HU Berlin Summer Term 2012

Exercises BMS Basic Course
Algebraic Geometry

Prof. Dr. J. Kramer

Solution to be presented on June 26th in the exercise class.

Exercise sheet 10

Exercise 10.1

Let $A = \bigoplus_{d\geq 0} A_d$ be a graded ring and $\mathrm{Proj}(A)$ the set of relevant homogeneous prime ideals of A. For a homogeneous ideal $\mathfrak{a} \subseteq A$, let

$$V_+(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Proj}(A) \,|\, \mathfrak{p} \supseteq \mathfrak{a}\}$$

denote the set of all relevant homogeneous prime ideals of A containing $\mathfrak{a}$.

(a) Show that the sets $V_+(\mathfrak{a})$, where $\mathfrak{a}$ ranges over the homogeneous ideals of A, satisfy the axioms for closed sets in $\mathrm{Proj}(A)$.

(b) Show that $V(\mathfrak{a}) \cap \mathrm{Proj}(A) = V_+(\mathfrak{a}^h)$, where $\mathfrak{a}^h$ is the homogeneous ideal generated by $\mathfrak{a}$. This shows that $\mathrm{Proj}(A)$ carries the topology induced from $\mathrm{Spec}(A)$.

Exercise 10.2

Let $A = \bigoplus_{d\geq 0} A_d$ be a graded ring, $A_+ := \bigoplus_{d\geq 1} A_d$, and $\mathfrak{a}_+ := \mathfrak{a} \cap A_+$ for a homogeneous ideal $\mathfrak{a} \subseteq A$. Further, for a subset $Y \subseteq \mathrm{Proj}(A)$, define

$$I_+(Y) := \Big(\bigcap_{\mathfrak{p}\in Y} \mathfrak{p}\Big) \cap A_+.$$

Prove the following assertions:

(a) If $\mathfrak{a} \subseteq A_+$ is a homogeneous ideal, then $I_+(V_+(\mathfrak{a})) = \sqrt{\mathfrak{a}}_+$. If $Y \subseteq \mathrm{Proj}(A)$ is a subset, then $V_+(I_+(Y)) = \overline{Y}$.

(b) The maps

$$Y \mapsto I_+(Y) \quad \text{and} \quad \mathfrak{a} \mapsto V_+(\mathfrak{a})$$

define mutually inverse, inclusion reversing bijections between the set of homogeneous ideals $\mathfrak{a} \subseteq A_+$ such that $\mathfrak{a} = \sqrt{\mathfrak{a}}_+$ and the set of closed subsets of $\mathrm{Proj}(A)$. Via this bijection, the closed irreducible subsets correspond to ideals of the form $\mathfrak{p}_+$, where $\mathfrak{p}$ is a relevant prime ideal.

(c) If $\mathfrak{a} \subseteq A_+$ is a homogeneous ideal, then $V_+(\mathfrak{a}) = \emptyset$ if and only if $\sqrt{\mathfrak{a}}_+ = A_+$. In particular, $\mathrm{Proj}(A) = \emptyset$ if and only if every element in A_+ is nilpotent.

(d) The sets
$$D_+(f) := \mathrm{Proj}(A) \setminus V_+(f)$$
for homogeneous elements $f \in A_+$ form a basis of the topology of $\mathrm{Proj}(A)$.

(e) Let $(f_i)_i$ be a family of homogeneous elements $f_i \in A_+$ and let $\mathfrak{a}$ be the ideal generated by the f_i. Then, we have
$$\bigcup_i D_+(f_i) = \mathrm{Proj}(A) \Longleftrightarrow \sqrt{\mathfrak{a}}_+ = A_+ \, .$$

Exercise 10.3

Let $A = \bigoplus_{d \geq 0} A_d$ be a graded ring. With the previous notations, we define a presheaf of rings on $\mathrm{Proj}(A)$ by setting
$$\mathcal{O}_{\mathrm{Proj}(A)}(D_+(f)) := A_{(f)}$$
for a homogeneous $f \in A_+$ and then defining
$$\mathcal{O}_{\mathrm{Proj}(A)}(U) := \varprojlim_{\substack{f \in A_+,\text{ homog.} \\ D_+(f) \subseteq U}} \mathcal{O}_{\mathrm{Proj}(A)}(D_+(f))$$
for an open subset $U \subseteq \mathrm{Proj}(A)$.

(a) Prove that $(\mathrm{Proj}(A), \mathcal{O}_{\mathrm{Proj}(A)})$ is a ringed space.

(b) Prove that $\mathrm{Proj}(A)$ is a seperated scheme.

Elaborate every step of your proof as detailed as possible.