate as in the proof of Lemma 3.6, assuming $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -N$,

$$\begin{aligned}
&\sum_{k=N+1}^{\infty} \left| \frac{(2\pi|m|)^k}{k!} P_m(z, s+k) \right| \\
&= \sum_{k=N+1}^{\infty} \left| \frac{(2\pi|m|)^{N+1}}{k!/(k-N-1)!} \cdot \frac{(2\pi|m|)^{k-N-1}}{(k-N-1)!} P_m\big(z, (s+N+1)+(k-N-1)\big) \right| \\
&\leq (2\pi|m|)^{N+1} \sum_{k=0}^{\infty} \left| \frac{(2\pi|m|)^k}{k!} P_m(z, s+N+1+k) \right| \\
&\leq (2\pi|m|)^{N+1} \cdot \exp(2\pi|m|yC^{-2}) \cdot \mathcal{E}_{\mathrm{par}}\big(z, \mathrm{Re}(s)+N+1\big).
\end{aligned}$$

This proves that the series in question converges absolutely and locally uniformly for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -N$, and hence the holomorphicity statement.

Since the finite sum $\sum_{k=0}^{N} (2\pi|m|)^k/k!\, P_m(z, s+k)$ is a meromorphic function on the whole $s$-plane by Proposition 3.4, we conclude that $V_m(z,s)$ has a meromorphic continuation to the half-plane $\mathcal{H}'_N$. Since $N$ was chosen arbitrarily, this proves the meromorphic continuation of $V_m(z,s)$ to the whole $s$-plane.

In order to determine the poles of $V_m(z,s)$, we calculate its poles in the strip

$$\mathcal{S}'_N := \{ s \in \mathbb{C} \mid -N < \mathrm{Re}(s) \leq -N+1 \}$$

for any $N \in \mathbb{N}$. By considering $V_m(z,s)$ with its decomposition (17) in the strip $\mathcal{S}'_N$, we see that the poles come from the finite sum $F_N(z,s) := \sum_{k=0}^{N} (2\pi|m|)^k/k!\, P_m(z, s+k)$. By Proposition 3.4, $F_N(z,s)$ has poles in the strip $\mathcal{S}'_N$ at $s = s_j - N$ and $s = -s_j - N + 1$. The explicit formula for the residues of $P_m(z,s)$ given in Proposition 3.4 now leads to the following residue of $F_N(z,s)$ at $s = s_j - N$:

$$\begin{aligned}
&\mathrm{res}_{s=s_j-N} F_N(z,s) = \sum_{k=0}^{N} \frac{(2\pi|m|)^k}{k!} \mathrm{res}_{s=s_j-(N-k)} P_m(z,s) \\
&= \sum_{k=0}^{N} \frac{(2\pi|m|)^k}{k!} \frac{(-1)^{N-k} 2^{-2s_j+2(N-k)+1} \pi^{-s_j+(N-k)+1} \Gamma(2s_j-(N-k)-1)}{(N-k)!\,\Gamma(s_j-(N-k))|m|^{s_j-(N-k)-1/2}} \sum_{s_\ell = s_j} \overline{\rho}_\ell(m)\, \psi_\ell(z) \\
&= \sum_{k=0}^{N} \frac{(-1)^{N-k} 2^{-2s_j+2N-k+1} \pi^{-s_j+N+1} \Gamma(2s_j-N+k-1)}{k!\,(N-k)!\,\Gamma(s_j-N+k)|m|^{s_j-N-1/2}} \sum_{s_\ell = s_j} \overline{\rho}_\ell(m)\, \psi_\ell(z) \\
&= \frac{(-1)^N\, 2^{N-1} \pi^{-s_j+N+1} \Gamma(s_j - N/2 - 1/2)}{N!\,\Gamma(-N/2+1/2)|m|^{s_j-N-1/2}} \sum_{s_\ell = s_j} \overline{\rho}_\ell(m)\, \psi_\ell(z).
\end{aligned}$$

This shows that the residue in question vanishes if $N$ is odd, and that the residue of $V_m(z,s)$ at $s = s_j - 2N$ is given by (15). Analoguously, it is shown that the residue of $F_N(z,s)$ at $s = -s_j - N + 1$ is zero if $N$ is odd, and that the residue of $V_m(z,s)$ at $s = -s_j - 2N + 1$ is given by (16). □

# 4 Elliptic Eisenstein series

**4.1. Definition.** For $z \in \mathbb{H}$ with $z \neq \gamma i$ for any $\gamma \in \Gamma$, and $s \in \mathbb{C}$, the *elliptic Eisenstein series* is defined by

$$\mathcal{E}_{\mathrm{ell}}(z,s) = \sum_{\gamma \in \Gamma_i \backslash \Gamma} \sinh\big(\varrho(\gamma z)\big)^{-s}.$$

**4.2. Lemma.** *(i) For $z \in \mathbb{H}$ with $z \neq \gamma i$ for any $\gamma \in \Gamma$, the elliptic Eisenstein series $\mathcal{E}_{\mathrm{ell}}(z,s)$ converges absolutely and locally uniformly for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, and hence defines a holomorphic function.*
*(ii) The elliptic Eisenstein series $\mathcal{E}_{\mathrm{ell}}(z,s)$ is invariant under the action of $\Gamma$, i.e., we have $\mathcal{E}_{\mathrm{ell}}(\gamma z,s) = \mathcal{E}_{\mathrm{ell}}(z,s)$ for any $\gamma \in \Gamma$.*
*(iii) For fixed $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the elliptic Eisenstein series $\mathcal{E}_{\mathrm{ell}}(z,s)$ converges absolutely and uniformly for $z$ in compacta $K \subseteq \mathbb{H}$ not containing any translate $\gamma i$ of $i$ by $\gamma \in \Gamma$.*

*Proof.* (i) To ease notation, we write $s = \sigma + it \in \mathbb{C}$; we assume that $\sigma = \mathrm{Re}(s) > 1$. We fix $z \in \mathbb{H}$ such that $z \neq \gamma i$ for any $\gamma \in \Gamma$. Since $\Gamma$ acts properly discontinuously on $\mathbb{H}$ and $z \neq \gamma i$ for any $\gamma \in \Gamma$, the minimum

$$R_1(z) := \min_{\gamma\in\Gamma} d_{\mathbb{H}}(i,\gamma z)$$

exists and is strictly positive. Introducing the quantity

$$C_1(z) := \frac{1-\exp\bigl(-2R_1(z)\bigr)}{2} > 0,$$

we derive the inequality

$$\frac{1-\exp\bigl(-2\varrho(\gamma z)\bigr)}{2} \geq C_1(z)$$

for all $\gamma \in \Gamma$. From this we obtain the estimate

$$\sinh\bigl(\varrho(\gamma z)\bigr) = \exp\bigl(\varrho(\gamma z)\bigr) \cdot \frac{1-\exp\bigl(-2\varrho(\gamma z)\bigr)}{2} \geq C_1(z) \cdot \exp\bigl(\varrho(\gamma z)\bigr),$$

again for all $\gamma \in \Gamma$. From this we derive the estimate

$$\sum_{\gamma\in\Gamma_i\backslash\Gamma} \Bigl|\sinh\bigl(\varrho(\gamma z)\bigr)^{-s}\Bigr| = \sum_{\gamma\in\Gamma_i\backslash\Gamma} \sinh\bigl(\varrho(\gamma z)\bigr)^{-\sigma} \leq C_1(z)^{-\sigma} \cdot \sum_{\gamma\in\Gamma_i\backslash\Gamma} \exp\bigl(-\sigma\varrho(\gamma z)\bigr).$$

In order to complete the proof of (i), we are left to show the local uniform convergence of the series

$$\sum_{\gamma\in\Gamma_i\backslash\Gamma} \exp\bigl(-\sigma\varrho(\gamma z)\bigr)$$

for $\sigma > 1$. To do this, we introduce for $r \in \mathbb{R}_{\geq 0}$ the quantities

$$G(r) := \bigl\{\gamma \in \Gamma_i\backslash\Gamma \bigm| \varrho(\gamma z) < r\bigr\}, \quad N(r) := \sharp G(r).$$

We note that the number $N(r)$ is finite, since $\Gamma$ acts properly discontinously on $\mathbb{H}$ and $z \neq \gamma i$ for any $\gamma \in \Gamma$; in particular, we have $N(r) = 0$ for $0 \leq r \leq R_1(z)$.
For fixed $r \in \mathbb{R}_{>0}$, we are next going to estimate the number $N(r)$. Let $\mathcal{B}_r(i)$ denote the open hyperbolic disk of radius $r$ centered at $i$ containing the finitely many translates $\gamma z$ of $z$ for $\gamma \in G(r)$. Then, there exists a constant $\varepsilon(z) > 0$, depending on $z$, such that the open hyperbolic disks $\mathcal{B}_{\varepsilon(z)}(\gamma z)$ of radius $\varepsilon(z)$ centered at $\gamma z$ do not intersect for all $\gamma \in G(r)$ and are contained in $\mathcal{B}_r(i)$. Consequently, we obtain

$$N(r) \cdot \mathrm{vol}_{\mathrm{hyp}}\bigl(\mathcal{B}_{\varepsilon(z)}(\gamma z)\bigr) \leq \mathrm{vol}_{\mathrm{hyp}}\bigl(\mathcal{B}_r(i)\bigr) \quad \bigl(\gamma \in G(r)\bigr).$$

This yields the estimate

$$\begin{aligned} N(r) &\leq \frac{4\pi\sinh^2\bigl(r/2\bigr)}{4\pi\sinh^2\bigl(\varepsilon(z)/2\bigr)} = \frac{\cosh(r)-1}{2\sinh^2\bigl(\varepsilon(z)/2\bigr)} = \frac{\exp(r)+\exp(-r)-2}{4\sinh^2\bigl(\varepsilon(z)/2\bigr)} \\ &< \exp(r) \cdot \frac{1+\exp(-2r)}{4\sinh^2\bigl(\varepsilon(z)/2\bigr)} < C_2(z) \cdot \exp(r) \qquad (18) \end{aligned}$$

with a suitable constant $C_2(z) > 0$ depending on $z$.

