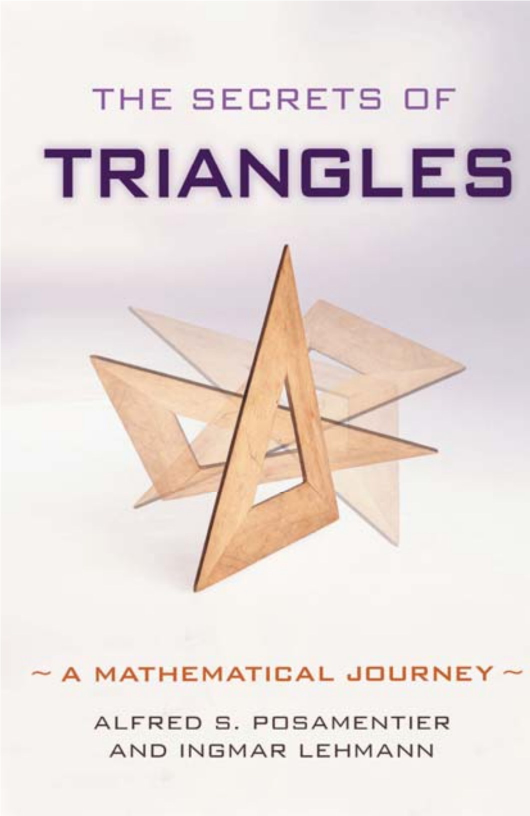
Errata and more:



The Secrets of Triangles

~ A Mathematical Journey ~

Alfred S. Posamentier & Ingmar Lehmann

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Preface

The triangle is one of the basic structures of geometry. We see it in many patterns, and we find that many geometric structures can be best analyzed by partitioning them down into triangles. Yet triangles provide one of the richest examples of geometric phenomena that allow us to admire the beauty of geometry. This is what we hope to achieve in this book, using nothing more than the geometric concepts presented in high school geometry courses.

When we hear the word *triangle* we tend to recall some of the special triangles that we encountered many times in the past, beginning in our earliest days of schooling with such triangles as the equilateral triangle, the isosceles triangle, and the scalene triangle, which were classified by their side lengths. At times we considered triangles classified by their angles, such as the right triangle, the acute triangle and the obtuse triangle. An enlightened teacher probably alerted us to the fact that these triangle descriptions were words that came from our everyday English language outside of mathematics. *Equilateral* means equal sided, as the word *lateral* refers to side. *Isosceles* comes from the Greek word *iso* which means equal; and the Greek word *isoskeles* means equal-legged. The term *scalene* stems from the Latin *scalenus* or the Greek *skalenos*, meaning unequal. A right triangle is one that is erect, coming from the German word *recht*, which comes from the Latin *rectus*, meaning upright, as in perpendicular to a horizontal line.

When we speak of an acute pain, we refer to a sharp pain, hence the word *acute* means sharp. And so an acute triangle is one which has all sharp angles. A dull person is often referred to as being *obtuse*, or not sharp and clear. Consequently, an obtuse triangle is one that has a dull angle.

This classification of triangles is essentially what many people recall about triangles from school days. Some may even recall that there were certain intriguing relationships that occurred in all triangles, such as that the three altitudes (the line segments from a vertex drawn perpendicular to the opposite side) are always concurrent (i.e. intersect at a common point), as are the three angle bisectors (the lines that divide an angle into two equal parts), and the three medians (lines that join a vertex with the midpoint of the opposite side). There are boundless other beautiful properties of triangles – many of which are truly amazing – that we shall explore in this book. Concurrencies arise when one would least expect them. For example, suppose we inscribe a circle in any randomly drawn triangle and then join the three points of tangency to each to the opposite vertices, then we find that these three lines are concurrent. And that is true for *all* triangles! This was first discovered by the French mathematician Joseph Gergonne (1771-1859). We will expand on this surprising relationship in the pages that follow.

Another truly amazing triangle relationship that will be among the many aspects of the triangle we will explore is called *Morley's Theorem*, named after Frank Morley (1860-1937), who was the father of famous American author, Christopher Morley. In 1899 he discovered that if one trisects each of the angles of a triangle – regardless of the shape or size of the triangle – the adjacent angle trisectors will *always* meet at three points forming an equilateral triangle. We will explore this astonishing property (and other related relationships) and even *prove* that it is really true for all triangles!

