# Exercises BMS Basic Course
# **Commutative Algebra**
Prof. Dr. J. Kramer

To be handed in on November 10th after the 1st lecture

**Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.**

**Exercise sheet 3 (40 points)**

**Exercise 1 (10 points)**
Let $A$ be a commutative ring with 1 and let $\mathfrak{N}(A)$ be its nilradical. Show that the following are equivalent:

(i) $A$ has exactly one prime ideal.

(ii) Every element of $A$ is either a unit or nilpotent.

(iii) The quotient ring $A/\mathfrak{N}(A)$ is a field.

**Exercise 2 (10 points)**
Let $A$ be a commutative ring with 1. Let $A[X]$ be the ring of polynomials in $X$ with coefficients in $A$ and let $f = a_n X^n + \ldots + a_1 X + a_0 \in A[X]$. Prove that

(a) $f$ is a unit in $A[X]$ if and only if $a_0$ is a unit in $A$ and $a_1, \ldots, a_n$ are nilpotent.

(b) $f$ is nilpotent if and only if $a_0, a_1, \ldots, a_n$ are nilpotent.

(c) $f$ is a zero-divisor if and only if there exists $a \in A$, $a \neq 0$, such that $af = 0$.

**Exercise 3 (10 points)**

(a) Identify the sets $\mathrm{Spec}(\mathbb{Z})$, $\mathrm{Spec}(\mathbb{Z}/3\mathbb{Z})$, $\mathrm{Spec}(\mathbb{Z}/6\mathbb{Z})$, $\mathrm{Spec}(\mathbb{C}[X])$, $\mathrm{Spec}(\mathbb{C}[X]/(X^2))$, and $\mathrm{Spec}(\mathbb{R}[X]/(X^2+1))$ and compute $V(\mathfrak{p})$ for every occuring prime ideal $\mathfrak{p}$.

(b) Prove that there is a bijection $\mathrm{Max}(\mathbb{C}[X]) \to \mathbb{C}$.

**Exercise 4 (10 points)**
Let $A$ be a commutative ring with 1. For $f \in A$, we define the *distinguished* or *basic set*

$$D(f) := \mathrm{Spec}(A) \setminus V(f)$$

to be the complement of $V(f)$ in $\mathrm{Spec}(A)$. Show that the sets $D(f)$ $(f \in A)$ are open and that they form a basis of open sets for the Zariski topology of $\mathrm{Spec}(A)$.

**Please turn over !**

Furthermore, prove that for $f, g \in A$ we have

(a) $D(f) \cap D(g) = D(f \cdot g)$.

(b) $D(f) = \emptyset \iff f$ is nilpotent.

(c) $D(f) = X \iff f \in A^\times$.

(d) $D(f) = D(g) \iff \mathfrak{r}(f) = \mathfrak{r}(g)$.
Here, $\mathfrak{r}(f)$ denotes the radical of the principal ideal $(f)$.