HU Berlin Winter Term 2011/12

Exercises BMS Basic Course
Commutative Algebra

Prof. Dr. J. Kramer

To be handed in on December 1st after the 1st lecture

Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.

Exercise sheet 6 (40 points)

Exercise 1 (10 points)

Let A be commutative ring with 1 and let M, M', M'' be A-modules.

(a) Show that a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

splits if and only if there exists an isomorphism $M \cong M' \oplus M''$.

(b) Let F be an additive functor mapping the category of A-modules to itself. Recall that F is called *additive* if, for any homomorphisms $f, g : M \longrightarrow M'$, we have $F(f+g) = F(f) + f(g) : F(M) \longrightarrow F(M')$. Show that the isomorphism $M \cong M' \oplus M''$ implies the isomorphism $F(M) \cong F(M') \oplus F(M'')$.

(c) If the functor F of part (b) is covariant and right-exact, show that the connecting homomorphisms

$$\delta_n : L_n F(M'') \longrightarrow L_{n-1} F(M')$$

in the long exact sequence of left-derived functors are zero for all $n \in \mathbb{N}$.

Exercise 2 (10 points)

Compute the $\mathbb{Z}$-module $\mathrm{Ext}^1_{\mathbb{Z}}\big((\mathbb{Z}/n\mathbb{Z})^2, \mathbb{Z}/n\mathbb{Z}\big)$ and give an interpretation of the elements as extensions of $(\mathbb{Z}/n\mathbb{Z})^2$ by $\mathbb{Z}/n\mathbb{Z}$ in the case of $n = 2$.

Exercise 3 (10 points)

Let A be a commutative ring with 1 and let M, N, P be A-modules. Construct the following isomorphisms by using the universal property of the tensor product:

(a) $M \otimes_A N \cong N \otimes_A M$.

(b) $(M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P)$.

(c) $(M \oplus N) \otimes_A P \cong (M \otimes_A P) \oplus (N \otimes_A P)$.

(d) $A \otimes_A M \cong M$.

Please turn over !

Exercise 4 (10 points)

Let A and B be commutative rings with 1. Let $\varphi : A \longrightarrow B$ be a ring homomorphism and let $\varphi^* : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ be given by $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$.

(a) Show that $\varphi^{*-1}(D(f)) = D(\varphi(f))$ for any $f \in A$ and deduce that the map φ^* is continuous (with respect to the Zariski topology).

(b) Prove that $\varphi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$ for any ideal $\mathfrak{a} \subseteq A$ and that $\overline{\varphi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$ for any ideal $\mathfrak{b} \subseteq B$.

(c) Let $\psi : B \longrightarrow C$ be a homomorphism of rings. Show that $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

By (a) and (c), "Spec" is a contravariant functor from the category of commutative rings with 1 to the category of topological spaces.