HU Berlin Winter Term 2011/12

Exercises BMS Basic Course
Commutative Algebra

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To be handed in on December 8th after the 1st lecture

Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.

Exercise sheet 7 (40 points)

Exercise 1 (10 points)

Let A be a commutative ring with 1, M an A-module, and $A \subseteq B$ a ring extension. Let $\mathfrak{a} \subseteq A$, $\mathfrak{b} \subseteq A$ be ideals. Prove the validity of the following $\mathbb{Z}$- resp. A-module isomorphisms:

(a) $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m,n)\mathbb{Z}$.

(b) $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

(c) $A/\mathfrak{a} \otimes_A A/\mathfrak{b} \cong A/(\mathfrak{a}+\mathfrak{b})$.

(d) $M \otimes_A A/\mathfrak{a} \cong M/\mathfrak{a}M$.

(e) $B \otimes_A A[X] \cong B[X]$.

(f) $A[X] \otimes_A A[Y] \cong A[X,Y]$.

Exercise 2 (10 points)

Let M be a finitely generated $\mathbb{Z}$-module. Show that, for any $\mathbb{Z}$-module N, we have

$$\operatorname{Tor}_n^{\mathbb{Z}}(M,N) = 0 \quad (n \in \mathbb{N}, n > 1).$$

What can be said about $\operatorname{Ext}^n_{\mathbb{Z}}(M,N)$, when $n \in \mathbb{N}, n > 1$?

Exercise 3 (10 points)

Let X be a topological space. A *presheaf* $\mathcal{F}$ *of abelian groups (resp. rings) on* X consists of the data:

(1) for every open subset $U \subseteq X$, an abelian group (resp. a ring) $\mathcal{F}(U)$,

(2) for every inclusion $V \subseteq U$ of open subsets of X, a morphism of abelian groups (resp. rings) $\varrho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$,

subject to the conditions:

Please turn over !

(i) $\mathcal{F}(\emptyset) = 0$,

(ii) if U is an open subset of X, then $\varrho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$,

(iii) if $W \subseteq V \subseteq U$ are three open subsets of X, then $\varrho_{UW} = \varrho_{VW} \circ \varrho_{UV}$.

The following problems have to be solved:

(a) Interpret a presheaf of abelian groups (resp. rings) as a contravariant functor from the category of open sets of X to the category of abelian groups (resp. rings).

(b) Let G be an abelian group. Show that the assignment

$$\mathcal{G}(U) := \big\{\varphi : U \longrightarrow G \,\big|\, \forall x \in U,\, \exists V \subseteq U, V \text{ open}\colon x \in V,\ \varphi|_V = \text{const.}\big\}$$

for every open subset $U \subseteq X$ gives rise to a presheaf $\mathcal{G}$ of abelian groups on X.

(c) Let $x \in X$ be fixed and let G be an abelian group. Show that the assignment

$$\mathcal{G}(U) := \begin{cases} \{0\}, & \text{if } x \notin U, \\ G, & \text{if } x \in U, \end{cases}$$

for every open subset $U \subseteq X$ gives rise to a presheaf $\mathcal{G}$ of abelian groups on X; it is called the *skyscraper presheaf* at x with stalk G.

(d) Let $X := \mathbb{R}^n$. Show that the assignment

$$\mathcal{C}(U) := \{f : U \longrightarrow \mathbb{R} \mid f \text{ smooth on } U\}$$

for every open subset $U \subseteq X$ gives rise to a presheaf $\mathcal{C}$ of rings on $\mathbb{R}^n$.

Exercise 4 (10 points)

Let $A = \prod_{j=1}^n A_j$ be the direct product of commutative rings A_j with 1. Show that

$$\mathrm{Spec}(A) = X_1 \,\dot\cup \ldots \dot\cup\, X_n,$$

where X_j is canonically homeomorphic to $\mathrm{Spec}(A_j)$ for $j = 1, \ldots, n$.

Conversely, let A be a commutative ring with 1. Show that the following statements are equivalent:

(i) $\mathrm{Spec}(A)$ is disconnected.

(ii) $A \cong A_1 \times A_2$, where neither of the commutative rings A_1, A_2 is the zero ring.

(iii) A contains an idempotent $a \neq 0, 1$.