Uniform sup-norm bounds on average for cusp forms of higher weights

Dedicated to the Memory of Friedrich Hirzebruch

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Abstract

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind acting by fractional linear transformations on the upper half-plane $\mathbb{H}$. Consider the d-dimensional space of cusp forms $\mathcal{S}_{2k}^{\Gamma}$ of weight $2k$ for Γ, and let $\{f_1, \ldots, f_d\}$ be an orthonormal basis of $\mathcal{S}_{2k}^{\Gamma}$ with respect to the Petersson inner product. In this paper we show that the sup-norm of the quantity $S_{2k}^{\Gamma}(z) := \sum_{j=1}^{d} |f_j(z)|^2 \operatorname{Im}(z)^{2k}$ is bounded as $O_\Gamma(k)$ in the cocompact setting, and as $O_\Gamma(k^{3/2})$ in the cofinite case, where the implied constants depend solely on Γ. We also show that the implied constants are uniform if Γ is replaced by a subgroup of finite index.

1 Introduction

1.1. Motivation. Let M denote a compact Riemann surface of genus $g \geq 2$. From the uniformization theorem there is a unique metric on M, which is compatible with its complex structure and which has constant Gauss curvature equal to -1. On the other hand, from complex algebraic geometry, there is a canonical metric on M obtained by pull-back through the Abel-Jacobi map from M into its Jacobian variety $\mathrm{Jac}(M)$. Let μ_{hyp} and μ_{can} denote the $(1,1)$-forms associated to the hyperbolic and canonical metrics, respectively. A natural question to consider is to compare μ_{hyp} with μ_{can}, in whatever manner possible for general compact Riemann surfaces M of genus $g \geq 2$. Since M has volume 1 with respect to μ_{can}, let us rescale the hyperbolic metric by a multiplicative constant so that the associated $(1,1)$-form μ_{shyp} also gives M volume 1. After some reflection upon the question in hand, one concludes that perhaps the most approachable manner in which one can compare the two metrics is to consider the sup-norm of the function $\mu_{\mathrm{can}}/\mu_{\mathrm{shyp}}$ on M viewed as a finite degree cover of some fixed base Riemann surface M_0.

In [9], the authors proved the following (optimal) result. If M is a compact Riemann surface of genus $g \geq 2$, which is a finite degree cover of a fixed compact Riemann surface M_0, then the bound

$$\sup_{z \in M} \left(\frac{\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{shyp}}(z)} \right) = O_{M_0}(1) \tag{1}$$

holds. To be precise, the main result of [9] applies whenever M is a finite degree cover of M_0, which has finite hyperbolic volume and need not necessarily be compact. In the setting of arithmetic geometry, the ratio $\sup_{z \in M} \mu_{\mathrm{can}}(z)/\mu_{\mathrm{shyp}}(z)$ appears as an analytic invariant in the Arakelov theory of algebraic curves, and the bound (1) playes an important role in [10], where the authors derived bounds for Faltings's delta function and, subsequently, for the Faltings height of Jacobians associated to modular curves. Further comments on the significance of (1) as well as related results will be given in subsection 1.3 below.

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From the point of view of automorphic forms, the ratio $\mu_{\mathrm{can}}/\mu_{\mathrm{shyp}}$ roughly equals the sum of squared norms of an orthonormal basis of cusp forms of weight 2 on M. From this point of view, we can extend the bound (1) in two regards: First, we can consider the sum of squared norms of an orthonormal basis of cusp forms of arbitrary weight $2k$ on M, and second, we can develop bounds which are uniform in the weight. The study of these two questions is the subject of the present article.

1.2. Statement of results. Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind acting by fractional linear transformations on the upper half-plane $\mathbb{H}$, and let $M := \Gamma\backslash\mathbb{H}$ be the corresponding quotient space. We then consider the $\mathbb{C}$-vector space $\mathcal{S}_{2k}^{\Gamma}$ of cusp forms of weight $2k$ for Γ, and let $\{f_1, \ldots, f_d\}$ be an orthonormal basis of $\mathcal{S}_{2k}^{\Gamma}$ with respect to the Petersson inner product; here $d := \dim_{\mathbb{C}}(\mathcal{S}_{2k}^{\Gamma})$. With these notations, we put for $z \in \mathbb{H}$

$$S_{2k}^{\Gamma}(z) := \sum_{j=1}^{d} |f_j(z)|^2 \, \mathrm{Im}(z)^{2k}.$$

In this article, we prove optimal L^∞-bounds for $S_{2k}^{\Gamma}(z)$ in two different directions, namely uniform L^∞-bounds with regard to the weight $2k$, as well as uniform L^∞-bounds through finite degree covers of M. More precisely, the following statement is proven:

Let $\Gamma_0 \subset \mathrm{PSL}_2(\mathbb{R})$ be a fixed Fuchsian subgroup of the first kind and let $\Gamma \subseteq \Gamma_0$ be any subgroup of finite index. For any $k \in \mathbb{N}_{>0}$, we then have the bound

$$\sup_{z\in M} \big(S_{2k}^{\Gamma}(z)\big) = O_{\Gamma_0}(k^{3/2}), \tag{2}$$

where the implied constant depends solely on Γ_0. Moreover, if Γ_0 is cocompact, then we have the improved bound

$$\sup_{z\in M} \big(S_{2k}^{\Gamma}(z)\big) = O_{\Gamma_0}(k), \tag{3}$$

where, again, the implied constant depends solely on Γ_0.

We were somewhat surprised to find different orders of growth in the weight comparing the cocompact to the general cofinite case. With regard to this phenomenon, we prove the following auxiliary result in Proposition 5.1, which indicates where the maximal values occur in the cofinite case:

For a cofinite Fuchsian subgroup Γ of the first kind and $k \in \mathbb{N}_{>0}$, let $\varepsilon > 0$ be such that the neighborhoods of area ε around the cusps of M are disjoint. Assuming that $0 < \varepsilon < 2\pi/k$, we have the bound

$$\sup_{z\in M} \big(S_{2k}^{\Gamma}(z)\big) = O_{\Gamma,\varepsilon}(k),$$

where the implied constant depends solely on Γ and ε.

Moreover, as far as the bounds (2) and (3) are concerned, we are able to show that the results are optimal in both cases, at least up to an additive term in the exponent of the form $-\varepsilon$ for any $\varepsilon > 0$.

1.3. Related results. As stated, the origin of the problems considered in the present article comes from [9], which studies the case of cusp forms of weight 2, i.e., $k = 1$ in the present notation. In this respect, we recall that in the case $\Gamma_0 = \mathrm{PSL}_2(\mathbb{Z})$ and $\Gamma = \Gamma_0(N)$, as a first step the main result of [1] proved for any $\varepsilon > 0$ that

$$\sup_{z\in M} \big(S_2^{\Gamma_0(N)}(z)\big) = O(N^{2+\varepsilon}),$$

which was improved to $O(N^{1+\varepsilon})$ in [16]. In [9], the bound was finally improved to $O_{\Gamma_0}(1)$, not only for the above mentioned setting, but also to the case when neither Γ nor Γ_0 possess any arithmetic properties. With this stated, the present article generalizes the results of [9] to cusp forms of arbitrary even weight and for arbitrary Fuchsian subgroups $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ of the first kind.

