Exercises BMS Basic Course
Commutative Algebra

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To be handed in on November 24th after the 1st lecture

Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.

Exercise sheet 5 (40 points)

Exercise 1 (10 points)

Let A be commutative ring with 1 and let P be an A-module. Show that the following are equivalent:

(i) P is projective.

(ii) The functor $\mathrm{Hom}_A(P, \cdot)$ is exact.

(iii) Every exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow P \longrightarrow 0$ of A-modules and A-module homomorphisms splits.

(iv) P is a direct summand of a free A-module.

Exercise 2 (10 points)

Does the following statement hold true?

Let $\mathbf{C}$ and $\mathbf{D}$ be two chain complexes such that $H_n(\mathbf{C}) \cong H_n(\mathbf{D})$ for all $n \in \mathbb{Z}$. Then there exists either a morphism of chain complexes $\mathbf{f} : \mathbf{C} \longrightarrow \mathbf{D}$ or a morphism of chain complexes $\mathbf{f} : \mathbf{D} \longrightarrow \mathbf{C}$ such that $H_n(\mathbf{f})$ is an isomorphism for all $n \in \mathbb{Z}$.

Give a proof or a counter-example.

Exercise 3 (10 points)

Let X be a non-empty topological space. X is called *irreducible* if for any closed subsets Y_1 and Y_2 of X the equality $X = Y_1 \cup Y_2$ implies $X = Y_1$ or $X = Y_2$.

(a) Show that X is irreducible if and only if every pair of non-empty open sets in X has a non-empty intersection. Further, show that X is irreducible if and only if every non-empty open subset of X is dense in X.

(b) Let A be a commutative ring with 1. Show that $\mathrm{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.

Please turn over !

Exercise 4 (10 points)

Let X be a non-empty topological space.

(a) Let $Y \subseteq X$ be a topological subspace which is irreducible. Show that the closure $\overline{Y}$ of Y in X is also irreducible.

(b) Show that every irreducible topological subspace Y of X is contained in a maximal irreducible topological subspace.

(c) Show that the maximal irreducible topological subspaces of X are closed and cover X. They are called the *irreducible components of* X.

(d) Let A be a commutative ring with 1. Show that the irreducible components of $\mathrm{Spec}(A)$ are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A.