HU Berlin Winter Term 2011/12

Exercises BMS Basic Course
Commutative Algebra

Prof. Dr. J. Kramer

To be handed in on January 12th after the 1st lecture

Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.

Exercise sheet 10 (40 points)

Exercise 1 (10 points)
Let A be a commutative ring with 1.

(a) Let
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$
be a short exact sequence of A-modules. Show that M is noetherian if and only if M' and M'' are noetherian.

(b) Prove that if M_i ($i = 1, \ldots, n$) are noetherian A-modules, then $\bigoplus_{i=1}^n M_i$ is a noetherian A-module.

Exercise 2 (10 points)
Let A be a commutative ring with 1.

(a) Show that every surjective endomorphism of a noetherian A-module is an isomorphism.

(b) Give an example of an injective endomorphism of a noetherian A-module which is not an isomorphism.

Exercise 3 (10 points)

(a) Show that an ideal $\mathfrak{q} \subset \mathbb{Z}$ is primary if and only if $\mathfrak{q} = (0)$ or $\mathfrak{q} = (p^n)$ for some prime p and some $n \in \mathbb{N}_{>0}$.

(b) Let A be a commutative ring with 1 and let $\mathfrak{p} \in \operatorname{Spec}(A)$. Show that a $\mathfrak{p}$-primary ideal $\mathfrak{q} \subset A$ need not be equal to some power of $\mathfrak{p}$. Reciprocally, show that a power $\mathfrak{p}^n$ ($n \in \mathbb{N}_{>0}$) of $\mathfrak{p}$ need not be primary, in general.

Exercise 4 (10 points)
Prove the following assertions:

(a) A noetherian topological space X which is Hausdorff must be a finite set with the discrete topology.

Please turn over !

(b) A topological space X is noetherian if and only if every open subset $U \subseteq X$ is quasi-compact.

(c) Let A be a commutative ring with 1. If A is a noetherian ring, then $\mathrm{Spec}(A)$ is a noetherian topological space.

(d) Give an example to show that the converse of (c) does not hold in general.