In a related direction, there has been considerable interest in obtaining sup-norm bounds for individual Hecke eigenforms, with the most recent results coming from the setting when the groups under consideration are arithmetic. For example, the holomorphic setting of the quantum unique ergodicity (QUE) problem has been studied in [17], [15], and [7]. In [7], it is proven for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ that normalized Hecke eigenforms of weight $2k$ converge weakly to the constant function $3/\pi$ as k tends to infinity. In another direction, the authors prove in [6] the so far best known bound for the L^∞-norm of L^2-normalized Hecke eigenforms for the congruence subgroups $\Gamma_0(N)$ for squarefree N. Specifically, it is shown that

$$\|f\|_\infty \ll_\varepsilon k^{\frac{11}{2}} N^{-\frac{1}{6}+\varepsilon},$$

with an implied constant which only depends on $\varepsilon > 0$. We refer to the introduction as well as the bibliography of the paper [6], which gives an excellent account on the improvements of the bounds for the L^∞-norm of L^2-normalized Hecke eigenforms for the congruence subgroups $\Gamma_0(N)$.

When comparing the results of the above articles to the main theorem of [9] and the present article, one comes to the conclusion that the various results are complementary. From the main result in the present paper in the case $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, one obtains a bound for individual cusp forms which is weaker than in the theorems of the above mentioned articles. When taking the average results from the above mentioned articles, one obtains an average bound which is weaker than the main theorem in the present paper.

More recently, a number of articles have appeared whose results are closely related to the contents of [9], [11], or the present article. In [13], [14], and [19], the authors study various fundamental problems in arithmetic, such as Shafarevich-type conjectures using certain aspects of Arakelov theory, including bounds for certain analytic invariants such as (1) as well as effective bounds for Faltings's delta function (see [11]). In [2], the authors prove an arithmetic analogue of the Hilbert-Samuel theorem, which has far-reaching potential; the main result of the present article is related to the Bergman measure studied in subsection 2.5 therein.

Finally, we refer the reader to the interesting article [21], in which the author proves the existence of cusp forms which, in the (not necessarily squarefree) level aspect, have large modulus, thus disproving a "folklore" conjecture asserting that all forms should be uniformly small.

1.4. Outline of the paper. In section 2, we establish notations and recall background material. In section 3 we prove technical results for the heat kernel associated to the Laplacian Δ_k acting on Maass forms of weight k for Γ. In section 4, we provide a proof of the bound (3) for $\Gamma = \Gamma_0$. By an additional investigation in the neighborhoods of the cusps, we arrive in section 5 at a proof of the bound (2), again in the case that $\Gamma = \Gamma_0$. Finally, in section 6, we are able to establish the uniformity of our bounds (2) and (3) with regard to finite index subgroups Γ in Γ_0. To complete the article, we show that our bounds are optimal, which is the content of section 7.

2 Background material

2.1. Hyperbolic metric. Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be any Fuchsian subgroup of the first kind acting by fractional linear transformations on the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \,|\, z = x + iy\,,\, y > 0\}$. Let M be the quotient space $\Gamma\backslash\mathbb{H}$ and g the genus of M. Denote by $\mathcal{T}$ the set of elliptic fixed points of M and by $\mathcal{C}$ the set of cusps of M; we put $t := |\mathcal{T}|$ and $c := |\mathcal{C}|$. If $p \in \mathcal{T}$, we let m_p denote the order of the elliptic fixed point p; we set $m_p = 1$, if p is a regular point of M. Locally, away from the elliptic fixed points, we identify M with its universal cover $\mathbb{H}$, and hence, denote the points on $M \setminus \mathcal{T}$ by the same letter as the points on $\mathbb{H}$.

We denote by $\mathrm{d}s^2_{\mathrm{hyp}}(z)$ the line element and by $\mu_{\mathrm{hyp}}(z)$ the volume form corresponding to the hyperbolic metric on M, which is compatible with the complex structure of M and has constant curvature equal to -1. Locally on $M \setminus \mathcal{T}$, we have

$$\mathrm{d}s^2_{\mathrm{hyp}}(z) = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} \quad \text{and} \quad \mu_{\mathrm{hyp}}(z) = \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}\,.$$

We denote the hyperbolic distance between $z, w \in M$ by $\mathrm{dist}_{\mathrm{hyp}}(z,w)$ and we recall that the hyperbolic volume $\mathrm{vol}_{\mathrm{hyp}}(M)$ of M is given by the formula

$$\mathrm{vol}_{\mathrm{hyp}}(M) = 2\pi\bigg(2g - 2 + c + \sum_{p \in \mathcal{T}} \Big(1 - \frac{1}{m_p}\Big)\bigg).$$

2.2. Cusp forms of higher weights. For $k \in \mathbb{N}_{>0}$, we let $\mathcal{S}^{\Gamma}_{2k}$ denote the space of cusp forms of weight $2k$ for Γ, i.e., the space of holomorphic functions $f : \mathbb{H} \longrightarrow \mathbb{C}$, which have the transformation behavior

$$f(\gamma z) = (cz + d)^{2k} f(z)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, and which vanish at all the cusps of M. The space $\mathcal{S}^{\Gamma}_{2k}$ is equipped with the inner product

$$\langle f_1, f_2 \rangle := \int_M f_1(z)\overline{f_2(z)}\, y^{2k} \mu_{\mathrm{hyp}}(z) \qquad (f_1, f_2 \in \mathcal{S}^{\Gamma}_{2k}).$$

By letting $d := \dim_{\mathbb{C}}(\mathcal{S}^{\Gamma}_{2k})$ and choosing an orthonormal basis $\{f_1, \ldots, f_d\}$ of $\mathcal{S}^{\Gamma}_{2k}$, we define the quantity

$$S^{\Gamma}_{2k}(z) := \sum_{j=1}^{d} |f_j(z)|^2\, y^{2k}.$$

The main result of this paper consists in giving optimal bounds for the quantity $S^{\Gamma}_{2k}(z)$ as z ranges through out M.

2.3. Maass forms of higher weights. Following [3] or [4], we introduce for any $k \in \mathbb{N}$ the space $\mathcal{V}^{\Gamma}_k$ of functions $\varphi : \mathbb{H} \longrightarrow \mathbb{C}$, which have the transformation behavior

$$\varphi(\gamma z) = \left(\frac{cz + d}{c\bar{z} + d}\right)^{k} \varphi(z) = e^{2ik \arg\,(cz+d)} \varphi(z)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. For $\varphi \in \mathcal{V}^{\Gamma}_k$, we set

$$\|\varphi\|^2 := \int_M |\varphi(z)|^2 \mu_{\mathrm{hyp}}(z),$$

whenever it is defined. We then introduce the Hilbert space

$$\mathcal{H}^{\Gamma}_k := \big\{\varphi \in \mathcal{V}^{\Gamma}_k \,\big|\, \|\varphi\| < \infty\big\}$$

equipped with the inner product

$$\langle \varphi_1, \varphi_2 \rangle := \int_M \varphi_1(z)\overline{\varphi_2(z)} \mu_{\mathrm{hyp}}(z) \qquad (\varphi_1, \varphi_2 \in \mathcal{H}^{\Gamma}_k).$$

The generalized Laplacian

$$\Delta_k := -y^2\bigg(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\bigg) + 2iky\frac{\partial}{\partial x}$$

acts on the smooth functions of $\mathcal{H}_k^\Gamma$ and extends to an essentially self-adjoint linear operator acting on a dense subspace of $\mathcal{H}_k^\Gamma$.

From [3] or [4], we quote that the eigenvalues for the equation

$$\Delta_k \varphi(z) = \lambda \varphi(z) \qquad (\varphi \in \mathcal{H}_k^\Gamma)$$

satisfy the inequality $\lambda \geq k(1-k)$.