For fixed $R \in \mathbb{R}_{>0}$, the monotone increasing step function $N : [0,R] \longrightarrow \mathbb{N}$ induces a Stieltjes measure $dN(r)$ on the interval $[0,R]$. Since the function $\exp(-\sigma r) : [0,R] \longrightarrow \mathbb{R}_{>0}$ is continuous and the function $N(r)$ is of bounded variation, the function $\exp(-\sigma r)$ is Riemann–Stieltjes integrable with respect to $N(r)$ on the interval $[0,R]$. Furthermore, since $N(r)$ and $\exp(-\sigma r)$ are bounded on $[0,R]$, the theorem of partial integration can be applied to give

$$\begin{aligned}\sum_{\substack{\gamma\in\Gamma_i\backslash\Gamma\\ \gamma\in G(R)}} \exp\Big(-\sigma\varrho(\gamma z)\Big) &= \int_0^R \exp(-\sigma r)dN(r)\\ &= \Big[N(r)\exp(-\sigma r)\Big]_0^R - \int_0^R N(r)d\big(\exp(-\sigma r)\big)\\ &= \Big[N(r)\exp(-\sigma r)\Big]_0^R + \int_0^R \sigma N(r)\exp(-\sigma r)dr. \qquad (19)\end{aligned}$$

Using (18), the first summand of (19) can be bounded as

$$\Big[N(r)\exp(-\sigma r)\Big]_0^R = N(R)\exp(-\sigma R) < C_2(z)\exp\big((1-\sigma)R\big).$$

On the other hand, again using (18), the integral in (19) can be bounded as

$$\int_0^R \sigma N(r)\exp(-\sigma r)dr < \sigma C_2(z)\int_0^R \exp\big((1-\sigma)r\big)dr = \frac{\sigma C_2(z)}{1-\sigma}\Big(\exp\big((1-\sigma)R\big)-1\Big).$$

Summing up, we arrive at

$$\begin{aligned}\sum_{\gamma\in\Gamma_i\backslash\Gamma} \exp\big(-\sigma\varrho(\gamma z)\big) &= \lim_{R\to\infty} \sum_{\substack{\gamma\in\Gamma_i\backslash\Gamma\\ \gamma\in G(R)}} \exp\big(-\sigma\varrho(\gamma z)\big)\\ &\leq \lim_{R\to\infty}\Big[C_2(z)\exp\big((1-\sigma)R\big) + \frac{\sigma C_2(z)}{1-\sigma}\Big(\exp\big((1-\sigma)R\big)-1\Big)\Big]\\ &= \frac{\sigma C_2(z)}{\sigma-1},\end{aligned}$$

keeping in mind that $\sigma > 1$. The absolute and local uniform convergence of the elliptic Eisenstein series $\mathcal{E}_{\mathrm{ell}}(z,s)$ now follows for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

(ii) From definition 4.1 we immediately deduce for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$,

$$\mathcal{E}_{\mathrm{ell}}(\gamma z,s) = \mathcal{E}_{\mathrm{ell}}(z,s)$$

for all $\gamma \in \Gamma$, provided that $z \neq \gamma i$ for any $\gamma \in \Gamma$.

(iii) Let finally $K \subseteq \mathbb{H}$ be a compact subset not containing any translate $\gamma i$ of $i$ by $\gamma \in \Gamma$. Then, the constants $C_1(z)$ and $C_2(z)$ constructed in the first part of the proof can be chosen uniformly for all $z \in K$. For fixed $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the series $\mathcal{E}_{\mathrm{ell}}(z,s)$ therefore converges absolutely and uniformly on $K \subseteq \mathbb{H}$. □

**4.3. Lemma.** *For $z = x + iy \in \mathbb{H}$ with $z \neq \gamma i$ for any $\gamma \in \Gamma$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the elliptic Eisenstein series $\mathcal{E}_{\mathrm{ell}}(z,s)$ is twice continuously differentiable with respect to $x, y$.*

*Proof.* In order to prove the claim, we have to show in a first step that the series of partial derivatives

$$\sum_{\gamma\in\Gamma_i\backslash\Gamma} \frac{\partial}{\partial x}\sinh\big(\varrho(\gamma z)\big)^{-s}, \quad \sum_{\gamma\in\Gamma_i\backslash\Gamma} \frac{\partial}{\partial y}\sinh\big(\varrho(\gamma z)\big)^{-s} \qquad (20)$$

converge absolutely and uniformly on compacta $K \subseteq \mathbb{H}$ not containing any translate $\gamma i$ of $i$ by $\gamma \in \Gamma$ provided that $\sigma = \mathrm{Re}(s) > 1$. To do this, we introduce for functions $f \in \mathcal{C}^1(\mathbb{H})$ the notation

$$\nabla_{\mathrm{hyp}} f(z) := y^2 \Big( \Big( \frac{\partial f(z)}{\partial x} \Big)^2 + \Big( \frac{\partial f(z)}{\partial y} \Big)^2 \Big).$$

Letting $\varphi(x, y) := (a^2 + c^2)(x^2 + y^2) + 2(ab + cd)x + (b^2 + d^2)$, we have by (2),

$$\sinh\big(\varrho(\gamma z)\big) = \sqrt{\cosh^2\big(\varrho(\gamma z)\big) - 1} = \sqrt{\Big(\frac{\varphi(x, y)}{2y}\Big)^2 - 1}\,,$$

from which we derive

$$\frac{\partial}{\partial x} \sinh\big(\varrho(\gamma z)\big) = \frac{\cosh\big(\varrho(\gamma z)\big)}{\sinh\big(\varrho(\gamma z)\big)} \cdot \frac{\partial}{\partial x} \frac{\varphi(x, y)}{2y} = \coth\big(\varrho(\gamma z)\big) \cdot \frac{(a^2 + c^2)x + (ab + cd)}{y}\,,$$

and

$$\frac{\partial}{\partial y} \sinh\big(\varrho(\gamma z)\big) = \coth\big(\varrho(\gamma z)\big) \cdot \Big( (a^2 + c^2) - \frac{\varphi(x, y)}{2y^2} \Big)\,.$$

A straightforward computation yields

$$\nabla_{\mathrm{hyp}} \sinh\big(\varrho(\gamma z)\big) = \cosh^2\big(\varrho(\gamma z)\big), \tag{21}$$

from which we deduce

$$\Big| \frac{\partial}{\partial x} \sinh\big(\varrho(\gamma z)\big) \Big| \le y^{-1} \sqrt{\nabla_{\mathrm{hyp}} \sinh\big(\varrho(\gamma z)\big)} = y^{-1} \cosh\big(\varrho(\gamma z)\big)\,,$$
$$\Big| \frac{\partial}{\partial y} \sinh\big(\varrho(\gamma z)\big) \Big| \le y^{-1} \sqrt{\nabla_{\mathrm{hyp}} \sinh\big(\varrho(\gamma z)\big)} = y^{-1} \cosh\big(\varrho(\gamma z)\big)\,.$$

By the choice of the compact set $K$, there is a positive constant $C_K$ such that the inequality $\cosh\big(\varrho(\gamma z)\big) \le C_K \cdot \sinh\big(\varrho(\gamma z)\big)$ holds for all $z \in K$. Therefore, we obtain for $z \in K$,

$$\Big| \frac{\partial}{\partial x} \sinh\big(\varrho(\gamma z)\big)^{-s} \Big| \le C_K \cdot |s| \cdot y^{-1} \cdot \sinh\big(\varrho(\gamma z)\big)^{-\sigma}\,,$$
$$\Big| \frac{\partial}{\partial y} \sinh\big(\varrho(\gamma z)\big)^{-s} \Big| \le C_K \cdot |s| \cdot y^{-1} \cdot \sinh\big(\varrho(\gamma z)\big)^{-\sigma}\,.$$

The absolute and locally uniform convergence for the series (20) now follows from Lemma 4.2 provided that $\sigma > 1$.

To ease notation, we put for the second step $x_1 := x$ and $x_2 := y$. We will then show that for $j, k = 1, 2$ the series

$$\sum_{\gamma \in \Gamma_i \backslash \Gamma} \frac{\partial^2}{\partial x_j \partial x_k} \sinh\big(\varrho(\gamma z)\big)^{-s} \tag{22}$$

converge absolutely and uniformly on compacta $K \subseteq \mathbb{H}$ not containing any translate $\gamma i$ of $i$ by $\gamma \in \Gamma$ provided that $\sigma = \mathrm{Re}(s) > 1$. Setting $f(z) := \sinh\big(\varrho(\gamma z)\big)$, we estimate for $z \in K$,

$$\begin{aligned}
&\Big| \frac{\partial^2}{\partial x_j \partial x_k} \sinh\big(\varrho(\gamma z)\big)^{-s} \Big| \\
&= \Big| (-s)(-s-1) \cdot f(z)^{-(s+2)} \cdot \frac{\partial f(z)}{\partial x_j} \cdot \frac{\partial f(z)}{\partial x_k} + (-s) f(z)^{-(s+1)} \cdot \frac{\partial^2 f(z)}{\partial x_j \partial x_k} \Big| \\
&\le |s^2 + s| \cdot f(z)^{-(\sigma+2)} \cdot \Big| \frac{\partial f(z)}{\partial x_j} \Big| \cdot \Big| \frac{\partial f(z)}{\partial x_k} \Big| + |s| \cdot f(z)^{-(\sigma+1)} \cdot \Big| \frac{\partial^2 f(z)}{\partial x_j \partial x_k} \Big| \\
&\le C_K^2 \cdot |s^2 + s| \cdot x_2^{-2} \cdot f(z)^{-\sigma} + |s| \cdot f(z)^{-(\sigma+1)} \cdot \Big| \frac{\partial^2 f(z)}{\partial x_j \partial x_k} \Big|\,.
\end{aligned}$$

