HU Berlin Winter Term 2012/13

Test Exam BMS Basic Course
Commutative Algebra

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Test Exam (60 points)

Problem 1 (10 points)

(a) Let $A \subseteq B$ be integral domains and let B be integral over A. Show that A is a field if and only if B is a field.

(b) Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. Show that $\mathfrak{m}/\mathfrak{m}^2 = 0$ if and only if A is a field.

Problem 2 (10 points)

(a) Show that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. Furthermore, for any $\mathbb{Z}$-module M, show that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, M) \cong M/nM$.

(b) For any finitely generated $\mathbb{Z}$-module M, show that

$$\mathrm{Tor}^{\mathbb{Z}}_n(M, N) = 0 \quad (n \in \mathbb{N}, n > 1).$$

Problem 3 (10 points)

Let A be a commutative ring with 1 and let M, N be A-modules.

(a) Formulate the universal property of the tensor product $M \otimes_A N$. Show that this determines $M \otimes_A N$ unique up to A-module isomorphism.

(b) If $m, n \in \mathbb{N}_{>0}$ are coprime, then show that

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0.$$

Problem 4 (10 points)

(a) Let k be a field and A the subring of $k[X, Y]$, given by

$$A := \big\{ a + Xf \,|\, a \in k,\ f \in k[X, Y] \big\}.$$

Show that A is not a noetherian ring by finding an ideal which is not finitely generated.

(b) Let k be an algebraically closed field and let $\mathfrak{a} \subsetneq k[X_1, \dots, X_n]$ be a proper ideal. Prove that

$$V(\mathfrak{a}) \neq \emptyset,$$

where $V(\mathfrak{a}) := \big\{ (a_1, \dots, a_n) \in k^n \,\big|\, f(a_1, \dots, a_n) = 0,\ f \in \mathfrak{a} \big\}$.

Please turn over!

Problem 5 (10 points)

(a) Let $A = k[X,Y,Z]/(XY - Z^2)$ and $\mathfrak{p} = (\bar{X}, \bar{Z})$, where $\bar{X}, \bar{Z}$ denote the classes of X, Z in A, respectively. Show that $\mathfrak{p}^2$ is not a primary ideal of A.

(b) Compute a minimal primary decomposition of the ideal

$$\mathfrak{a} := (XY, YZ, XZ) \subseteq \mathbb{C}[X,Y,Z].$$

Determine the associated prime ideals and determine the isolated as well as the embedded prime ideals for this primary decomposition.

Problem 6 (10 points)

(a) Show that the ring $\mathbb{Z}[\sqrt{-3}]$ is not integrally closed in its field of fractions.

(b) Show that ring $\mathbb{C}[X,Y]/(Y^2 - X^3 - X^2)$ is not integrally closed in its field of fractions.