Furthermore, if $\lambda = k(1-k)$, then the corresponding eigenfunction φ is of the form $\varphi(z) = f(z)y^k$, where f is a cusp form of weight $2k$ for Γ, i.e., we have an isomorphism of $\mathbb{C}$-vector spaces

$$\ker\big(\Delta_k - k(1-k)\big) \cong \mathcal{S}_{2k}^\Gamma.$$

2.4. Heat kernels of higher weights. The heat kernel on $\mathbb{H}$ associated to Δ_k is computed in [18] and corrects a corresponding formula in [3]. It is given by

$$K_k(t;\rho) = \frac{\sqrt{2}\, e^{-t/4}}{(4\pi t)^{3/2}} \int_\rho^\infty \frac{r e^{-r^2/(4t)}}{\sqrt{\cosh(r) - \cosh(\rho)}}\, T_{2k}\left(\frac{\cosh(r/2)}{\cosh(\rho/2)}\right) \mathrm{d}r\,,$$

where

$$T_{2k}(X) := \cosh(2k \operatorname{arccosh}(X))$$

denotes the $2k$-th Chebyshev polynomial.

The heat kernel on M associated to Δ_k is defined by (see [3], p. 153)

$$K_k^\Gamma(t;z,w) := \sum_{\gamma\in\Gamma} \left(\frac{c\bar{w}+d}{cw+d}\right)^k \left(\frac{z-\gamma\bar{w}}{\gamma w - \bar{z}}\right)^k K_k(t;\rho_{\gamma;z,w}),$$

where $\rho_{\gamma;z,w} := \mathrm{dist}_{\mathrm{hyp}}(z,\gamma w)$. If $z = w$, we put $\rho_{\gamma;z} := \rho_{\gamma;z,z}$ and $K_k^\Gamma(t;z) := K_k^\Gamma(t;z,z)$.

2.5. Spectral expansions. The resolvent kernel on M associated to Δ_k is the integral kernel $G_k^\Gamma(s;z,w)$, which inverts the operator $\Delta_k - s(1-s)$ (see [4], p. 27, Theorem 1.4.10). The heat kernel and the resolvent kernel on M associated to Δ_k are related through the expression

$$G_k^\Gamma(s;z,w) = \int_0^\infty e^{-(s-1/2)^2 t} e^{t/4} K_k^\Gamma(t;z,w), \tag{4}$$

which holds for $s \in \mathbb{C}$ such that $\mathrm{Re}((s-1/2)^2)$ is sufficiently large. In other words, (4) expresses the resolvent kernel on M associated to Δ_k as the Laplace transform of the heat kernel on M associated to Δ_k, with an appropriate change of variables. Conversely, one then can express the heat kernel on M as an inverse Laplace transform, with an appropriate change of variables, of the resolvent kernel on M.

The spectral expansion of the resolvent kernel on M associated to Δ_k is given on p. 40 of [4], which is established as an example of a more general spectral expansion theorem given on p. 37 of [4]. Using the inverse Laplace transform, one then obtains the spectral expansion for the heat kernel on M associated to Δ_k; we leave the details for the derivation to the interested reader. For the purposes of the present article, we derive from the spectral expansion of $K_k^\Gamma(t;z)$ and the fact that the smallest eigenvalue of Δ_k is given by $k(1-k)$ and that the corresponding eigenfunctions are related to $\mathcal{S}_{2k}^\Gamma$, the important relation

$$S_{2k}^\Gamma(z) = \lim_{t\to\infty} e^{-k(k-1)t} K_k^\Gamma(t;z).$$

Furthermore, it is evident from the spectral expansion of the heat kernel that $e^{-k(k-1)t}K_k^\Gamma(t;z)$ is a monotone decreasing function for any $t > 0$, hence we arrive at the estimate

$$e^{k(k-1)t} S_{2k}^\Gamma(z) \leq K_k^\Gamma(t;z) \tag{5}$$

for any $t > 0$ and $z \in \mathbb{H}$.

3 Heat kernel analysis

3.1. Lemma. *For $t > 0$, $\rho > 0$, and $r \geq \rho$, let*

$$F_k(t;\rho,r) := \frac{re^{-r^2/(4t)}}{\sinh(r)} T_{2k}\left(\frac{\cosh(r/2)}{\cosh(\rho/2)}\right).$$

Then, for all values of t, ρ, r in the given range, we have

$$\sinh(r)\frac{\partial}{\partial\rho}F_k(t;\rho,r) + \sinh(\rho)\frac{\partial}{\partial r}F_k(t;\rho,r) < 0.$$

Proof. We put

$$X := \frac{\cosh(r/2)}{\cosh(\rho/2)},$$

and compute

$$\begin{aligned}
&\sinh(r)\frac{\partial}{\partial\rho}F_k(t;\rho,r) + \sinh(\rho)\frac{\partial}{\partial r}F_k(t;\rho,r) = \\
&\sinh(\rho)\, F_k(t;\rho,r)\left(\frac{1}{r} - \frac{r}{2t} - \frac{\cosh(r)}{\sinh(r)}\right) + \frac{re^{-r^2/(4t)}}{\sinh(r)} T'_{2k}(X)\left(\sinh(r)\frac{\partial X}{\partial\rho} + \sinh(\rho)\frac{\partial X}{\partial r}\right).
\end{aligned}$$

It is now easy to see that

$$\frac{1}{r} - \frac{r}{2t} - \frac{\cosh(r)}{\sinh(r)} < 0$$

for all $t > 0$ and $r > 0$. Since $r \geq \rho$, we have $X \geq 1$, and hence

$$T_{2k}(X) = \cosh(2k\,\mathrm{arccosh}(X)) \geq 1,$$

from which we conclude that

$$\sinh(\rho)\, F_k(t;\rho,r)\left(\frac{1}{r} - \frac{r}{2t} - \frac{\cosh(r)}{\sinh(r)}\right) < 0.$$

Furthermore, since $T_{2k}(X)$ is an increasing, positive function, its derivative $T'_{2k}(X)$ is again a positive function. To complete the proof of the lemma, we are therefore left to show that

$$\sinh(r)\frac{\partial X}{\partial\rho} + \sinh(\rho)\frac{\partial X}{\partial r} \leq 0.$$

For this we compute

$$\begin{aligned}
&\sinh(r)\frac{\partial X}{\partial\rho} + \sinh(\rho)\frac{\partial X}{\partial r} = \\
&-\sinh(r)\frac{\cosh(r/2)\sinh(\rho/2)}{2\cosh^2(\rho/2)} + \sinh(\rho)\frac{\sinh(r/2)}{2\cosh(\rho/2)} = \\
&\frac{1}{2\cosh^2(\rho/2)}\big(-\sinh(r)\cosh(r/2)\sinh(\rho/2) + \sinh(\rho)\cosh(\rho/2)\sinh(r/2)\big) = \\
&\frac{1}{2\cosh^2(\rho/2)}\big(-2\sinh(r/2)\cosh^2(r/2)\sinh(\rho/2) + 2\sinh(\rho/2)\cosh^2(\rho/2)\sinh(r/2)\big) = \\
&\frac{\sinh(r/2)\sinh(\rho/2)}{\cosh^2(\rho/2)}\big(-\cosh^2(r/2) + \cosh^2(\rho/2)\big),
\end{aligned}$$

which is negative for $r > \rho$ and vanishes for $r = \rho$. □

3.2. Proposition. *For any $t > 0$, the heat kernel $K_k(t;\rho)$ on $\mathbb{H}$ associated to Δ_k is strictly monotone decreasing for $\rho > 0$.*