We are left to estimate the term $|\partial^2 f(z)/\partial x_j \partial x_k|$. For this, we use the fact that for real functions $g(z) = g(x_1, x_2)$ defined on $\mathbb{H}$ with continuous first- and second-order partial derivatives, the inequality

$$\Big|\frac{\partial^2 g(z)}{\partial x_j \partial x_k}\Big| \leq x_2^{-2}\Big(\sqrt{\nabla_{\mathrm{hyp}} g(z)} + \frac{\sqrt{\nabla^2_{\mathrm{hyp}} g(z)}}{\sqrt{\nabla_{\mathrm{hyp}} g(z)}} + |\Delta_{\mathrm{hyp}} g(z)|\Big)$$

holds for all $z = x_1 + ix_2 \in \mathbb{H}$ provided that $\nabla_{\mathrm{hyp}} g(z) \neq 0$ (see [5]). Using (21), we obtain

$$\nabla^2_{\mathrm{hyp}} \sinh\big(\varrho(\gamma z)\big) = \nabla_{\mathrm{hyp}} \cosh^2\big(\varrho(\gamma z)\big) = 4\cosh^2\big(\varrho(\gamma z)\big)\sinh^2\big(\varrho(\gamma z)\big),$$

which yields

$$\frac{\nabla^2_{\mathrm{hyp}} \sinh\big(\varrho(\gamma z)\big)}{\nabla_{\mathrm{hyp}} \sinh\big(\varrho(\gamma z)\big)} = 4\sinh^2\big(\varrho(\gamma z)\big).$$

This, together with the relation $|\Delta_{\mathrm{hyp}} \sinh\big(\varrho(\gamma z)\big)| = 2\sinh\big(\varrho(\gamma z)\big) + \sinh\big(\varrho(\gamma z)\big)^{-1}$, leads to

$$\sqrt{\nabla_{\mathrm{hyp}} f(z)} + \frac{\sqrt{\nabla^2_{\mathrm{hyp}} f(z)}}{\sqrt{\nabla_{\mathrm{hyp}} f(z)}} + |\Delta_{\mathrm{hyp}} f(z)| = \cosh\big(\varrho(\gamma z)\big) + 4\sinh\big(\varrho(\gamma z)\big) + \sinh\big(\varrho(\gamma z)\big)^{-1}.$$

Therefore, by the choice of the compact set $K$, there is a positive constant $C'_K$ such that the inequality

$$\Big|\frac{\partial^2 f(z)}{\partial x_j \partial x_k}\Big| \leq C'_K \cdot x_2^{-2} \cdot \sinh\big(\varrho(\gamma z)\big)$$

holds for $z \in K$. Again, the absolute and locally uniform convergence for the series (22) now follows from Lemma 4.2 provided that $\sigma > 1$.

This concludes the proof of the lemma. □

**4.4. Lemma.** *For $z \in \mathbb{H}$ with $z \neq \gamma i$ for any $\gamma \in \Gamma$, and $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > 1$, *the elliptic Eisenstein series $\mathcal{E}_{\mathrm{ell}}(z,s)$ satisfies the differential equation*

$$\big(\Delta_{\mathrm{hyp}} - s(1-s)\big)\mathcal{E}_{\mathrm{ell}}(z,s) = -s^2\mathcal{E}_{\mathrm{ell}}(z,s+2).$$

*Proof.* Since the differential operator

$$\Delta_{\mathrm{hyp}} = -\frac{\partial^2}{\partial\varrho^2} - \frac{1}{\tanh(\varrho)}\frac{\partial}{\partial\varrho} - \frac{1}{\sinh^2(\varrho)}\frac{\partial^2}{\partial\vartheta^2}$$

is invariant under the action of $\Gamma$, it suffices by Lemma 4.3 to prove the equality

$$\big(\Delta_{\mathrm{hyp}} - s(1-s)\big)\sinh(\varrho)^{-s} = -s^2\sinh(\varrho)^{-(s+2)}.$$

This follows immediately from the subsequent calculation

$$\begin{aligned}\Delta_{\mathrm{hyp}} \sinh(\varrho)^{-s} &= s(-s-1)\sinh(\varrho)^{-(s+2)}\cosh^2(\varrho) + s\sinh(\varrho)^{-s} + s\sinh(\varrho)^{-(s+2)}\cosh^2(\varrho)\\ &= (-s^2 - s + s)\sinh(\varrho)^{-(s+2)}\big(1+\sinh^2(\varrho)\big) + s\sinh(\varrho)^{-s}\\ &= -s^2\sinh(\varrho)^{-(s+2)} + s(1-s)\sinh(\varrho)^{-s}.\end{aligned}$$

□

# 5 Fourier expansion of the elliptic Eisenstein series

**5.1. Lemma.** *For $z \in \mathbb{H}$ with $\mathrm{Im}(z) \neq \mathrm{Im}(\gamma^{-1}i)$ for any $\gamma \in \Gamma$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the elliptic Eisenstein series $\mathcal{E}_{\mathrm{ell}}(z,s)$ admits the Fourier expansion*

$$\mathcal{E}_{\mathrm{ell}}(z,s) = \sum_{m \in \mathbb{Z}} a_m(y,s) e(mx),$$

*where*

$$a_m(y,s) = \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_i \backslash \Gamma / \Gamma_\infty} e\Big(m \frac{ab+cd}{a^2+c^2}\Big) \int\limits_{-\infty}^{\infty} \Big(-1 + \Big(\frac{a^2+c^2}{2y} t^2 + \frac{(a^2+c^2)^2 y^2 + 1}{2(a^2+c^2)y}\Big)^2\Big)^{-s/2} e(-mt)dt.$$

*Proof.* Since $\mathcal{E}_{\mathrm{ell}}(z+1,s) = \mathcal{E}_{\mathrm{ell}}(z,s)$, the series $\mathcal{E}_{\mathrm{ell}}(z,s)$ admits the Fourier expansion

$$\mathcal{E}_{\mathrm{ell}}(z,s) = \sum_{m \in \mathbb{Z}} a_m(y,s) e(mx),$$

where

$$\begin{aligned} a_m(y,s) = \int\limits_0^1 \mathcal{E}_{\mathrm{ell}}(z,s) e(-mx)dx &= \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_i \backslash \Gamma} \int\limits_0^1 \sinh\big(\varrho(\gamma z)\big)^{-s} e(-mx)dx \\ &= \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_i \backslash \Gamma / \Gamma_\infty} \sum_{n \in \mathbb{Z}} \int\limits_0^1 \sinh\big(\varrho(\gamma(z+n))\big)^{-s} e(-mx)dx \\ &= \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_i \backslash \Gamma / \Gamma_\infty} \int\limits_{-\infty}^{\infty} \sinh\big(\varrho(\gamma z)\big)^{-s} e(-mx)dx. \end{aligned}$$

Now, writing $\sinh^2\big(\varrho(\gamma z)\big) = -1 + \cosh^2\big(\varrho(\gamma z)\big)$, using (2), and substituting $t := x + \frac{ab+cd}{a^2+c^2}$, we obtain

$$\begin{aligned} \cosh\big(\varrho(\gamma z)\big) &= \frac{1}{2y}\Big((a^2+c^2)t^2 + (a^2+c^2)y^2 + \frac{1}{a^2+c^2}\Big) \\ &= \frac{a^2+c^2}{2y} t^2 + \frac{(a^2+c^2)^2 y^2 + 1}{2(a^2+c^2)y}. \end{aligned}$$

From this the claimed formula for $a_m(y,s)$ follows immediately. □

**5.2. Proposition.** *For $z \in \mathbb{H}$ with $\mathrm{Im}(z) > 1$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have*

$$a_0(y,s) = \frac{2^s \sqrt{\pi}\, \Gamma(s-1/2)}{\Gamma(s)} \cdot y^{1-s} \sum_{k=0}^{\infty} \frac{(s-\frac{1}{2})_k \cdot (\frac{s}{2})_k}{(\frac{s}{2}+\frac{1}{2})_k \cdot k!} \cdot y^{-2k} \cdot V_0(s+2k)\,,$$

*where*

$$V_0(s) := \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_i \backslash \Gamma / \Gamma_\infty} \frac{1}{(a^2+c^2)^s}.$$

*Proof.* Letting $m=0$, we derive from Lemma 5.1

$$a_0(y,s) = \sum_{\gamma=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\Gamma_i\backslash\Gamma/\Gamma_\infty} b_{0,\gamma}(y,s),$$

where

$$\begin{aligned} b_{0,\gamma}(y,s) &:= 2\int_0^\infty \Big(-1+\Big(\frac{a^2+c^2}{2y}t^2+\frac{(a^2+c^2)^2y^2+1}{2(a^2+c^2)y}\Big)^2\Big)^{-s/2}dt \\ &= \int_0^\infty \Big(-1+\Big(\frac{a^2+c^2}{2y}t+\frac{(a^2+c^2)^2y^2+1}{2(a^2+c^2)y}\Big)^2\Big)^{-s/2}\frac{dt}{\sqrt{t}}. \end{aligned}$$

Substituting

$$r := \frac{((a^2+c^2)y+1)^2}{(a^2+c^2)^2}\Big(t+\frac{((a^2+c^2)y+1)^2}{(a^2+c^2)^2}\Big)^{-1},$$

we obtain

$$b_{0,\gamma}(y,s) = \frac{2^s y^s (a^2+c^2)^{s-1}}{((a^2+c^2)y+1)^{2s-1}}\int_0^1 r^{s-3/2}(1-r)^{-1/2}\Big(1-\frac{4(a^2+c^2)y}{((a^2+c^2)y+1)^2}\cdot r\Big)^{-s/2}dr.$$

Now using the integral representation (3) of Gauss's hypergeometric function $F(a',b';c';w)$ with

$$a' := \frac{s}{2},\quad b' := s-\frac{1}{2},\quad c' := s,\quad \text{and}\quad w := \frac{4(a^2+c^2)y}{((a^2+c^2)y+1)^2},$$

which is justified since $\mathrm{Re}(c') > \mathrm{Re}(b') = \mathrm{Re}(s)-1/2>0$, we obtain

$$b_{0,\gamma}(y,s) = \frac{2^s y^s (a^2+c^2)^{s-1}}{((a^2+c^2)y+1)^{2s-1}}\cdot\frac{\sqrt{\pi}\,\Gamma(s-1/2)}{\Gamma(s)}\cdot F\Big(\frac{s}{2},s-\frac{1}{2};s;\frac{4(a^2+c^2)y}{((a^2+c^2)y+1)^2}\Big).$$

Since $\Gamma=\mathrm{PSL}_2(\mathbb{Z})$ and $y>1$, we have

$$\frac{4(a^2+c^2)y}{((a^2+c^2)y+1)^2}<1,$$

and so the hypergeometric function in question can be represented as a series, which shows that

$$F\Big(\frac{s}{2},s-\frac{1}{2};s;\frac{4(a^2+c^2)y}{((a^2+c^2)y+1)^2}\Big) = F\Big(s-\frac{1}{2},\frac{s}{2};s;\frac{4(a^2+c^2)y}{((a^2+c^2)y+1)^2}\Big).$$

