

## Exercises BMS Basic Course

# Commutative Algebra

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To be handed in on November 3rd after the 1st lecture

**Please hand in every exercise solution on a separate sheet and do not forget to put your name and student ID on every sheet.**

### Exercise sheet 2 (40 points)

#### Exercise 1 (10 points)

Let  $A, B$  be commutative rings with 1 and let  $f : A \rightarrow B$  be a ring homomorphism. Let  $\mathfrak{a} \subseteq A$  and  $\mathfrak{b} \subseteq B$  be ideals.

- (a) Show that  $f^{-1}(\mathfrak{b})$  is an ideal of  $A$ , called the *contraction*  $\mathfrak{b}^c$  of  $\mathfrak{b}$ . Show that  $\mathfrak{b}^c$  is a prime ideal, if  $\mathfrak{b}$  is a prime ideal.
- (b) Show that  $f(\mathfrak{a})$  is not necessarily an ideal of  $B$ . We define the *extension*  $\mathfrak{a}^e$  of  $\mathfrak{a}$  to be the ideal generated by  $f(\mathfrak{a})$  in  $B$ . Show that  $\mathfrak{a}^e$  need not be a prime ideal, if  $\mathfrak{a}$  is a prime ideal.
- (c) Show that  $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$  and  $\mathfrak{b} \supseteq \mathfrak{b}^{ec}$ . Further, prove that  $\mathfrak{a}^e = \mathfrak{a}^{ec}$  and  $\mathfrak{b}^c = \mathfrak{b}^{ec}$ .

#### Exercise 2 (10 points)

Let  $A$  be a commutative ring with 1.

- (a) Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$  ( $n \in \mathbb{N}$ ,  $n > 1$ ) be prime ideals of  $A$  and let  $\mathfrak{a}$  be an ideal contained in  $\bigcup_{j=1}^n \mathfrak{p}_j$ . Prove that  $\mathfrak{a} \subseteq \mathfrak{p}_j$  for at least one  $j$ .
- (b) Let  $\mathfrak{a}_1, \dots, \mathfrak{a}_n$  ( $n \in \mathbb{N}$ ,  $n > 1$ ) be ideals of  $A$  and let  $\mathfrak{p}$  be a prime ideal containing  $\bigcap_{j=1}^n \mathfrak{a}_j$ . Prove that  $\mathfrak{a}_j \subseteq \mathfrak{p}$  for at least one  $j$ . Further, if  $\mathfrak{p} = \bigcap_{j=1}^n \mathfrak{a}_j$ , then  $\mathfrak{p} = \mathfrak{a}_j$  for some  $j$ .

#### Exercise 3 (10 points)

Let  $A := C([0, 1], \mathbb{R})$  denote the ring of continuous, real-valued functions on  $[0, 1]$ .

- (a) Let  $x \in [0, 1]$ . Show that the set

$$\mathfrak{m}_x := \{ f \in A \mid f(x) = 0 \}$$

is a maximal ideal in  $A$ .

- (b) Let  $f_1, \dots, f_n \in A$  ( $n \in \mathbb{N}$ ,  $n > 1$ ) be functions without any common zero. Show that  $f_1^2 + \dots + f_n^2$  is a unit in  $A$ .

(c) Let  $\mathfrak{a} \subset A$  be a proper ideal and let

$$V(\mathfrak{a}) := \{a \in [0, 1] \mid f(a) = 0 \text{ for every } f \in \mathfrak{a}\}$$

be the set of common zeros of all functions in  $\mathfrak{a}$ . Prove that  $V(\mathfrak{a}) \neq \emptyset$ .

(d) Deduce using (c) that  $\text{Max}(A) = \{\mathfrak{m}_x \mid x \in [0, 1]\}$  and show that the map

$$[0, 1] \rightarrow \text{Max}(A), \quad x \mapsto \mathfrak{m}_x$$

is a bijection.

#### Exercise 4 (10 points)

Let  $A$  be a commutative ring with 1. We consider the set

$$\text{Spec}(A) = \{\mathfrak{p} \subsetneq A \mid \mathfrak{p} \text{ prime ideal}\}$$

of all prime ideals of  $A$ . For a subset  $S \subseteq A$ , let

$$V(S) := \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \supseteq S\}$$

denote the set of all prime ideals of  $A$  which contain  $S$ .

- (a) Prove that  $V(S) = V(\mathfrak{a}) = V(\mathfrak{r}(\mathfrak{a}))$ , where  $\mathfrak{a}$  denotes the ideal generated by  $S$  in  $A$ .
- (b) Show that the sets  $V(S)$ , where  $S$  ranges over the subsets of  $A$ , satisfy the axioms for closed sets in  $\text{Spec}(A)$ . Conclude that  $\text{Spec}(A)$  is a topological space. (The resulting topology on  $\text{Spec}(A)$  is called the *Zariski topology*; the topological space  $\text{Spec}(A)$  is called the *prime spectrum of  $A$* .)