Proof. We will prove that $\partial/\partial\rho\, K_k(t;\rho) < 0$ for $\rho > 0$. To simplify notations, we put

$$c(t) := \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}}.$$

In the notation of Lemma 3.1, we then have, using integration by parts,

$$\begin{aligned} K_k(t;\rho) &= c(t)\int_{\rho}^{\infty} F_k(t;\rho,r)\frac{\sinh(r)}{\sqrt{\cosh(r)-\cosh(\rho)}}\,\mathrm{d}r \\ &= -2c(t)\int_{\rho}^{\infty}\frac{\partial}{\partial r}F_k(t;\rho,r)\sqrt{\cosh(r)-\cosh(\rho)}\,\mathrm{d}r\,. \end{aligned}$$

We now apply the Leibniz rule of differentiation to write

$$\begin{aligned} \frac{\partial}{\partial\rho}K_k(t;\rho) = &-2c(t)\int_{\rho}^{\infty}\frac{\partial^2}{\partial r\,\partial\rho}F_k(t;\rho,r)\sqrt{\cosh(r)-\cosh(\rho)}\,\mathrm{d}r \\ &+ c(t)\int_{\rho}^{\infty}\frac{\partial}{\partial r}F_k(t;\rho,r)\frac{\sinh(\rho)}{\sqrt{\cosh(r)-\cosh(\rho)}}\,\mathrm{d}r\,. \end{aligned}$$

Using integration by parts on the first term once again, yields the identity

$$\frac{\partial}{\partial\rho}K_k(t;\rho) = c(t)\int_{\rho}^{\infty}\left(\sinh(r)\frac{\partial}{\partial\rho}F_k(t;\rho,r) + \sinh(\rho)\frac{\partial}{\partial r}F_k(t;\rho,r)\right)\frac{\mathrm{d}r}{\sqrt{\cosh(r)-\cosh(\rho)}}\,.$$

With Lemma 3.1 we conclude that $\partial/\partial\rho\, K_k(t;\rho) < 0$ for $\rho > 0$, which proves the claim. □

3.3. Proposition. *For given Γ, $k \in \mathbb{N}$, and $t > 0$, the heat kernel $K_k^{\Gamma}(t;z)$ on M associated to Δ_k converges absolutely and uniformly on compact subsets K of M.*

Proof. Let $K \subseteq M$ be a compact subset. In order to prove the absolute and uniform convergence of the heat kernel $K_k^{\Gamma}(t;z)$ on M associated to Δ_k for $t > 0$ and $z \in K$, we have to show the convergence of

$$\sum_{\gamma\in\Gamma} K_k(t;\rho_{\gamma;z})$$

for $t > 0$ and $z \in K$. To do this, we introduce for $\rho > 0$ and $z \in K$ the counting function

$$N(\rho;z) := \#\big\{\gamma\in\Gamma \,|\, \rho_{\gamma;z} = \mathrm{dist_{hyp}}(z,\gamma z) < \rho\big\}. \tag{6}$$

By arguing as in the proof of Lemma 2.3 (a) of [12], one proves that

$$N(\rho;z) = O_{\Gamma,K}(e^{\rho}), \tag{7}$$

uniformly for all $z \in K$ with an implied constant depending solely on Γ and K. The dependence on Γ is given by the maximal order of elliptic elements of Γ.

By means of the counting function $N(\rho;z)$, we obtain the following Stieltjes integral representation of the quantity under consideration

$$\sum_{\gamma\in\Gamma} K_k(t;\rho_{\gamma;z}) = \int_0^{\infty} K_k(t;\rho)\,\mathrm{d}N(\rho;z).$$

Since $K_k(t;\rho)$ is a non-negative, continuous, and, by Proposition 3.2, monotone decreasing function of ρ, an elementary argument allows one to derive from (7) the bound

$$\int\limits_0^\infty K_k(t;\rho)\,\mathrm{d}N(\rho;z) = O_{\Gamma,K}\biggl(\int\limits_0^\infty K_k(t;\rho)\,e^{\rho}\,\mathrm{d}\rho\biggr), \tag{8}$$

again uniformly for all $z\in K$ with an implied constant depending solely on Γ and K.
We are thus left to find a suitable bound for $K_k(t;\rho)$. For this we observe the inequality

$$\frac{r^2}{4t}\geq\frac{r^2}{8t}+\frac{\rho^2}{8t}$$

for $r\geq\rho$, which gives

$$\begin{aligned}
K_k(t;\rho) &= \frac{\sqrt{2}\,e^{-t/4}}{(4\pi t)^{3/2}}\int\limits_\rho^\infty\frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r)-\cosh(\rho)}}\,T_{2k}\biggl(\frac{\cosh(r/2)}{\cosh(\rho/2)}\biggr)\mathrm{d}r\\
&\leq e^{-\rho^2/(8t)}\,\frac{\sqrt{2}\,e^{-t/4}}{(4\pi t)^{3/2}}\int\limits_\rho^\infty\frac{re^{-r^2/(8t)}}{\sqrt{\cosh(r)-\cosh(\rho)}}\,T_{2k}\biggl(\frac{\cosh(r/2)}{\cosh(\rho/2)}\biggr)\mathrm{d}r\\
&\leq e^{-\rho^2/(8t)}\,\frac{\sqrt{2}\,e^{-t/4}}{(4\pi t)^{3/2}}\int\limits_0^\infty\frac{re^{-r^2/(8t)}}{\sqrt{\cosh(r)-1}}\,T_{2k}(\cosh(r/2))\,\mathrm{d}r\,;
\end{aligned}\tag{9}$$

for the last inequality we used that the preceding integral is monotone decreasing in ρ, which follows along the same lines as the proof of Proposition 3.2. Using the equalities

$$\cosh(r)-1=2\sinh^2(r/2)\quad\text{and}\quad T_{2k}(\cosh(r/2))=\cosh(kr)\,,$$

the estimate (9) leads to the bound

$$K_k(t;\rho)\leq e^{-\rho^2/(8t)}\,G_k(t) \tag{10}$$

with the function $G_k(t)$ given by

$$G_k(t):=\frac{e^{-t/4}}{(4\pi t)^{3/2}}\int\limits_0^\infty\frac{re^{-r^2/(8t)}}{\sinh(r/2)}\cosh(kr)\,\mathrm{d}r.$$

Introducing the function

$$H(t):=\int\limits_0^\infty e^{-\rho^2/(8t)}e^{\rho}\,\mathrm{d}\rho\,,$$

the bound (10) in combination with (8) yields

$$\sum_{\gamma\in\Gamma}K_k(t;\rho_{\gamma;z})=O_{\Gamma,K}\bigl(G_k(t)H(t)\bigr),$$

where the implied constant equals the implied constant in (8). From this the claim of the proposition follows. ☐

3.4. Corollary. *For any Fuchsian subgroup Γ of the first kind and $k\in\mathbb{N}_{>0}$, we have the bound*

$$S_{2k}^{\Gamma}(z)\leq\sum_{\gamma\in\Gamma}K_k(t;\rho_{\gamma;z})$$

for any $t>0$ and $z\in\mathbb{H}$, where the right-hand side converges uniformly on compact subsets of M.