Now, the hypergeometric function under consideration is of the form $F(b',a';2a';w)$, which allows us to apply the following formula (see [1], formula 15.3.17):

$$F(b',a';2a';w) = 2^{2b'}(1+\sqrt{1-w})^{-2b'}F\Big(b',b'-a'+\frac{1}{2};a'+\frac{1}{2};\Big(\frac{1-\sqrt{1-w}}{1+\sqrt{1-w}}\Big)^2\Big). \tag{23}$$

Again, since $y>1$, we have

$$\sqrt{1-\frac{4(a^2+c^2)y}{((a^2+c^2)y+1)^2}} = \frac{(a^2+c^2)y-1}{(a^2+c^2)y+1},$$

which leads to

$$\begin{aligned} &F\Big(s-\frac{1}{2},\frac{s}{2};s;\frac{4(a^2+c^2)y}{((a^2+c^2)y+1)^2}\Big) \\ &= 2^{2s-1}\Big(\frac{2(a^2+c^2)y}{(a^2+c^2)y+1}\Big)^{-2s+1}F\Big(s-\frac{1}{2},\frac{s}{2};\frac{s}{2}+\frac{1}{2};\frac{1}{(a^2+c^2)^2y^2}\Big) \\ &= ((a^2+c^2)y)^{-2s+1}((a^2+c^2)y+1)^{2s-1}F\Big(s-\frac{1}{2},\frac{s}{2};\frac{s}{2}+\frac{1}{2};\frac{1}{(a^2+c^2)^2y^2}\Big). \end{aligned} \tag{24}$$

Adding up, we obtain

$$b_{0,\gamma}(y,s)=\frac{2^s\sqrt{\pi}\,\Gamma(s-1/2)}{\Gamma(s)}\cdot\frac{y^{1-s}}{(a^2+c^2)^s}\cdot F\Big(s-\frac{1}{2},\frac{s}{2};\frac{s}{2}+\frac{1}{2};\frac{1}{(a^2+c^2)^2y^2}\Big).$$

Introducing the notation

$$g(s):=\frac{2^s\sqrt{\pi}\,\Gamma(s-1/2)}{\Gamma(s)},$$

we arrive at

$$\begin{aligned}a_0(y,s)&=g(s)\cdot y^{1-s}\sum_{\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_i\backslash\Gamma/\Gamma_\infty}\frac{1}{(a^2+c^2)^s}\,F\Big(s-\frac{1}{2},\frac{s}{2};\frac{s}{2}+\frac{1}{2};\frac{1}{(a^2+c^2)^2y^2}\Big)\\&=g(s)\cdot y^{1-s}\sum_{\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_i\backslash\Gamma/\Gamma_\infty}\frac{1}{(a^2+c^2)^s}\sum_{k=0}^{\infty}\frac{(s-\frac{1}{2})_k\cdot(\frac{s}{2})_k}{(\frac{s}{2}+\frac{1}{2})_k\cdot k!}\Big(\frac{1}{(a^2+c^2)^2y^2}\Big)^k\\&=g(s)\cdot y^{1-s}\sum_{k=0}^{\infty}\frac{(s-\frac{1}{2})_k\cdot(\frac{s}{2})_k}{(\frac{s}{2}+\frac{1}{2})_k\cdot k!}\cdot y^{-2k}\cdot V_0(s+2k).\end{aligned}$$

This completes the proof of the proposition. □

**5.3. Remark.** The statement of Proposition 5.2 can easily be generalized to the case $z\in\mathbb{H}$ with $\operatorname{Im}(z)\neq\operatorname{Im}(\gamma^{-1}i)$ for any $\gamma\in\Gamma$ as follows: When applying formula (23) in the case $y<(a^2+c^2)^{-1}$, formula (24) becomes

$$F\Big(s-\frac{1}{2},\frac{s}{2};s;\frac{4(a^2+c^2)y}{((a^2+c^2)y+1)^2}\Big)=((a^2+c^2)y+1)^{2s-1}F\Big(s-\frac{1}{2},\frac{s}{2};s+\frac{1}{2};(a^2+c^2)^2y^2\Big).$$

Therefore, we arrive at

$$\begin{aligned}a_0(y,s)=\,&g(s)\cdot y^{1-s}\sum_{\substack{\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_i\backslash\Gamma/\Gamma_\infty\\ y>(a^2+c^2)^{-1}}}(a^2+c^2)^{-s}\sum_{k=0}^{\infty}\frac{(s-\frac{1}{2})_k\cdot(\frac{s}{2})_k}{(\frac{s}{2}+\frac{1}{2})_k\cdot k!}\cdot\big((a^2+c^2)y\big)^{-2k}\\&+g(s)\cdot y^{s}\sum_{\substack{\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_i\backslash\Gamma/\Gamma_\infty\\ y<(a^2+c^2)^{-1}}}(a^2+c^2)^{s-1}\sum_{k=0}^{\infty}\frac{(s-\frac{1}{2})_k\cdot(\frac{s}{2})_k}{(s+\frac{1}{2})_k\cdot k!}\cdot\big((a^2+c^2)y\big)^{2k}.\end{aligned}$$

**5.4. Proposition.** *For $z\in\mathbb{H}$ with $\operatorname{Im}(z)>1$, $s\in\mathbb{C}$ with $\operatorname{Re}(s)>1$, and $m\neq0$, we have*

$$a_m(y,s)=2^sy^s\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(\frac{s}{2})_{k_1}\cdot(\frac{s}{2})_{k_2}}{k_1!\cdot k_2!}\cdot I_{-m}(y,s;k_1,k_2)\cdot V_{-m}(s+2k_1+2k_2),$$

*where*

$$I_m(y,s;k_1,k_2):=\int_{-\infty}^{\infty}(y+it)^{-s-2k_1}(y-it)^{-s-2k_2}e(mt)\,dt$$

*and*

$$V_m(s)=\sum_{\gamma\in\Gamma_i\backslash\Gamma/\Gamma_\infty}\operatorname{Im}(\gamma^{-1}i)^se\big(m\operatorname{Re}(\gamma^{-1}i)\big).$$

*Proof.* For $m \neq 0$, we derive from Lemma 5.1

$$a_m(y,s) = \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_i \backslash \Gamma / \Gamma_\infty} e\Big(m \frac{ab+cd}{a^2+c^2}\Big) b_{m,\gamma}(y,s)\,,$$

where

$$b_{m,\gamma}(y,s) := \int_{-\infty}^{\infty} \Big(-1 + \Big(\frac{a^2+c^2}{2y}t^2 + \frac{(a^2+c^2)^2y^2+1}{2(a^2+c^2)y}\Big)^2\Big)^{-s/2} e(-mt)\, dt\,. \tag{25}$$

We write

$$\begin{aligned}
&-1 + \Big(\frac{a^2+c^2}{2y}t^2 + \frac{(a^2+c^2)^2y^2+1}{2(a^2+c^2)y}\Big)^2 \\
&= \frac{(a^2+c^2)^2}{(2y)^2}\Big(t^2 + \Big(y + \frac{1}{a^2+c^2}\Big)^2\Big)\Big(t^2 + \Big(y - \frac{1}{a^2+c^2}\Big)^2\Big) \\
&= \frac{(a^2+c^2)^2}{(2y)^2}\Big(it + y + \frac{1}{a^2+c^2}\Big)\Big(-it + y + \frac{1}{a^2+c^2}\Big)\Big(it + y - \frac{1}{a^2+c^2}\Big)\Big(-it + y - \frac{1}{a^2+c^2}\Big) \\
&= \frac{(a^2+c^2)^2}{(2y)^2}(y+it)^2(y-it)^2\Big(1 - \frac{1}{(a^2+c^2)^2(y+it)^2}\Big)\Big(1 - \frac{1}{(a^2+c^2)^2(y-it)^2}\Big).
\end{aligned} \tag{26}$$

Since $y > 1$, we have the estimate

$$\max_{-\infty < t < \infty}\Big(\frac{1}{|(a^2+c^2)^2(y \pm it)^2|}\Big) = \frac{1}{(a^2+c^2)^2y^2} < 1,$$

and hence we can write

$$\Big(1 - \frac{1}{(a^2+c^2)^2(y \pm it)^2}\Big)^{-s/2} = \sum_{k=0}^{\infty} \frac{(\frac{s}{2})_k}{k!(a^2+c^2)^{2k}} \cdot (y \pm it)^{-2k}\,.$$

Therefore, we obtain

$$\begin{aligned}
b_{m,\gamma}(y,s) = \frac{(2y)^s}{(a^2+c^2)^s} &\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(\frac{s}{2})_{k_1} \cdot (\frac{s}{2})_{k_2}}{k_1! \cdot k_2! \cdot (a^2+c^2)^{2(k_1+k_2)}} \\
&\times \int_{-\infty}^{\infty} (y+it)^{-s-2k_1}(y-it)^{-s-2k_2} e(-mt)\, dt\,,
\end{aligned} \tag{27}$$

from which the statement follows. □

**5.5. Remark.** The statement of Proposition 5.4 can be generalized to the case $z \in \mathbb{H}$ with $\mathrm{Im}(z) \neq \mathrm{Im}(\gamma^{-1}i)$ for any $\gamma \in \Gamma$. In this case the Fourier coefficient in question becomes

$$a_m(y,s) = \sum_{\substack{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_i \backslash \Gamma / \Gamma_\infty \\ y > (a^2+c^2)^{-1}}} e\Big(m\frac{ab+cd}{a^2+c^2}\Big)\, b^{(>)}_{m,\gamma}(y,s) \;+ \sum_{\substack{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_i \backslash \Gamma / \Gamma_\infty \\ y < (a^2+c^2)^{-1}}} e\Big(m\frac{ab+cd}{a^2+c^2}\Big)\, b^{(<)}_{m,\gamma}(y,s)\,,$$

where

$$b^{(>)}_{m,\gamma}(y,s) = \frac{2^sy^s}{(a^2+c^2)^{1-s}} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(\frac{s}{2})_{k_1} \cdot (\frac{s}{2})_{k_2}}{k_1! \cdot k_2!} \cdot I_{-m/(a^2+c^2)}\Big((a^2+c^2)y, s; k_1, k_2\Big), \tag{28}$$

$$b^{(<)}_{m,\gamma}(y,s) = \frac{2^sy^{1-s}}{(a^2+c^2)^s} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(\frac{s}{2})_{k_1} \cdot (\frac{s}{2})_{k_2}}{k_1! \cdot k_2!} \cdot I_{-my}\Big(\frac{1}{(a^2+c^2)y}, s; k_1, k_2\Big). \tag{29}$$