A remarkable relationship between the interior lines of a triangle (such as the ones we described above) and the sides of the triangle was discovered by the Italian mathematician Giovanni Ceva (1647-1734) in 1678. This theorem makes proving concurrency almost trivial, where traditional proofs – not using Ceva's theorem – would be quite cumbersome. We will also consider the analogue of this lovely relationship – discovered by Menelaus of Alexandria (70-130 CE) – to easily establish if three given points lie on the same straight line.

Besides exploring the multitude of surprising relationships connected with triangles – both special and general triangles – we will also show how and when triangles can be constructed using a straightedge and compasses. This is perhaps the one opportunity in geometry where genuine problem solving techniques are best and most simply exhibited. It is today a rather neglected aspect of geometric explorations, yet one that will appeal to all by the cleverness of the approaches used in doing these relatively elementary constructions – the simplest of which is likely the one most readers will recall from their high school geometry course, the construction of a triangle, given the lengths of its three sides. Yet, we can also – and very cleverly – construct a triangle given only the lengths of its three altitudes. Fun with such triangle constructions will sharpen problem-solving skills.

To make our book reader friendly, we will use a very simple language – one that was used in high school geometry books in past years. We will avoid using some of the more modern (and more precise) nomenclature; again, to make it easier to read. We will call a line $\overleftrightarrow{AB}$, a line segment $\overline{AB}$, a ray $\overrightarrow{AB}$, and the measure of a line segment *AB*, all with the designation *AB* to make the reading a bit less cumbersome. We also do not expect the reader to be familiar with the accepted designations for various triangle parts, such as the center of the inscribed circle of a triangle usually being designated by the letter *I*, or the centroid usually being designated by the letter *G*. We use convenient letters for each diagram that we feel will be reader friendly. To further make our discussions clear to the reader, we provide diagrams for all these discussions – something not necessarily common to all geometry books. We are truly focusing on concept clarity!

As we are about to embark on a journey of exploration of triangle properties that are possessed by special triangles, such as the right triangle (yes, also including the famous Pythagorean theorem), the equilateral triangle, the isosceles triangle, and, of course, the general triangle. We will construct triangles and then we will admire the brilliance of those who discovered the many hidden treasures of geometry. So join us now on this bountiful exploration of all aspects of one of the most common, yet mighty, of geometric figures: the triangle!

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Errata

Page 36: Line 1 from the bottom: Change "$AT_b = AT_c$, $BT_c = BT_a$, $CT_b = CT_a$" to

"$AP_b = AP_c$, $BP_c = BP_a$, $CP_b = CP_a$".

Page 275: Line 6 and line 4 from the bottom: Change "$PA + PB + PB > \frac{1}{2}(AB + AC + BC)$" to

"$PA + PB + PC > \frac{1}{2}(AB + AC + BC)$".

Page 288: Line 4 from the bottom: Change "which states that $\frac{R}{2r} \geq \sqrt{2} + 1$." to

"which states that $\frac{R}{r} \geq \sqrt{2} + 1$, if the triangle has one angle which is at least 90 degrees."

Page 360: Line 3: Change "$c = \phi b = \frac{\sqrt{5}+1}{2} \cdot b$, and $a = \phi c = \phi^2 b = \left(\frac{\sqrt{5}+1}{2}\right)^2 b$, " to

"$c = \phi b = \left(\frac{\sqrt{5}+1}{2}\right) b$, and $a = \phi c = \phi^2 b = \left(\frac{\sqrt{5}+3}{2}\right) b$"

Page 360: Line 5: Change "$a = \phi c = \frac{\sqrt{5}+1}{2} c$; and $b = \phi^{-1} \cdot c = \frac{1}{\phi} \cdot c = \frac{\sqrt{5}-1}{2} \cdot c$. " to

"$a = \phi c = \left(\frac{\sqrt{5}+1}{2}\right) c$ and $b = \phi^{-1} c = \frac{1}{\phi} c = \left(\frac{\sqrt{5}-1}{2}\right) c$."

Page 360: Line 7: Change "$= \frac{1}{f^2} \cdot a = \frac{3-\sqrt{5}}{2} \cdot a$." to "$= \frac{1}{\phi^2} \cdot a = \frac{3-\sqrt{5}}{2} \cdot a$."

We appreciate any comments about the book as well as any typographical errors that have not yet been detected so that they can be incorporated in future printings of the book.

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