Proof. Since $k \in \mathbb{N}_{>0}$ and

$$\left| \left(\frac{c\bar{z}+d}{cz+d} \right)^{k} \left(\frac{z-\gamma\bar{z}}{\gamma z-\bar{z}} \right)^{k} \right| = 1$$

for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we deduce for any $t > 0$ and $z \in \mathbb{H}$ from (5) that

$$S_{2k}^{\Gamma}(z) \leq e^{k(k-1)t}\, S_{2k}^{\Gamma}(z) \leq K_{k}^{\Gamma}(t;z) \leq \sum_{\gamma \in \Gamma} K_{k}(t;\rho_{\gamma;z}), \tag{11}$$

where the right-hand side of (11) converges uniformly on compact subsets by Proposition 3.3. This proves the claim. ☐

4 Bounds in the cocompact setting

4.1. Proposition. *For any $\delta > 0$, there is a constant $C_{\delta} > 0$, such that for any Fuchsian subgroup Γ of the first kind and $k \in \mathbb{N}_{>0}$, we have the bound*

$$S_{2k}^{\Gamma}(z) \leq k \sum_{\substack{\gamma \in \Gamma \\ \rho_{\gamma;z} < \delta}} \frac{2\sqrt{2}}{\cosh^{2k}(\rho_{\gamma;z}/2)} + C_{\delta}\, k \sum_{\substack{\gamma \in \Gamma \\ \rho_{\gamma;z} \geq \delta}} \frac{\rho_{\gamma;z}\, e^{-\rho_{\gamma;z}}}{\cosh^{2k}(\rho_{\gamma;z}/2)},$$

where we recall that $\rho_{\gamma;z} = \mathrm{dist}_{\mathrm{hyp}}(z, \gamma z)$ with $z \in \mathbb{H}$ and $\gamma \in \Gamma$.

Proof. From Corollary 3.4, we recall for any $t > 0$ and $z \in \mathbb{H}$ the inequality

$$S_{2k}^{\Gamma}(z) \leq \sum_{\gamma \in \Gamma} K_{k}(t;\rho_{\gamma;z}). \tag{12}$$

We proceed by estimating the right-hand side of (12), i.e., by giving a suitable bound for

$$K_{k}(t;\rho_{\gamma;z}) = \frac{\sqrt{2}\, e^{-t/4}}{(4\pi t)^{3/2}} \int_{\rho_{\gamma;z}}^{\infty} \frac{r e^{-r^{2}/(4t)}}{\sqrt{\cosh(r) - \cosh(\rho_{\gamma;z})}}\, T_{2k}\!\left(\frac{\cosh(r/2)}{\cosh(\rho_{\gamma;z}/2)} \right) \mathrm{d}r\,.$$

We start with some elementary bounds for the Chebyshev polynomials $T_{2k}(X) = \cosh(2k \operatorname{arccosh}(X))$. Using that

$$\operatorname{arccosh}(X) = \log\big(X + \sqrt{X^{2}-1} \big),$$

we find

$$\begin{aligned} \operatorname{arccosh}\!\left(\frac{\cosh(r/2)}{\cosh(\rho_{\gamma;z}/2)} \right) &= \log\left(\frac{1}{\cosh(\rho_{\gamma;z}/2)} \left(\cosh(r/2) + \sqrt{\cosh^{2}(r/2) - \cosh^{2}(\rho_{\gamma;z}/2)} \right) \right) \\ &\leq \log\left(\frac{1}{\cosh(\rho_{\gamma;z}/2)} \left(\cosh(r/2) + \sqrt{\cosh^{2}(r/2) - 1} \right) \right) \\ &= r/2 - \log\big(\cosh(\rho_{\gamma;z}/2) \big). \end{aligned}$$

Therefore, we obtain the bound

$$T_{2k}\!\left(\frac{\cosh(r/2)}{\cosh(\rho_{\gamma;z}/2)} \right) = \cosh\left(2k \operatorname{arccosh}\!\left(\frac{\cosh(r/2)}{\cosh(\rho_{\gamma;z}/2)} \right) \right) \leq \frac{e^{kr}}{\cosh^{2k}(\rho_{\gamma;z}/2)}\,,$$

and hence arrive at

$$\begin{aligned} S_{2k}^{\Gamma}(z) &\leq \frac{\sqrt{2}\, e^{-t/4}}{(4\pi t)^{3/2}} \sum_{\gamma \in \Gamma} \int_{\rho_{\gamma;z}}^{\infty} \frac{r e^{-r^{2}/(4t)}}{\sqrt{\cosh(r) - \cosh(\rho_{\gamma;z})}}\, \frac{e^{kr}}{\cosh^{2k}(\rho_{\gamma;z}/2)}\, \mathrm{d}r \\ &= \frac{\sqrt{2}\, e^{-t/4}}{(4\pi t)^{3/2}} \sum_{\gamma \in \Gamma} \frac{1}{\cosh^{2k}(\rho_{\gamma;z}/2)} \int_{\rho_{\gamma;z}}^{\infty} \frac{r e^{-r^{2}/(4t)+kr}}{\sqrt{\cosh(r) - \cosh(\rho_{\gamma;z})}}\, \mathrm{d}r. \end{aligned} \tag{13}$$

We next multiply both sides of inequality (13) by $te^{-s(s-1)t}$ with $s \in \mathbb{R}$, $s > k$, and integrate from $t = 0$ to $t = \infty$. Recalling form [5], formula 3.325, namely

$$\int_0^\infty e^{-a^2 t} e^{-b^2/(4t)}\, t^{1/2} \frac{\mathrm{d}t}{t} = \frac{\sqrt{\pi}}{a} e^{-ab},$$

we arrive with $a = s - 1/2$ and $b = r$ at the bound

$$\frac{S_{2k}^{\Gamma}(z)}{(s(s-1) - k(k-1))^2} \leq \frac{\sqrt{2\pi}}{(4\pi)^{3/2}(s-1/2)} \sum_{\gamma \in \Gamma} \frac{1}{\cosh^{2k}(\rho_{\gamma;z}/2)} \int_{\rho_{\gamma;z}}^{\infty} \frac{re^{-(s-1/2)r+kr}}{\sqrt{\cosh(r) - \cosh(\rho_{\gamma;z})}}\, \mathrm{d}r.$$

Now, let $s = k + 1$, to get

$$S_{2k}^{\Gamma}(z) \leq \frac{\sqrt{2}}{2\pi} \frac{k^2}{k+1/2} \sum_{\gamma \in \Gamma} \frac{1}{\cosh^{2k}(\rho_{\gamma;z}/2)} \int_{\rho_{\gamma;z}}^{\infty} \frac{re^{-r/2}}{\sqrt{\cosh(r) - \cosh(\rho_{\gamma;z})}}\, \mathrm{d}r. \tag{14}$$

To finish, we will estimate the integral in (14) in a manner similar to the proof of Lemma 4.2 in [11]. We start by first considering the case, where $\rho \geq \delta$. Let us then use the decomposition

$$\int_{\rho_{\gamma;z}}^{\infty} \ldots = \int_{\rho_{\gamma;z}}^{\rho_{\gamma;z}+\log(4)} \ldots + \int_{\rho_{\gamma;z}+\log(4)}^{\infty} \ldots$$

For $r \in [\rho_{\gamma;z}, \rho_{\gamma;z} + \log(4)]$, we have the bound

$$\cosh(r) - \cosh(\rho_{\gamma;z}) = (r - \rho_{\gamma;z}) \sinh(r_*) \geq (r - \rho_{\gamma;z}) \sinh(\rho_{\gamma;z}),$$

where $r_* \in [\rho_{\gamma;z}, \rho_{\gamma;z} + \log(4)]$. With this in mind, we have the estimate

$$\begin{aligned}\int_{\rho_{\gamma;z}}^{\rho_{\gamma;z}+\log(4)} \frac{re^{-r/2}}{\sqrt{\cosh(r) - \cosh(\rho_{\gamma;z})}}\, \mathrm{d}r &\leq \frac{(\rho_{\gamma;z} + \log(4))e^{-\rho_{\gamma;z}/2}}{\sqrt{\sinh(\rho_{\gamma;z})}} \int_{\rho_{\gamma;z}}^{\rho_{\gamma;z}+\log(4)} \frac{\mathrm{d}r}{\sqrt{r - \rho_{\gamma;z}}} \\ &= 2\sqrt{\log(4)}\, \frac{(\rho_{\gamma;z} + \log(4))e^{-\rho_{\gamma;z}/2}}{\sqrt{\sinh(\rho_{\gamma;z})}}\,. \end{aligned} \tag{15}$$