Here $b^{(>)}_{m,\gamma}(y,s)$ is obtained as in the proof of Proposition 5.4 (after a suitable substitution in (27)), whereas $b^{(<)}_{m,\gamma}(y,s)$ is obtained by rewriting (26) as

$$-1+\Big(\frac{a^2+c^2}{2y}t^2+\frac{(a^2+c^2)^2y^2+1}{2(a^2+c^2)y}\Big)^2=\frac{(a^2+c^2)^2y^2}{4}\Big(\frac{1}{(a^2+c^2)y}+\frac{it}{y}\Big)^2\Big(\frac{1}{(a^2+c^2)y}-\frac{it}{y}\Big)^2$$
$$\times\left(1-\Big(\frac{1}{(a^2+c^2)y}+\frac{it}{y}\Big)^{-2}\right)\left(1-\Big(\frac{1}{(a^2+c^2)y}-\frac{it}{y}\Big)^{-2}\right),$$

which, after using the expansion

$$\left(1-\Big(\frac{1}{(a^2+c^2)y}\pm\frac{it}{y}\Big)^{-2}\right)^{-s/2}=\sum_{k=0}^{\infty}\frac{(\frac{s}{2})_k}{k!}\cdot\Big(\frac{1}{(a^2+c^2)y}\pm\frac{it}{y}\Big)^{-2k},$$

yields the claimed formula (again after a suitable substitution in the corresponding integral).

**5.6. Remark.** The series $V_m(s)$ $(m\in\mathbb{Z})$ of Propositions 5.2 and 5.4 can be rewritten as follows: Consider the anti-isomorphism $\phi:\Gamma\to\Gamma$ given by $\gamma\mapsto\gamma^{-1}$. Since $\phi(\Gamma_\infty)=\Gamma_\infty$ and $\phi(\Gamma_i)=\Gamma_i$, we have $\phi(\Gamma_\infty\backslash\Gamma/\Gamma_i)=\Gamma_i\backslash\Gamma/\Gamma_\infty$. Therefore, we obtain

$$V_m(s)=\sum_{\gamma\in\Gamma_i\backslash\Gamma/\Gamma_\infty}\operatorname{Im}(\gamma^{-1}i)^s e\big(m\operatorname{Re}(\gamma^{-1}i)\big)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma/\Gamma_i}\operatorname{Im}(\gamma i)^s e\big(m\operatorname{Re}(\gamma i)\big)=\frac{1}{2}V_m(i,s)$$

with the Poincaré series (14) evaluated at $z=i$. Note that the series $V_0(s)$ multiplied by $\zeta(2s)$ equals the Dedekind zeta function associated to the field of $\mathbb{Q}(i)$.

# 6 Meromorphic continuation of the elliptic Eisenstein series

**6.1. Lemma.** *The series*

$$V_0(s)=\sum_{\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_i\backslash\Gamma/\Gamma_\infty}\frac{1}{(a^2+c^2)^s}$$

*converges absolutely and locally uniformly for $s\in\mathbb{C}$ with $\operatorname{Re}(s)>1$, and hence defines a holomorphic function. It has a meromorphic continuation to the whole $s$-plane with a simple pole at $s=1$ and poles at $s=\rho/2$, where $\rho$ is a non-trivial zero of $\zeta(s)$. Furthermore, we have $V_0(1/2)=0$.*

*Proof.* Since $V_0(s)=\mathcal{E}_{\mathrm{par}}(i,s)/2$, the claimed assertions immediately follow from the known properties of the parabolic Eisenstein series $\mathcal{E}_{\mathrm{par}}(z,s)$ recalled in Section 2.4. In particular, the vanishing of $V_0(s)$ at $s=1/2$ follows from the functional equation (6) by observing that $\varphi(1/2)=-1$. □

**6.2. Lemma.** *For $z\in\mathbb{H}$ with $\operatorname{Im}(z)>1$, and $N\in\mathbb{N}$, the series*

$$\sum_{k=N+1}^{\infty}\frac{(s-\frac{1}{2})_k\cdot(\frac{s}{2})_k}{\Gamma(\frac{s}{2}+\frac{1}{2}+k)\cdot k!}\cdot y^{-2k}\cdot V_0(s+2k)$$

*converges absolutely and locally uniformly for $s\in\mathbb{C}$ with $\operatorname{Re}(s)>-2N-1$, and hence defines a holomorphic function.*

*Proof.* Fix $N\in\mathbb{N}$, and let $s\in\mathbb{C}$ with $\operatorname{Re}(s)>-2N-1$. Then, for $k\in\mathbb{N}$, we define the functions

$$f_k(y,s):=g_k(y,s)\cdot V_0(s+2k),\ \text{ where }g_k(y,s):=\frac{(s-\frac{1}{2})_k\cdot(\frac{s}{2})_k}{\Gamma(\frac{s}{2}+\frac{1}{2}+k)\cdot k!}\cdot y^{-2k}.$$

If $k\geq N+1$, we have $\operatorname{Re}(s+2k)\geq\operatorname{Re}(s)+2N+2>1$, whence the functions $V_0(s+2k)$ are holomorphic. Since the functions $g_k(y,s)$ are also holomorphic in the range under consideration, the functions $f_k(y,s)$ are holomorphic for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>-2N-1$ as long as $k\geq N+1$. We now estimate

$$\sum_{k=N+1}^{\infty}\bigl|f_k(y,s)\bigr|\leq V_0\bigl(\mathrm{Re}(s)+2N\bigr)\sum_{k=N+1}^{\infty}\bigl|g_k(y,s)\bigr|.$$

Since the ratio of successive terms in the latter series has limit

$$\lim_{k\to\infty}\frac{|g_{k+1}(y,s)|}{|g_k(y,s)|}=\lim_{k\to\infty}\left|\frac{\bigl(s-\frac{1}{2}+k\bigr)\bigl(\frac{s}{2}+k\bigr)}{\bigl(\frac{s}{2}+\frac{1}{2}+k\bigr)(1+k)}\cdot\frac{1}{y^2}\right|=\frac{1}{y^2}<1,$$

we derive from d'Alembert's criterion that the series $\sum_{k=N+1}^{\infty}f_k(y,s)$ converges absolutely and locally uniformly for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>-2N-1$, which proves the claim. □

**6.3. Proposition.** *For $z\in\mathbb{H}$ with $\mathrm{Im}(z)>1$, the function $a_0(y,s)$ has a meromorphic continuation to the whole $s$-plane with possible poles at $s=1-2N$, $s=\rho/2-2N$, $s=1/2-2N$, and $s=-1/2-2N$ ($N\in\mathbb{N}$), where $\rho$ is a non-trivial zero of $\zeta(s)$.*

*Proof.* We start by proving that the function $a_0(y,s)$ has a meromorphic continuation to the half-plane

$$\mathcal{H}_N:=\{s\in\mathbb{C}\mid\mathrm{Re}(s)>-2N-1\}$$

for any $N\in\mathbb{N}$. By Proposition 5.2 and the duplication formula for the $\Gamma$-function, we can write, using the notation from the proof of Lemma 6.2,

$$\begin{aligned}a_0(y,s)&=\frac{2^s\sqrt{\pi}\,\Gamma(s-\frac{1}{2})\Gamma(\frac{s}{2}+\frac{1}{2})}{\Gamma(s)}\cdot y^{1-s}\left(\sum_{k=0}^{N}f_k(y,s)+\sum_{k=N+1}^{\infty}f_k(y,s)\right)\\&=\frac{2\pi\,\Gamma(s-\frac{1}{2})}{\Gamma(\frac{s}{2})}\cdot y^{1-s}\left(\sum_{k=0}^{N}f_k(y,s)+\sum_{k=N+1}^{\infty}f_k(y,s)\right).\end{aligned}\tag{30}$$

Since $\mathrm{Re}(s)>-2N-1$ by assumption, Lemma 6.2 proves that the series $\sum_{k=N+1}^{\infty}f_k(y,s)$ is a holomorphic function on the half-plane $\mathcal{H}_N$. Since the finite sum $\sum_{k=0}^{N}f_k(y,s)$ is a meromorphic function on the whole $s$-plane by Lemma 6.1, we conclude that $a_0(y,s)$ has a meromorphic continuation to the half-plane $\mathcal{H}_N$. Since $N$ was chosen arbitrarily, this proves the meromorphic continuation of $a_0(y,s)$ to the whole $s$-plane.

In order to determine the poles of $a_0(y,s)$, we calculate its poles in the strip

$$\mathcal{S}_N:=\{s\in\mathbb{C}\mid-2N-1<\mathrm{Re}(s)\leq-2N+1\}$$

for any $N\in\mathbb{N}$. By considering $a_0(y,s)$ with its decomposition (30) in the strip $\mathcal{S}_N$, we see that the poles come from the finite sum $\sum_{k=0}^{N}f_k(y,s)$, which has poles in the strip $\mathcal{S}_N$ arising from the function $f_N(y,s)$, more precisely from the factor $V_0(s+2N)$ at $s=1-2N$ and $s=\rho/2-2N$, where $\rho$ is a non-trivial zero of $\zeta(s)$, and from the $\Gamma$-factor $\Gamma(s-1/2)$ at $s=1/2-2N$ and $s=-1/2-2N$. Therefore, the possible poles of $a_0(y,s)$ in the strip $\mathcal{S}_N$ are located at $s=1-2N$, $s=\rho/2-2N$, $s=1/2-2N$, and $s=-1/2-2N$, as claimed. □

**6.4. Remark.** Using Remark 5.3, one can establish the meromorphic continuation of $a_0(y,s)$ to the whole $s$-plane in the more general case $\mathrm{Im}(z)\neq\mathrm{Im}(\gamma^{-1}i)$ for any $\gamma\in\Gamma$, using the same techniques as in Lemma 6.2 and Proposition 6.3 applied accordingly to the modified situation. The poles of $a_0(y,s)$ turn out to be same as in the case $\mathrm{Im}(z)>1$.

