Exercises BMS Basic Course
Commutative Algebra

Prof. Dr. J. Kramer

To be handed in on January 26th after the 1st lecture

Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.

Exercise sheet 12 (40 points)

See overleaf for the definition of the terms integral and integral closure.

Exercise 1 (10 points)

Let $A \subseteq B$ be commutative rings with 1. Show that the integral closure of A in B is a subring of B containing A.

Exercise 2 (10 points)

Let $A \subseteq B \subseteq C$ be commutative rings with 1. Let C be integral over B and B integral over A. Conclude that C is integral over A.

Exercise 3 (10 points)

Let $A \subseteq B$ be commutative rings with 1. Let B be integral over A.

(a) If $\mathfrak{b} \subseteq B$ is an ideal and $\mathfrak{a} := \mathfrak{b}^c = A \cap \mathfrak{b}$, then $B/\mathfrak{b}$ is integral over $A/\mathfrak{a}$.

(b) If $S \subseteq A$ is a multiplicatively closed subset, then $S^{-1}B$ is integral over $S^{-1}A$.

Exercise 4 (10 points)

(a) Let A be a commutative ring with 1, $\mathfrak{a} \subseteq A$ an ideal, and $b \in A$. Show that $b \in \mathfrak{r}(\mathfrak{a})$ if and only if $1 \in (\mathfrak{a}, 1 - bX) \subseteq A[X]$.

(b) Let k be an algebraically closed field. Give an alternative proof of the Strong Nullstellensatz, i.e.

$$\mathfrak{a} \subseteq k[X_1, \ldots, X_n] \text{ ideal} \Longrightarrow I(V(\mathfrak{a})) = \mathfrak{r}(\mathfrak{a}),$$

adding a new variable X and using (a). This is the so-called *"trick" of Rabinowitch*. Use the fact that, if $\mathfrak{b} \subset k[X_1, \ldots, X_n, X]$ is a proper ideal, then $V(\mathfrak{b}) \neq \emptyset$.

(c) Let k be an algebraically closed field and $\mathfrak{a} := (XY, YZ, ZX, (X - Y)(X + 1)) \subseteq k[X, Y, Z]$ an ideal. Determine the irreducible components of the algebraic set $V(\mathfrak{a})$.

Please turn over !

Definitions

Let $A \subseteq B$ be commutative rings with 1. An element $b \in B$ is called *integral* over A, if b is a root of a monic polynomial with coefficients in A, that is $f(b) = 0$ for some $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in A[X]$. The set of elements of B which are integral over A is called the *integral closure* of A in B. If the integral closure of A in B is equal to B, we say that B is *integral over* A. If the integral closure of A in B is equal to A, we say that A is *integrally closed* in B.