If $r \geq \rho_{\gamma;z} + \log(4)$, we have

$$\frac{\cosh(r)}{2} \geq \frac{\cosh(\rho_{\gamma;z} + \log(4))}{2} \geq \frac{\cosh(\rho_{\gamma;z})\cosh(\log(4))}{2} \geq \cosh(\rho_{\gamma;z}),$$

so then

$$\cosh(r) - \cosh(\rho_{\gamma;z}) \geq \frac{1}{2}\cosh(r) \geq \frac{e^r}{4},$$

hence

$$\int_{\rho_{\gamma;z}+\log(4)}^{\infty} \frac{re^{-r/2}}{\sqrt{\cosh(r) - \cosh(\rho_{\gamma;z})}}\, \mathrm{d}r \leq 2 \int_{\rho_{\gamma;z}+\log(4)}^{\infty} re^{-r}\, \mathrm{d}r = \frac{(\rho_{\gamma;z} + \log(4) + 1)e^{-\rho_{\gamma;z}}}{2}\,. \tag{16}$$

Combining inequalities (15) and (16), we find for $\rho_{\gamma;z} \geq \delta$ a suitable constant $C_\delta > 0$ depending on δ such that

$$\begin{aligned}&\int_{\rho_{\gamma;z}}^{\infty} \frac{re^{-r/2}}{\sqrt{\cosh(r) - \cosh(\rho_{\gamma;z})}}\, \mathrm{d}r \leq \\ &2\sqrt{\log(4)}\, \frac{(\rho_{\gamma;z} + \log(4))e^{-\rho_{\gamma;z}/2}}{\sqrt{\sinh(\rho_{\gamma;z})}} + \frac{(\rho_{\gamma;z} + \log(4) + 1)e^{-\rho_{\gamma;z}}}{2} \leq C_\delta\, \rho_{\gamma;z}\, e^{-\rho_{\gamma;z}}.\end{aligned}$$

From inequality (14), we thus obtain the bound

$$S_{2k}^{\Gamma}(z) \leq k \sum_{\substack{\gamma\in\Gamma\\ \rho_{\gamma;z}<\delta}} \frac{1}{\cosh^{2k}(\rho_{\gamma;z}/2)} \int\limits_{\rho_{\gamma;z}}^{\infty} \frac{re^{-r/2}}{\sqrt{\cosh(r)-\cosh(\rho_{\gamma;z})}}\,\mathrm{d}r + C_\delta\, k \sum_{\substack{\gamma\in\Gamma\\ \rho_{\gamma;z}\geq\delta}} \frac{\rho_{\gamma;z}\, e^{-\rho_{\gamma;z}}}{\cosh^{2k}(\rho_{\gamma;z}/2)}\,. \tag{17}$$

In order to estimate the finite sum in (17), we introduce the function

$$h(\rho) := \int\limits_{\rho}^{\infty} \frac{re^{-r/2}}{\sqrt{\cosh(r)-\cosh(\rho)}}\,\mathrm{d}r = -2\int\limits_{\rho}^{\infty} \sqrt{\cosh(r)-\cosh(\rho)}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{re^{-r/2}}{\sinh(r)}\right)\mathrm{d}r.$$

We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}h(\rho) &= \int\limits_{\rho}^{\infty} \frac{\sinh(\rho)}{\sqrt{\cosh(r)-\cosh(\rho)}}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{re^{-r/2}}{\sinh(r)}\right)\mathrm{d}r \\ &= \int\limits_{\rho}^{\infty} \frac{\sinh(\rho)}{\sqrt{\cosh(r)-\cosh(\rho)}}\,\frac{re^{-r/2}}{\sinh(r)}\left(\frac{1}{r}-\frac{1}{2}-\coth(r)\right)\mathrm{d}r.\end{aligned}$$

Since $\tanh(r) \leq r$, we have that $\coth(r) \geq 1/r$, so then $1/r - 1/2 - \coth(r) \leq -1/2 < 0$, hence the function $h(\rho)$ is monotone decreasing. Therefore, (17) simplifies to

$$S_{2k}^{\Gamma}(z) \leq k \sum_{\substack{\gamma\in\Gamma\\ \rho_{\gamma;z}<\delta}} \frac{1}{\cosh^{2k}(\rho_{\gamma;z}/2)} \int\limits_{0}^{\infty} \frac{re^{-r/2}}{\sqrt{\cosh(r)-1}}\,\mathrm{d}r + C_\delta\, k \sum_{\substack{\gamma\in\Gamma\\ \rho_{\gamma;z}\geq\delta}} \frac{\rho_{\gamma;z}\, e^{-\rho_{\gamma;z}}}{\cosh^{2k}(\rho_{\gamma;z}/2)}\,.$$

Using that $\sinh(r) \geq r$, we have that

$$\int\limits_{0}^{\infty} \frac{re^{-r/2}}{\sqrt{\cosh(r)-1}}\,\mathrm{d}r = \int\limits_{0}^{\infty} \frac{re^{-r/2}}{\sqrt{2}\sinh(r/2)}\,\mathrm{d}r \leq \sqrt{2}\int\limits_{0}^{\infty} e^{-r/2}\,\mathrm{d}r = 2\sqrt{2}.$$

Therefore, we arrive at the bound

$$S_{2k}^{\Gamma}(z) \leq k \sum_{\substack{\gamma\in\Gamma\\ \rho_{\gamma;z}<\delta}} \frac{2\sqrt{2}}{\cosh^{2k}(\rho_{\gamma;z}/2)} + C_\delta\, k \sum_{\substack{\gamma\in\Gamma\\ \rho_{\gamma;z}\geq\delta}} \frac{\rho_{\gamma;z}\, e^{-\rho_{\gamma;z}}}{\cosh^{2k}(\rho_{\gamma;z}/2)},$$

as claimed. □

4.2. Theorem. *For any Fuchsian subgroup Γ of the first kind, $k \in \mathbb{N}_{>0}$, and any compact subset $K \subseteq M$, we have the bound*

$$\sup_{z\in K}\left(S_{2k}^{\Gamma}(z)\right) = O_{\Gamma,K}(k),$$

where the implied constant depends solely on Γ and K.

Proof. From Proposition 4.1, we have the bound

$$\begin{aligned}S_{2k}^{\Gamma}(z) &\leq k \sum_{\substack{\gamma\in\Gamma\\ \rho_{\gamma;z}<\delta}} \frac{2\sqrt{2}}{\cosh^{2k}(\rho_{\gamma;z}/2)} + C_\delta\, k \sum_{\substack{\gamma\in\Gamma\\ \rho_{\gamma;z}\geq\delta}} \frac{\rho_{\gamma;z}\, e^{-\rho_{\gamma;z}}}{\cosh^{2k}(\rho_{\gamma;z}/2)} \\ &\leq k \sum_{\substack{\gamma\in\Gamma\\ \rho_{\gamma;z}<\delta}} \frac{2\sqrt{2}}{\cosh^{2}(\rho_{\gamma;z}/2)} + C_\delta\, k \sum_{\substack{\gamma\in\Gamma\\ \rho_{\gamma;z}\geq\delta}} \frac{\rho_{\gamma;z}\, e^{-\rho_{\gamma;z}}}{\cosh^{2}(\rho_{\gamma;z}/2)}\,. \end{aligned} \tag{18}$$