**6.5. Lemma.** *For $y>1$, $s\in\mathbb{C}$, $m\neq0$, and $k_1,k_2\in\mathbb{N}$, let $I_m(y,s;k_1,k_2)$ denote the integral*

$$I_m(y,s;k_1,k_2):=\int_{-\infty}^{\infty}(y+it)^{-s-2k_1}(y-it)^{-s-2k_2}e(mt)\,dt\,.$$

*Then, the following assertions hold:*

*(i) The integral* $I_m(y,s;k_1,k_2)$ *converges absolutely and locally uniformly for* $s\in\mathbb{C}$ *with* $\mathrm{Re}(s)>1/2-k_1-k_2$*, and hence defines a holomorphic function.*

*(ii) The integral* $I_m(y,s;k_1,k_2)$ *admits a holomorphic continuation to the whole* $s$*-plane.*

*(iii) Let* $\Omega\subseteq\mathbb{C}$ *be a compact subset and let* $d\in\mathbb{N}$ *be such that* $\Omega\subseteq\{s\in\mathbb{C}\,|\,\mathrm{Re}(s)>1/2-k_1-k_2-d/2\}$*. Then, we have for all* $s\in\Omega$ *the bound*

$$\big|I_m(y,s;k_1,k_2)\big|\ll\frac{(k_1+k_2)^d}{|m|^d}\cdot y^{-2(\mathrm{Re}(s)+k_1+k_2+d/2)+1},$$

*where the implied constant depends on* $\Omega$ *and* $d$*, but is independent of* $m$ *and* $k_1$*,* $k_2$*.*

*Proof.* (i) For $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1/2-k_1-k_2$, we have the estimate

$$\begin{aligned}\big|I_m(y,s;k_1,k_2)\big|&\leq\int_{-\infty}^{\infty}\big|(y+it)^{-s-2k_1}(y-it)^{-s-2k_2}\,e(mt)\big|\,dt\\&=\int_{-\infty}^{\infty}(y^2+t^2)^{-\mathrm{Re}(s)-k_1-k_2}\,dt\\&=y^{-2(\mathrm{Re}(s)+k_1+k_2)+1}\int_{-\infty}^{\infty}\Big(1+\frac{t^2}{y^2}\Big)^{-\mathrm{Re}(s)-k_1-k_2}\frac{dt}{y}\\&=y^{-2(\mathrm{Re}(s)+k_1+k_2)+1}\,\frac{\pi\,\Gamma\big(\mathrm{Re}(s)-1/2+k_1+k_2\big)}{\Gamma\big(\mathrm{Re}(s)+k_1+k_2\big)}\\&\leq\pi\cdot y^{-2(\mathrm{Re}(s)+k_1+k_2)+1}.\end{aligned}\tag{31}$$

For all $s\in\Omega$, where $\Omega\subseteq\{s\in\mathbb{C}\,|\,\mathrm{Re}(s)>1/2-k_1-k_2\}$ is a compact subset, we therefore obtain the bound

$$\big|I_m(y,s;k_1,k_2)\big|\leq\pi,$$

which shows that the integral $I_m(y,s;k_1,k_2)$ converges absolutely and locally uniformly for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1/2-k_1-k_2$.

(ii) Let $\mathrm{Re}(s)>1/2-k_1-k_2$. Integration by parts yields

$$\begin{aligned}I_m(y,s;k_1,k_2)&=\Big[(y+it)^{-s-2k_1}(y-it)^{-s-2k_2}\frac{e(mt)}{2\pi im}\Big]_{t=-\infty}^{\infty}\\&\quad+\frac{1}{2\pi m}\int_{-\infty}^{\infty}(s+2k_1)(y+it)^{-s-2k_1-1}(y-it)^{-s-2k_2}e(mt)\,dt\\&\quad-\frac{1}{2\pi m}\int_{-\infty}^{\infty}(s+2k_2)(y+it)^{-s-2k_1}(y-it)^{-s-2k_2-1}e(mt)\,dt\,.\end{aligned}$$

In absolute values, the boundary term equals

$$\left|(y+it)^{-s-2k_1}(y-it)^{-s-2k_2}\frac{e(mt)}{2\pi im}\right|=\frac{(y^2+t^2)^{-\mathrm{Re}(s)-k_1-k_2}}{2\pi|m|}\,.$$

Since $\mathrm{Re}(s)>1/2-k_1-k_2$ and $y^2+t^2>1$ for $t\in(-\infty,\infty)$, we have

$$(y^2+t^2)^{-\mathrm{Re}(s)-k_1-k_2}<(y^2+t^2)^{-1/2},$$

from which we conclude that the boundary term vanishes. Therefore, we obtain the recurrence formula

$$I_m(y,s;k_1,k_2)=\frac{(s+2k_1)}{2\pi m}I_m\Big(y,s;k_1+\frac{1}{2},k_2\Big)-\frac{(s+2k_2)}{2\pi m}I_m\Big(y,s;k_1,k_2+\frac{1}{2}\Big). \tag{32}$$

By part (i), both terms on the right-hand side are holomorphic for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>-k_1-k_2$. In this way we obtain the holomorphic continuation of $I_m(y,s;k_1,k_2)$ to the half-plane $\{s\in\mathbb{C}\,|\,\mathrm{Re}(s)>-k_1-k_2\}$.

Let $d\in\mathbb{N}$. Applying relation (32) $d$ times, we arrive at a formula of the type

$$I_m(y,s;k_1,k_2)=\frac{1}{(2\pi m)^d}\sum_{j=0}^{d}P_{d,j}(s;k_1,k_2)\cdot I_m\Big(y,s;k_1+\frac{j}{2},k_2+\frac{d-j}{2}\Big), \tag{33}$$

where $P_{d,j}(s;k_1,k_2)$ is a polynomial in $s$ and $k_1,k_2$ of degree $d$. In fact, one can prove by induction on $d$ that

$$P_{d,j}(s;k_1,k_2)=(-1)^{d-j}\cdot\binom{d}{d-j}\cdot(s+2k_1)_j\cdot(s+2k_2)_{d-j}\qquad(0\leq j\leq d).$$

Now all the terms in (33) are holomorphic for $s\in\mathbb{C}$ with

$$\mathrm{Re}(s)>1/2-(k_1+j/2)-(k_2+(d-j)/2)=1/2-k_1-k_2-d/2.$$

Therefore, formula (33) yields the holomorphic continuation of $I_m(y,s;k_1,k_2)$ to the half-plane $\{s\in\mathbb{C}\,|\,\mathrm{Re}(s)>1/2-k_1-k_2-d/2\}$. Since $d\in\mathbb{N}$ was chosen arbitrarily, this proves the holomorphic continuation of $I_m(y,s;k_1,k_2)$ to the whole $s$-plane.

(iii) Let $\Omega\subseteq\mathbb{C}$ be a compact subset and let $d\in\mathbb{N}$ be such that $\Omega\subseteq\{s\in\mathbb{C}\,|\,\mathrm{Re}(s)>1/2-k_1-k_2-d/2\}$. For $s\in\Omega$, the function $I_m(y,s;k_1,k_2)$ is given by formula (33). Since $\mathrm{Re}(s)>1/2-k_1-k_2-d/2=1/2-(k_1+j/2)-(k_2+(d-j)/2)$, the bound (31) provides the estimate

$$\Big|I_m\Big(y,s;k_1+\frac{j}{2},k_2+\frac{d-j}{2}\Big)\Big|\ll y^{-2(\mathrm{Re}(s)+k_1+k_2+d/2)+1},$$

where the implied constant is universal. Furthermore, letting $s\in\Omega$, we have the bound

$$\big|P_{d,j}(s;k_1,k_2)\big|\ll k_1^j\cdot k_2^{d-j}\ll(k_1+k_2)^d\qquad(0\leq j\leq d),$$

where the implied constant depends on $\Omega$ and $d$, but is independent of $m$ and $k_1$, $k_2$. Altogether, as long as $s\in\Omega$, we have the bound

$$\begin{aligned}\big|I_m(y,s;k_1,k_2)\big|&\ll(d+1)\cdot(k_1+k_2)^d\cdot\frac{y^{-2(\mathrm{Re}(s)+k_1+k_2+d/2)+1}}{(2\pi|m|)^d}\\&\ll\frac{(k_1+k_2)^d}{|m|^d}\cdot y^{-2(\mathrm{Re}(s)+k_1+k_2+d/2)+1},\end{aligned}$$

where the implied constant depends on $\Omega$ and $d$, but is independent of $m$ and $k_1$, $k_2$. □

**6.6. Lemma.** *For $m\neq 0$, the following assertions hold:*

*(i) The function $V_m(s)$ admits a meromorphic continuation to the whole $s$-plane with possible simple poles at $s=s_j-2N$ and $s=-s_j-2N+1$ ($N\in\mathbb{N}$).*

*(ii) Let $N\in\mathbb{N}$ and $\Omega\subseteq\{s\in\mathbb{C}\,|\,\mathrm{Re}(s)>-2N-1\}$ a compact subset not containing any pole of $V_m(s)$. Then, for all $s\in\Omega$, we have the bound*

$$\big|V_m(s)\big|\ll|m|^{2N+2},$$

*where the implied constant depends on $\Omega$ and $N$, but is independent of $m$.*

*(iii) Let $N \in \mathbb{N}$ and $\tilde{s}$ a pole of $V_m(s)$ with $\operatorname{Re}(\tilde{s}) = -2N + 1/2$. Then, the residue of $V_m(s)$ at $\tilde{s}$ is bounded by*

$$\left|\operatorname{res}_{s=\tilde{s}} V_m(s)\right| \ll |m|^{2N},$$

*where the implied constant depends on $\tilde{s}$ and $N$, but is independent of $m$.*

*Proof.* (i) Since we have $V_m(s) = V_m(i,s)/2$ by Remark 5.5, the claim follows immediately from Proposition 3.7.

