HU Berlin Winter Term 2011/12

Exercises BMS Basic Course
Commutative Algebra

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To be handed in on December 15th after the 1st lecture

Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.

Exercise sheet 8 (40 points)

Exercise 1 (10 points)

Let A be commutative ring with 1 and M a free A-module with ordered basis $\{b_1, \ldots, b_n\}$.

(a) Let $m_j = \sum_{k=1}^{n} a_{j,k} \cdot b_k \in M$ $(j = 1, \ldots, n)$. Show that

$$m_1 \wedge \ldots \wedge m_n = \det\big((a_{j,k})_{1 \le j,k \le n}\big) b_1 \wedge \ldots \wedge b_n.$$

Use this to prove that there is an A-module isomorphism

$$\bigwedge^n M \cong A.$$

(b) In general, show that for $1 \le k \le n$, the A-module $\bigwedge^k M$ is free of rank $\binom{n}{k}$.

Exercise 2 (10 points)

Let A be a commutative ring with 1 and let M, N be A-modules. A property P of M is said to be a *local property* of M, if the following holds: M has P if and only if $M_{\mathfrak{p}}$ has P for every $\mathfrak{p} \in \operatorname{Spec}(A)$. Prove the following assertions:

(a) Being trivial is a local property of M.

(b) Flatness is a local property of M.

(c) Being injective is a local property of an A-homomorphism $f : M \longrightarrow N$.

(d) Being surjective is a local property of an A-homomorphism $f : M \longrightarrow N$.

Exercise 3 (10 points)

Let A be a commutative ring with 1. A multiplicatively closed subset $S \subseteq A$ is said to be *saturated* if

$$xy \in S \Longleftrightarrow x \in S \text{ and } y \in S.$$

Please turn over !

(a) Prove that S is saturated if and only if $A \setminus S$ is a union of prime ideals.

(b) If $S \subseteq A$ is any multiplicatively closed subset, show that there is a unique smallest saturated multiplicatively closed subset $\overline{S} \subseteq A$ containing S and that

$$\overline{S} = A \setminus \bigcup_{\substack{\mathfrak{p} \in \operatorname{Spec}(A) \\ \mathfrak{p} \cap S = \emptyset}} \mathfrak{p}\,.$$

(c) Consider $S = 1 + \mathfrak{a}$ for any ideal $\mathfrak{a} \subseteq A$ and compute $\overline{S}$.

Exercise 4 (10 points)

Let A be a commutative ring with 1. Let $f \in A$ and let $\mathfrak{p} \in \operatorname{Spec}(A)$.

(a) Show that the natural ring homomorphism $A \longrightarrow A_f$ induces an homeomorphism

$$\operatorname{Spec}(A_f) \cong D(f) \subseteq \operatorname{Spec}(A).$$

(b) Show that the natural ring homomorphism $A \longrightarrow A_{\mathfrak{p}}$ induces an homeomorphism

$$\operatorname{Spec}(A_{\mathfrak{p}}) \cong \big\{ \mathfrak{q} \in \operatorname{Spec}(A) \,\big|\, \mathfrak{q} \subseteq \mathfrak{p} \big\} \subseteq \operatorname{Spec}(A).$$