HU Berlin Winter Term 2011/12

Exercises BMS Basic Course
Commutative Algebra

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To be handed in on January 5th after the 1st lecture

Please hand in every exercise solution on a seperate sheet and do not forget to put your name and student ID on every sheet.

Exercise sheet 9 (20+20 points)

See overleaf for the definition of the terms directed set, direct system, and direct limit.

Exercise 1* (10 points)

Let X be a set and $\{M_i\}_{i\in I}$ a family of subsets of X, which is closed under finite unions and intersections. Endowing I with the partial order $\leq$ defined by $i \leq j$ if and only if $M_i \subseteq M_j$ makes I into a directed set. For $i \leq j$ in I, let $\varphi_{ij} : M_i \longrightarrow M_j$ be the inclusion map. Then, $\{M_i\}_{i\in I}$ together with the maps $\{\varphi_{ij}\}_{i\leq j\in I}$ is a direct system in the category of sets. Prove that the direct limit of this direct system exists and describe it.

Exercise 2 (20 points)

Let A be a commutative ring with 1 and $\mathfrak{M}_A$ denote the category of A-modules and A-module-homomorphisms.

(a) Let $(I, \leq)$ be a directed set, let $\{M_i\}_{i\in I}$ be a family of A-modules, and let $\{\varphi_{ij} : M_i \longrightarrow M_j\}_{i\leq j\in I}$ be a family of A-module-homomorphisms constituting a direct system in $\mathfrak{M}_A$. Construct a direct limit of this direct system by defining

$$\varinjlim_{i\in I} M_i = \coprod_{i\in I} M_i / \sim ,$$

where

$$x_i \sim x_j \Longleftrightarrow \exists k \in I \ : \ \varphi_{ik}(x_i) = \varphi_{jk}(x_j).$$

(b) Show that if $\mu_i(x_i) = 0$, then there exists $i \leq j$ such that $\varphi_{ij}(x_i) = 0$ in M_j.

(c) Let $A = \mathbb{Z}$ and let I be the set of positive integers equipped with the partial order given by $m \leq n$ if and only if m divides n. Set $M_n = \mathbb{Z}/n\mathbb{Z}$ and for $m \leq n$ define $\varphi_{mn} : \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$ to be the multiplication by n/m. Prove that the direct limit $\varinjlim_{n\in I} M_n$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$.

Exercise 3* (10 points)

Let $\mathcal{F}$ be a presheaf of rings on a topological space X. Let $x \in X$. The *stalk* $\mathcal{F}_x$ *of* $\mathcal{F}$ *at* x is defined as

$$\mathcal{F}_x := \varinjlim_{\substack{U\subseteq X \text{ open} \\ U\ni x}} \mathcal{F}(U),$$

Please turn over !

where the direct limit is taken over all neighborhoods U of x, via the restriction maps of the presheaf $\mathcal{F}$.
Now, consider the presheaf $\mathcal{F}$ of differentiable real-valued functions on the open unit disc $\{z \in \mathbb{C} \,|\, |z| < 1\}$ (in the classical topology). Show that the stalk of $\mathcal{F}$ at the origin is a local ring.

Definitions

A *directed set* is a partially ordered set $(I, \leq)$ such that for any $i, j \in I$ there exists a $k \in I$ such that $i \leq k$ and $j \leq k$.

Let $\mathcal{C}$ be a category and $(I, \leq)$ a directed set. A *direct system in $\mathcal{C}$ indexed by I* consists of a family $\{M_i\}_{i \in I}$ of objects of $\mathcal{C}$ and a family of morphisms $\{\varphi_{ij} : M_i \longrightarrow M_j\}_{i \leq j \in I}$ satisfying

(1) $\varphi_{ii} = \mathrm{id}_{M_i}$ for each $i \in I$, where $\mathrm{id}_{M_i} : M_i \longrightarrow M_i$ is the identity morphism;

(2) $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ whenever $i \leq j \leq k$.

Let $\mathcal{C}$ be a category, $(I, \leq)$ a directed set, and $\big(\{M_i\}_{i \in I}, \{\varphi_{ij} : M_i \longrightarrow M_j\}_{i \leq j \in I}\big)$ a direct system in $\mathcal{C}$. The *direct limit* of such a direct system, if it exists, is an object $\varinjlim\limits_{i \in I} M_i$ of $\mathcal{C}$ together with morphisms $\mu_i : M_i \longrightarrow \varinjlim\limits_{i \in I} M_i$ satisfying the following universal property:

(D1) $\mu_i = \mu_j \circ \varphi_{ij}$ whenever $i \leq j$;

(D2) Let N be an object of $\mathcal{C}$ together with morphisms $\nu_i : M_i \longrightarrow N$ for all $i \in I$ such that $\nu_i = \nu_j \circ \varphi_{ij}$ for all $i \leq j \in I$. Then there exists a unique morphism $\psi : \varinjlim\limits_{i \in I} M_i \longrightarrow N$ such that $\nu_i = \psi \circ \mu_i$ for all $i \in I$.

$$
\begin{tikzcd}
M_i \arrow[dd, "\varphi_{ij}"'] \arrow[dr, "\mu_i"] \arrow[drr, "\nu_i"] & & \\
& \varinjlim\limits_{i \in I} M_i \arrow[r, dashed, "\exists!\psi"] & N \\
M_j \arrow[ur, "\mu_j"] \arrow[urr, "\nu_j"'] & &
\end{tikzcd}
$$

Merry Christmas

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Happy New Year !