(ii) We will prove the claim more generally for the Poincaré series $V_m(z,s)$ for any $z \in \mathbb{H}$. For $s \in \Omega$, we then consider the decomposition

$$V_m(z,s) = \sum_{k=0}^{2N+1} \frac{(2\pi|m|)^k}{k!} P_m(z,s+k) + \sum_{k=2N+2}^{\infty} \frac{(2\pi|m|)^k}{k!} P_m(z,s+k). \tag{34}$$

From the proof of Proposition 3.7 we recall that the series on the right-hand side converges absolutely for $s \in \Omega$. Hence, we can rearrange the summation and find for $s \in \Omega$,

$$\begin{aligned}
&\left|\sum_{k=2N+2}^{\infty} \frac{(2\pi|m|)^k}{k!} P_m(z,s+k)\right| = \left|\sum_{k=0}^{\infty} \frac{(2\pi|m|)^{2N+k+2}}{(2N+k+2)!} P_m(z,s+2N+k+2)\right| \\
&= (2\pi|m|)^{2N+2} \left|\sum_{k=0}^{\infty} \frac{(2\pi|m|)^k}{(2N+k+2)!} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \operatorname{Im}(\gamma z)^{s+2N+k+2} \exp\bigl(-2\pi|m| \operatorname{Im}(\gamma z)\bigr) e\bigl(m \operatorname{Re}(\gamma z)\bigr)\right| \\
&= (2\pi|m|)^{2N+2} \left|\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \operatorname{Im}(\gamma z)^{s+2N+2} \exp\bigl(-2\pi|m| \operatorname{Im}(\gamma z)\bigr) e\bigl(m \operatorname{Re}(\gamma z)\bigr) \sum_{k=0}^{\infty} \frac{(2\pi|m| \operatorname{Im}(\gamma z))^k}{(2N+k+2)!}\right| \\
&\leq (2\pi|m|)^{2N+2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \operatorname{Im}(\gamma z)^{\operatorname{Re}(s)+2N+2} \exp\bigl(-2\pi|m| \operatorname{Im}(\gamma z)\bigr) \sum_{k=0}^{\infty} \frac{(2\pi|m| \operatorname{Im}(\gamma z))^k}{k!} \\
&= (2\pi|m|)^{2N+2} \cdot \mathcal{E}_{\mathrm{par}}(z, \operatorname{Re}(s)+2N+2) \ll |m|^{2N+2},
\end{aligned}$$

where the implied constant depends on $z$, $\Omega$, and $N$, but is independent of $m$.

In order to estimate the finite sum in the decomposition (34), we multiply the bounds (10) and (13) by the factor $2^{-2s+1}\pi^{-s+1}\Gamma(s)^{-1}|m|^{-s+1/2}$, and derive from the spectral expansion (9) of $P_m(z,s)$ for all $s \in \Omega$ the bound

$$\left|P_m(z,s+k)\right| \ll |m|^{-\operatorname{Re}(s)-k+1/2} \ll |m|^{2N-k+3/2} \qquad (k = 0, \ldots, 2N+1),$$

where the implied constant depends on $z$ and $\Omega$, but is independent of $m$. Hence, for all $s \in \Omega$, we obtain

$$\left|V_m(z,s)\right| \ll \sum_{k=0}^{2N+1} |m|^k \cdot |m|^{2N-k+3/2} + |m|^{2N+2} \ll |m|^{2N+2},$$

where the implied constant depends on $z$, $\Omega$, and $N$, but is independent of $m$. This proves the second claim.

(iii) In order to prove the third claim, we recall formulas (15), resp. (16), together with the bound (see [10], p. 86, adapted to our situation)

$$\left|\rho_\ell(m)\right|^2 \ll |t_j| \exp(\pi|t_j|) \qquad (\ell \in \mathbb{N}:\ s_\ell = s_j = 1/2 + it_j),$$

where the implied constant is universal. Then, we obtain

$$\left|\operatorname{res}_{s=s_j-2N} V_m(z,s)\right| \ll |m|^{-\operatorname{Re}(s_j)+2N+1/2} \ll |m|^{2N},$$

resp.

$$\left|\operatorname{res}_{s=-s_j-2N+1} V_m(z,s)\right| \ll |m|^{\operatorname{Re}(s_j)+2N-1/2} \ll |m|^{2N},$$

where the implied constants depend on $z$, $s_j$, and $N$, but are independent of $m$. □

**6.7. Lemma.** *For $z \in \mathbb{H}$ with $\mathrm{Im}(z) > 1$, $m \neq 0$, and $N \in \mathbb{N}$, the series*

$$2^s y^s \sum_{n=N+1}^{\infty} \sum_{k_1+k_2=n} \frac{(\frac{s}{2})_{k_1} \cdot (\frac{s}{2})_{k_2}}{k_1! \cdot k_2!} \cdot I_{-m}(y,s;k_1,k_2) \cdot V_{-m}(s+2k_1+2k_2)$$

*converges absolutely and locally uniformly for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -2N-1$, and hence defines a holomorphic function.*

*Proof.* Let $\Omega \subseteq \{s \in \mathbb{C} \,|\, \mathrm{Re}(s) > -2N-1\}$ be a compact subset. For $s \in \Omega$ and $k_1, k_2 \in \mathbb{N}$, we define the functions

$$f_{m;k_1,k_2}(y,s) := 2^s y^s \frac{(\frac{s}{2})_{k_1} \cdot (\frac{s}{2})_{k_2}}{k_1! \cdot k_2!} \cdot I_{-m}(y,s;k_1,k_2) \cdot V_{-m}(s+2k_1+2k_2).$$

If $k_1 + k_2 \geq N+1$, we have $\mathrm{Re}(s+2k_1+2k_2) \geq \mathrm{Re}(s) + 2N + 2 > 1$, whence the functions $V_{-m}(s+2k_1+2k_2)$ are holomorphic for $s \in \Omega$. By Lemma 6.5 (ii), the functions $I_{-m}(y,s;k_1,k_2)$ are holomorphic for $s \in \mathbb{C}$. Therefore, the functions $f_{m;k_1,k_2}(y,s)$ are holomorphic for $s \in \Omega$, as long as $k_1 + k_2 \geq N+1$. Now choose $d \in \mathbb{N}$ with $d > 2N+1$; then we have $\Omega \subseteq \{s \in \mathbb{C} \,|\, \mathrm{Re}(s) > 1/2 - k_1 - k_2 - d/2\}$, as long as $k_1 + k_2 \geq N+1$. Using Lemma 6.5 (iii), we estimate for $s \in \Omega$,

$$\begin{aligned}
&\sum_{n=N+1}^{\infty} \sum_{k_1+k_2=n} \big| f_{m;k_1,k_2}(y,s) \big| \\
&\ll V_0(\mathrm{Re}(s)+2N+2) \, \frac{2^{\mathrm{Re}(s)} y^{-\mathrm{Re}(s)-d+1}}{|m|^d} \sum_{n=N+1}^{\infty} \sum_{k_1+k_2=n} \frac{|(\frac{s}{2})_{k_1}| \cdot |(\frac{s}{2})_{k_2}|}{k_1! \cdot k_2!} \cdot \frac{(k_1+k_2)^d}{y^{2(k_1+k_2)}} \\
&\ll \frac{y^{-\mathrm{Re}(s)-d+1}}{|m|^d} \sum_{n=N+1}^{\infty} \frac{n^d}{y^{2n}} \sum_{k_1=0}^{n} \frac{(|\frac{s}{2}|)_{k_1} \cdot (|\frac{s}{2}|)_{n-k_1}}{k_1! \cdot (n-k_1)!} = \frac{y^{-\mathrm{Re}(s)-d+1}}{|m|^d} \sum_{n=N+1}^{\infty} \frac{n^d}{y^{2n}} \frac{(|s|)_n}{n!},
\end{aligned}$$

where the implied constants depend on $\Omega$, $d$, and $N$, but are independent of $m$ and $k_1$, $k_2$. Since the ratio of successive terms in the latter series has limit

$$\lim_{n\to\infty} \left| \frac{(n+1)^d \cdot (n+|s|)}{n^d \cdot (n+1)} \cdot \frac{1}{y^2} \right| = \frac{1}{y^2} < 1,$$

we derive from d'Alembert's criterion that the series in question converges absolutely and locally uniformly for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -2N-1$, which proves the claim.
For later purposes, we note for $s \in \Omega$ the bound

$$\left| \sum_{n=N+1}^{\infty} \sum_{k_1+k_2=n} f_{m;k_1,k_2}(y,s) \right| \ll |m|^{-d}, \tag{35}$$

where $d \in \mathbb{N}$ with $d > 2N+1$, and where the implied constant depends on $z$, $\Omega$, $d$, and $N$, but is independent of $m$. □

**6.8. Proposition.** *For $z \in \mathbb{H}$ with $\mathrm{Im}(z) > 1$, and $m \neq 0$, the following assertions hold:*

*(i) The function $a_m(y,s)$ admits a meromorphic continuation to the whole $s$-plane with possible simple poles at $s = s_j - 2N$ and $s = -s_j - 2N + 1$ ($N \in \mathbb{N}$).*

*(ii) Let $N \in \mathbb{N}$ and $\Omega \subseteq \{s \in \mathbb{C} \,|\, \mathrm{Re}(s) > -2N-1\}$ a compact subset not containing any pole of $a_m(y,s)$. Then, for all $s \in \Omega$, we have the bound*

$$\big| a_m(y,s) \big| \ll |m|^{-d},$$

*where $d \in \mathbb{N}$ with $d > 2N+1$, and where the implied constant depends on $z$, $\Omega$, $d$, and $N$, but is independent of $m$.*

*(iii) Let $N \in \mathbb{N}$ and $\tilde{s}$ a pole of $a_m(y,s)$ with $\mathrm{Re}(\tilde{s}) = -2N + 1/2$. Then, the residue of $a_m(y,s)$ at $\tilde{s}$ is bounded by*

$$\big|\operatorname{res}_{s=\tilde{s}} a_m(y,s)\big| \ll |m|^{-d},$$

*where $d \in \mathbb{N}$ with $d > 2N+3$, and where the implied constant depends on $z$, $\tilde{s}$, $d$, and $N$, but is independent of $m$.*

*Proof.* (i) As before, we obtain the meromorphic continuation of $a_m(y,s)$ to the whole $s$-plane by constructing its meromorphic continuations to the half-planes

$$\mathcal{H}_N := \{s \in \mathbb{C} \,|\, \mathrm{Re}(s) > -2N - 1\}$$

for any $N \in \mathbb{N}$. Applying Proposition 5.4 and using the notation from the proof of Lemma 6.7, we can write

$$a_m(y,s) = \sum_{n=0}^{N} \sum_{k_1+k_2=n} f_{m;k_1,k_2}(y,s) + \sum_{n=N+1}^{\infty} \sum_{k_1+k_2=n} f_{m;k_1,k_2}(y,s). \tag{36}$$