In order to estimate the first summmand in (18), we observe that the sum is finite and hence is a well-defined continuous function on M, which has a maximum $C'_{\Gamma,K,\delta}>0$ on K, depending solely on Γ, K, and δ. For $z\in K$, we thus have

$$k\sum_{\substack{\gamma\in\Gamma\\ \rho_{\gamma;z}<\delta}}\frac{2\sqrt{2}}{\cosh^2(\rho_{\gamma;z}/2)}\leq C'_{\Gamma,K,\delta}\,k. \tag{19}$$

To finish, we use the counting function $N(\rho;z)$ defined by (6) and its bound (7). For the second summmand in (18), we then find a constant $C''_{\Gamma,K,\delta}>0$ depending solely on Γ, K, and δ such that

$$C_\delta\,k\sum_{\substack{\gamma\in\Gamma\\ \rho_{\gamma;z}\geq\delta}}\frac{\rho_{\gamma;z}\,e^{-\rho_{\gamma;z}}}{\cosh^2(\rho_{\gamma;z}/2)}\leq 4\,C_\delta\,k\sum_{\substack{\gamma\in\Gamma\\ \rho_{\gamma;z}\geq\delta}}\rho_{\gamma;z}\,e^{-2\rho_{\gamma;z}}\leq C''_{\Gamma,K,\delta}\,k\int\limits_0^\infty\rho\,e^{-2\rho}e^{\rho}\,\mathrm{d}\rho=C''_{\Gamma,K,\delta}\,k. \tag{20}$$

Adding up inequalities (19) and (20) yields the claim keeping in mind that δ can be chosen universally. □

4.3. Corollary. *For any cocompact Fuchsian subgroup Γ of the first kind and $k\in\mathbb{N}_{>0}$, we have the bound*

$$\sup_{z\in M}\big(S_{2k}^{\Gamma}(z)\big)=O_\Gamma(k),$$

where the implied constant depends solely on Γ.

Proof. The proof is an immediate consequence of Theorem 4.2. □

5 Bounds in the cofinite setting

5.1. Proposition. *For a cofinite Fuchsian subgroup Γ of the first kind and $k\in\mathbb{N}_{>0}$, let $\varepsilon>0$ be such that the neighborhoods of area ε around the cusps of M are disjoint. Assuming that $0<\varepsilon<2\pi/k$, we have the bound*

$$\sup_{z\in M}\big(S_{2k}^{\Gamma}(z)\big)=O_{\Gamma,\varepsilon}(k),$$

where the implied constant depends solely on Γ and ε.

Proof. For a cusp $p\in\mathcal{C}$, we denote by $U_\varepsilon(p)$ the neighborhood of area ε centered at p. By means of the neighborhoods $U_\varepsilon(p)$, we have the compact subset

$$K_\varepsilon:=M\setminus\bigcup_{p\in\mathcal{C}}U_\varepsilon(p)$$

of M. We will now estimate the quantity $S_{2k}^{\Gamma}(z)$ for z ranging through K_ε and $U_\varepsilon(p)$ ($p\in\mathcal{C}$), respectively.

In the first case, we obtain from Theorem 4.2 that

$$\sup_{z\in K_\varepsilon}\big(S_{2k}^{\Gamma}(z)\big)=O_{\Gamma,K_\varepsilon}(k),$$

where the implied constant depends solely on Γ and K_ε.

In order to prove the claim in the second case, we may assume without loss of generality that p is the cusp at infinity and the neighborhood $U_\varepsilon(p)$ is given by the strip

$$\mathcal{S}_{1/\varepsilon}:=\{z\in\mathbb{H}\,|\,0\leq x<1,\,y>1/\varepsilon\}.$$

For a cusp form $f \in \mathcal{S}_{2k}^{\Gamma}$ of weight $2k$ for Γ, we then consider the expression

$$|f(z)|^2\, y^{2k} = \left|\frac{f(z)}{e^{2\pi i z}}\right|^2 \frac{y^{2k}}{e^{4\pi y}}.$$

The function $|f(z)/e^{2\pi i z}|^2$ is subharmonic and bounded in the strip $\mathcal{S}_{1/\varepsilon}$ and, hence, takes its maximum on the boundary

$$\partial\mathcal{S}_{1/\varepsilon} = \{z \in \mathbb{H} \,|\, 0 \leq x < 1,\, y = 1/\varepsilon\}$$

of $\mathcal{S}_{1/\varepsilon}$, by the strong maximum principle for subharmonic functions. On the other hand, an elementary calculation shows that the function $y^{2k}/e^{4\pi y}$ takes its maximum at

$$y = \frac{k}{2\pi} < \frac{1}{\varepsilon},$$

and is monotone decreasing for $y > k/(2\pi)$. Therefore, we have

$$\sup_{z\in\mathcal{S}_{1/\varepsilon}} \big(|f(z)|^2\, y^{2k}\big) = \sup_{z\in\partial\mathcal{S}_{1/\varepsilon}} \big(|f(z)|^2\, y^{2k}\big).$$

From this we conclude that

$$\sup_{z\in M} \big(S_{2k}^{\Gamma}(z)\big) = \sup_{z\in K_\varepsilon} \big(S_{2k}^{\Gamma}(z)\big) = O_{\Gamma,K_\varepsilon}(k).$$

Since the compact subset K_ε depends only on M, i.e., on Γ, and on ε, the claim of the proposition follows. □

5.2. Theorem. *For a cofinite Fuchsian subgroup Γ of the first kind and $k \in \mathbb{N}_{>0}$, we have the bound*

$$\sup_{z\in M} \big(S_{2k}^{\Gamma}(z)\big) = O_{\Gamma}(k^{3/2}),$$

where the implied constant depends solely on Γ.

Proof. As in the proof of Proposition 5.1, we choose $\varepsilon > 0$ such that the neighborhoods $U_\varepsilon(p)$ of area ε around the cusps $p \in \mathcal{C}$ are disjoint. These neighborhoods give rise to the compact subset

$$K_\varepsilon := M \setminus \bigcup_{p\in\mathcal{C}} U_\varepsilon(p)$$

of M. As before, we will estimate the quantity $S_{2k}^{\Gamma}(z)$ for z ranging through K_ε and $U_\varepsilon(p)$ ($p \in \mathcal{C}$), respectively. As in the proof of Proposition 5.1, we obtain

$$\sup_{z\in K_\varepsilon} \big(S_{2k}^{\Gamma}(z)\big) = O_{\Gamma,K_\varepsilon}(k), \tag{21}$$

where the implied constant depends solely on Γ and K_ε. Since the choice of ε depends only on M, the implied constant depends in the end solely on Γ.

In order to establish the claimed bound for the cuspidal neighborhoods, we distinguish two cases.

(i) If $0 < \varepsilon < 2\pi/k$, the bound for $S_{2k}^{\Gamma}(z)$ in the cuspidal neighborhoods $U_\varepsilon(p)$ ($p \in \mathcal{C}$) is reduced to the bound (21) as in the proof of Proposition 5.1. The proof of the theorem follows in this case.

