HU Berlin Winter Term 2011/12

Exercises BMS Basic Course
Commutative Algebra

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To be handed in on January 19th after the 1st lecture

Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.

Exercise sheet 11 (40 points)

Exercise 1 (10 points)

(a) Compute a minimal primary decomposition of the ideal (6) in the ring $\mathbb{Z}[\sqrt{-5}]$.

(b) Find two different minimal primary decompositions of the ideal (X^2, XY) in the ring $k[X, Y]$, where k is a field. Illustrate on the basis of this example the first and the second uniqueness theorem for minimal primary decompositions.

Exercise 2 (10 points)

(a) Compute a minimal primary decomposition of the ideal $\mathfrak{a} := (XZ - Y^2, YW - Z^2)$ in the ring $k[X, Y, Z, W]$, where k is a field. Determine the prime ideals which are associated with $\mathfrak{a}$ and determine the isolated and the embedded prime ideals.

(b) Solve (a) for the ideal $\mathfrak{b} := (XW - YZ, X^3 - Y^2, Z^3 - W^2) \subseteq k[X, Y, Z, W]$.

Exercise 3 (10 points)

Let A be a commutative ring with 1 and $\mathfrak{a} \subseteq A$ a decomposable ideal. Let $\pi : A \longrightarrow A/\mathfrak{a}$ denote the canonical projection.

(a) Show that $\mathfrak{a} = \bigcap_{i=1}^{n} q_i$ is a minimal primary decomposition of $\mathfrak{a}$ if and only if $(\overline{0}) = \pi(\mathfrak{a}) = \bigcap_{i=1}^{n} \pi(q_i)$ is a minimal primary decomposition of $(\overline{0})$.

(b) Show that $\mathrm{Spec}(A/\mathfrak{a})$ has only finitely many irreducible components.

Exercise 4 (10 points)

Let k be an algebraically closed field. For a subset $T \subseteq k[X_1, \ldots, X_n]$, we call

$$V(T) := \left\{(x_1, \ldots, x_n) \in k^n \;\middle|\; f(x_1, \ldots, x_n) = 0 \text{ for every } f \in T\right\}$$

an *algebraic set* in k^n.

Please turn over !

(a) Show that by defining the algebraic sets of k^n to be closed, one obtains a topology of k^n. This topology is called the *Zariski topology* of k^n.

(b) Show that there is a bijection between the algebraic sets of k^n and the ideals of $k[X_1, \ldots, X_n]$ which coincide with their radicals, the so-called *radical ideals*.

(c) Show that there is a bijection between the irreducible algebraic sets of k^n, the so-called *algebraic varieties*, and the prime ideals of $k[X_1, \ldots, X_n]$.