Since $\mathrm{Re}(s) > -2N-1$ by assumption, Lemma 6.7 proves that the series

$$\sum_{n=N+1}^{\infty} \sum_{k_1+k_2=n} f_{m;k_1,k_2}(y,s)$$

is a holomorphic function on the half-plane $\mathcal{H}_N$. Since the first double sum in (36) is a meromorphic function on the whole $s$-plane, we conclude that $a_m(y,s)$ has a meromorphic continuation to the half-plane $\mathcal{H}_N$.
In order to determine the poles of $a_m(y,s)$, we calculate its poles in the strip

$$\mathcal{S}_N := \{s \in \mathbb{C} \,|\, -2N - 1 < \mathrm{Re}(s) \leq -2N + 1\}$$

for any $N \in \mathbb{N}$. By considering $a_m(y,s)$ with its decomposition (36) in the strip $\mathcal{S}_N$, we see that the poles come from the finite sum

$$\sum_{n=0}^{N} \sum_{k_1+k_2=n} f_{m;k_1,k_2}(y,s) = 2^s y^s \sum_{n=0}^{N} V_{-m}(s+2n) \sum_{k_1=0}^{n} \frac{(\frac{s}{2})_{k_1} \cdot (\frac{s}{2})_{n-k_1}}{k_1! \cdot (n-k_1)!} \cdot I_{-m}(y,s;k_1,n-k_1),$$

which has possible simple poles at $s = s_j - 2N$ and $s = -s_j - 2N + 1$ in the strip $\mathcal{S}_N$ arising from the factors $V_{-m}(s+2n)$ ($n = 0, \ldots, N$). Therefore, the possible poles of $a_m(y,s)$ in the strip $\mathcal{S}_N$ are located at $s = s_j - 2N$ and $s = -s_j - 2N + 1$ ($N \in \mathbb{N}$).
(ii) In order to prove the second claim, we let $s \in \Omega$, where $\Omega \subseteq \mathcal{H}_N$ is a compact subset not containing any pole of $a_m(y,s)$, and we decompose $a_m(y,s)$ as in (36). Choosing now $d' \in \mathbb{N}$ with $d' > 4N+3$ and applying the bounds obtained in Lemma 6.5 (iii) (note that $\mathrm{Re}(s) > 1/2 - n - d'/2$ for $n = 0, \ldots, N$) and Lemma 6.6 (ii) (note that $\mathrm{Re}(s) + 2n > -2(N-n) - 1$ for $n = 0, \ldots, N$) to the finite double sum in (36) and the bound (35) to the remaining series in (36), we obtain the estimate

$$\begin{aligned} \big|a_m(y,s)\big| &\ll \sum_{n=0}^{N} \big|V_{-m}(s+2n)\big| \sum_{k_1=0}^{n} \big|I_{-m}(y,s;k_1,n-k_1)\big| + |m|^{-d'} \\ &\ll \sum_{n=0}^{N} |m|^{2(N-n)+2} \cdot |m|^{-d'} + |m|^{-d'} \ll |m|^{-(d'-2N-2)}, \end{aligned}$$

where the implied constants depend on $z$, $\Omega$, $d'$, and $N$, but are independent of $m$. Setting $d := d' - 2N - 2$ and observing that $d > 2N+1$, the proof of part (ii) is complete.

(iii) As in the proof of (ii), we work from the decomposition (36). We let $\tilde{s}$ be a pole of $a_m(y,s)$ with $\tilde{s} = -2N + 1/2$, i.e., $\tilde{s} \in \mathcal{S}_N$. As before, choosing $d' \in \mathbb{N}$ with $d' > 4N + 3$, the bounds obtained in Lemmas 6.5 (iii) and 6.6 (iii) give the estimate

$$\begin{aligned}\big|\operatorname{res}_{s=\tilde{s}} a_m(y,s)\big| &\ll \sum_{n=0}^{N} \big|\operatorname{res}_{s=\tilde{s}} V_{-m}(s+2n)\big| \sum_{k_1=0}^{n} \big|I_{-m}(y,\tilde{s};k_1,n-k_1)\big| \\ &\ll \sum_{n=0}^{N} |m|^{2(N-n)} \cdot |m|^{-d'} \ll |m|^{-(d'-2N)},\end{aligned}$$

where the implied constants depend on $z$, $\tilde{s}$, $d$, and $N$, but are independent of $m$. Setting $d := d' - 2N$ and observing that $d > 2N + 3$, the proof of part (iii) is also complete. □

**6.9. Remark.** By means of Remark 5.5, one can establish the meromorphic continuation of $a_m(y,s)$ $(m \neq 0)$ to the whole $s$-plane in the more general case $\mathrm{Im}(z) \neq \mathrm{Im}(\gamma^{-1}i)$ for any $\gamma \in \Gamma$ by applying Lemma 6.5 (noting that this lemma also holds for $y > 0$ and $m \in \mathbb{R}$, $m \neq 0$) as well as by using the same techniques as in Lemma 6.7 and Proposition 6.8 applied accordingly to the modified situation. The poles of $a_m(y,s)$ and their residues turn out to be same as in the case $\mathrm{Im}(z) > 1$. Moreover, also the statements (ii) and (iii) of Proposition 6.8 generalize to the case $\mathrm{Im}(z) \neq \mathrm{Im}(\gamma^{-1}i)$ for any $\gamma \in \Gamma$.

**6.10. Theorem.** *For $z \in \mathbb{H}$ with $\mathrm{Im}(z) > 1$, the elliptic Eisenstein series $\mathcal{E}_{\mathrm{ell}}(z,s)$ has a meromorphic continuation to the whole $s$-plane with possible poles at $s = s_\Gamma - 2N$, $s = s_j - 2N$, and $s = -s_j - 2N + 1$ $(N \in \mathbb{N})$, where $s_\Gamma$ is a pole of $\Gamma(s-1/2)\mathcal{E}_{\mathrm{par}}(i,s)$, and $s_j = 1/2 + it_j$ with $t_j > 0$ and $s_j(1-s_j) = \lambda_j$ a discrete eigenvalue of $\Delta_{\mathrm{hyp}}$.*

*Proof.* Let $z \in \mathbb{H}$ with $\mathrm{Im}(z) > 1$. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we represent the elliptic Eisenstein series $\mathcal{E}_{\mathrm{ell}}(z,s)$ by its Fourier expansion

$$\mathcal{E}_{\mathrm{ell}}(z,s) = \sum_{m \in \mathbb{Z}} a_m(y,s) e(mx), \tag{37}$$

where the coefficients $a_m(y,s)$ are explicitly given by Propositions 5.2 and 5.4 for $m = 0$ and $m \neq 0$, respectively. By Propositions 6.3 and 6.8, the functions $a_m(y,s)$ admit a meromorphic continuation to the whole $s$-plane.

In order to prove the meromorphic continuation of $\mathcal{E}_{\mathrm{ell}}(z,s)$ to the whole $s$-plane, let $s \in \Omega$, where $\Omega \subseteq \{s \in \mathbb{C} \,|\, \mathrm{Re}(s) > -2N - 1\}$ for some $N \in \mathbb{N}$ is a compact subset not containing any pole of $a_m(y,s)$ for all $m \in \mathbb{Z}$. Choosing $d \in \mathbb{N}$, $d > 2N + 2$, we have by Proposition 6.8 (ii) the bound

$$\sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \big|a_m(y,s) e(mx)\big| \ll \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} |m|^{-2},$$

where the implied constant depends on $z$, $\Omega$, $d$, and $N$, but is independent of $m$. Therefore, the Fourier expansion (37) converges absolutely and uniformly in $\Omega$. This proves that $\mathcal{E}_{\mathrm{ell}}(z,s)$ is holomorphic in $s \in \mathbb{C}$ away form the poles of $a_m(y,s)$ for $m \in \mathbb{Z}$.

Let now $\tilde{s} \in \mathbb{C}$ be a pole of $a_m(y,s)$ for $m \neq 0$ as in Proposition 6.8 (i); then, $\mathrm{Re}(\tilde{s}) = -2N + 1/2$ for some $N \in \mathbb{N}$. Choosing $d \in \mathbb{N}$, $d > 2N + 3$, we estimate using Proposition 6.8 (iii),

$$\lim_{s \to \tilde{s}} (s - \tilde{s}) \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} a_m(y,s) e(mx) \ll \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \big|\operatorname{res}_{s=\tilde{s}} a_m(y,s)\big| \ll \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} |m|^{-2},$$

where the implied constant depends on $z$, $\tilde{s}$, $d$, and $N$, but is independent of $m$.

In this way we obtain the meromorphic continuation of $\mathcal{E}_{\mathrm{ell}}(z,s)$ to the whole $s$-plane with possible poles at $s = s_\Gamma - 2N$, $s = s_j - 2N$, and $s = -s_j - 2N + 1$ $(N \in \mathbb{N})$. The poles at $s = s_\Gamma - 2N$ are contributed by $a_0(y,s)$; here $s_\Gamma$ denotes a pole of $\Gamma(s-1/2)\mathcal{E}_{\mathrm{par}}(i,s)$. □

**6.11. Remark.** Using Remark 6.9, one can establish the meromorphic continuation of $\mathcal{E}_{\mathrm{ell}}(z,s)$ to the whole $s$-plane in the more general case $\mathrm{Im}(z) \neq \mathrm{Im}(\gamma^{-1}i)$ for any $\gamma \in \Gamma$. The poles of $\mathcal{E}_{\mathrm{ell}}(z,s)$ and their residues turn out the be same as in the case $\mathrm{Im}(z) > 1$.

**6.12. Remark.** The elliptic Eisenstein series $\mathcal{E}_{\mathrm{ell}}(z,s)$ has a simple pole at $s = 1$ with residue

$$\mathrm{res}_{s=1}\,\mathcal{E}_{\mathrm{ell}}(z,s) = \mathrm{res}_{s=1}\,a_0(y,s) = 2\pi\,\mathrm{res}_{s=1}\left(\frac{V_0(s)}{\Gamma(s/2+1/2)}\right) = \pi\,\mathrm{res}_{s=1}\,\mathcal{E}_{\mathrm{par}}(i,s) = 3;$$

here we used the decomposition (30) for $a_0(y,s)$ with $N = 0$.

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