(ii) If $\varepsilon \geq 2\pi/k$, we have to modify the estimates for $S_{2k}^{\Gamma}(z)$ in the cuspidal neighborhoods $U_\varepsilon(p)$ ($p \in \mathcal{C}$). As before, we may assume without loss of generality that p is the cusp at infinity and the neighborhood $U_\varepsilon(p)$ is given by the strip

$$\mathcal{S}_{1/\varepsilon} := \{z \in \mathbb{H} \,|\, 0 \leq x < 1,\, y > 1/\varepsilon\}.$$

From the argument given in the proof of Proposition 5.1, we find that

$$\sup_{z\in\mathcal{S}_{k/(2\pi)}}\big(S_{2k}^{\Gamma}(z)\big)=\sup_{z\in\partial\mathcal{S}_{k/(2\pi)}}\big(S_{2k}^{\Gamma}(z)\big),$$

where $\mathcal{S}_{k/(2\pi)}$ is the subset of $\mathcal{S}_{1/\varepsilon}$ given by

$$\mathcal{S}_{k/(2\pi)}:=\{z\in\mathbb{H}\,|\,0\leq x<1,\,y>k/(2\pi)\}.$$

Therefore, we are reduced to estimate the quantity $S_{2k}^{\Gamma}(z)$ for z ranging through the set

$$\mathcal{S}_{1/\varepsilon}\setminus\mathcal{S}_{k/(2\pi)}=\{z\in\mathbb{H}\,|\,0\leq x<1,\,1/\varepsilon<y\leq k/(2\pi)\}.$$

For this, we will use the bound

$$S_{2k}^{\Gamma}(z)\leq k\sum_{\substack{\gamma\in\Gamma\\ \rho_{\gamma;z}<\delta}}\frac{2\sqrt{2}}{\cosh^{2k}(\rho_{\gamma;z}/2)}+C_{\delta}\,k\sum_{\substack{\gamma\in\Gamma\\ \rho_{\gamma;z}\geq\delta}}\frac{\rho_{\gamma;z}\,e^{-\rho_{\gamma;z}}}{\cosh^{2k}(\rho_{\gamma;z}/2)} \tag{22}$$

obtained in Proposition 4.1 with an arbitrarily, but fixed chosen $\delta>0$. By means of the stabilizer subgroup

$$\Gamma_{\infty}:=\left\{\begin{pmatrix}1&n\\0&1\end{pmatrix}\,\middle|\,n\in\mathbb{Z}\right\}$$

of the cusp at infinity, we can rewrite inequality (22) as

$$\begin{aligned}S_{2k}^{\Gamma}(z)\leq k&\sum_{\substack{\gamma\in\Gamma_{\infty}\\ \rho_{\gamma;z}<\delta}}\frac{2\sqrt{2}}{\cosh^{2k}(\rho_{\gamma;z}/2)}+C_{\delta}\,k\sum_{\substack{\gamma\in\Gamma_{\infty}\\ \rho_{\gamma;z}\geq\delta}}\frac{\rho_{\gamma;z}\,e^{-\rho_{\gamma;z}}}{\cosh^{2k}(\rho_{\gamma;z}/2)}+\\ &k\sum_{\substack{\gamma\in\Gamma\setminus\Gamma_{\infty}\\ \rho_{\gamma;z}<\delta}}\frac{2\sqrt{2}}{\cosh^{2k}(\rho_{\gamma;z}/2)}+C_{\delta}\,k\sum_{\substack{\gamma\in\Gamma\setminus\Gamma_{\infty}\\ \rho_{\gamma;z}\geq\delta}}\frac{\rho_{\gamma;z}\,e^{-\rho_{\gamma;z}}}{\cosh^{2k}(\rho_{\gamma;z}/2)}\,.\end{aligned} \tag{23}$$

Using the formula

$$\cosh^{2}\left(\frac{\mathrm{dist}_{\mathrm{hyp}}(z,w)}{2}\right)=\frac{|z-\bar{w}|^{2}}{4\,\mathrm{Im}(z)\,\mathrm{Im}(w)}\,,$$

the first two summands on the right-hand-side of (23) can be bounded as

$$\begin{aligned}&k\sum_{\substack{\gamma\in\Gamma_{\infty}\\ \rho_{\gamma;z}<\delta}}\frac{2\sqrt{2}}{\cosh^{2k}(\rho_{\gamma;z}/2)}+C_{\delta}\,k\sum_{\substack{\gamma\in\Gamma_{\infty}\\ \rho_{\gamma;z}\geq\delta}}\frac{\rho_{\gamma;z}\,e^{-\rho_{\gamma;z}}}{\cosh^{2k}(\rho_{\gamma;z}/2)}\leq\\ &k(2\sqrt{2}+C_{\delta}/e)+2k\sum_{n=1}^{\infty}\frac{2\sqrt{2}+C_{\delta}/e}{\big((n/2y)^{2}+1\big)^{k}}\,.\end{aligned}$$

By an integral test, we have (recalling formula 3.251.2 from [5])

$$\sum_{n=1}^{\infty}\frac{1}{\big((n/2y)^{2}+1\big)^{k}}\,\frac{1}{2y}\leq\int\limits_{0}^{\infty}\frac{1}{\big(1+\eta^{2}\big)^{k}}\,\mathrm{d}\eta=\frac{\sqrt{\pi}\,\Gamma(k-1/2)}{2\,\Gamma(k)},$$

which leads to the bound

$$k\sum_{\substack{\gamma\in\Gamma_{\infty}\\ \rho_{\gamma;z}<\delta}}\frac{2\sqrt{2}}{\cosh^{2k}(\rho_{\gamma;z}/2)}+C_{\delta}\,k\sum_{\substack{\gamma\in\Gamma_{\infty}\\ \rho_{\gamma;z}\geq\delta}}\frac{\rho_{\gamma;z}\,e^{-\rho_{\gamma;z}}}{\cosh^{2k}(\rho_{\gamma;z}/2)}=O\bigg(k\,y\frac{\Gamma(k-1/2)}{\Gamma(k)}\bigg)=O\big(k^{3/2}\big),$$

keeping in mind that $y \leq k/(2\pi)$ and using Stirling's formula.

We now turn to estimate the third summand on the right-hand-side of (23). For fixed $z \in \mathcal{S}_{1/\varepsilon} \setminus \mathcal{S}_{k/(2\pi)}$, the sum in question is finite and bounded by the corresponding sum with $k = 1$. Letting z more generally range across the compact subset given by the closure of $\mathcal{S}_{1/\varepsilon}$, the latter sum takes its maximum on that compact set, which depends solely on Γ, ε, and δ. In summary, we obtain

$$k \sum_{\substack{\gamma \in \Gamma \setminus \Gamma_\infty \\ \rho_{\gamma;z} < \delta}} \frac{2\sqrt{2}}{\cosh^{2k}(\rho_{\gamma;z}/2)} = O_\Gamma(k), \tag{24}$$

where the implied constant depends solely on Γ.

We are left to estimate the fourth summand on the right-hand-side of (23). Eventually, by shrinking ε, we may assume that we have $\mathrm{Im}(\gamma z) < 1/\varepsilon$ for all $\gamma \in \Gamma \setminus \Gamma_\infty$; this process depends only on Γ. We then find

$$C_\delta\, k \sum_{\substack{\gamma \in \Gamma \setminus \Gamma_\infty \\ \rho_{\gamma;z} \geq \delta}} \frac{\rho_{\gamma;z}\, e^{-\rho_{\gamma;z}}}{\cosh^{2k}(\rho_{\gamma;z}/2)} \leq C_\delta\, k \sum_{\substack{\gamma \in \Gamma \setminus \Gamma_\infty \\ \rho_{\gamma;z} \geq \delta}} \frac{e^{-\rho_{\gamma;z}/2}}{\cosh^{2k}(\rho_{\gamma;z}/2)} \leq C_\delta\, k \sum_{\gamma \in \Gamma \setminus \Gamma_\infty} \frac{e^{-\rho_{\gamma;z,\varepsilon}/2}}{\cosh^{2}(\rho_{\gamma;z,\varepsilon}/2)}\,, \tag{25}$$

where

$$\rho_{\gamma;z,\varepsilon} := \mathrm{dist}_{\mathrm{hyp}}\big(\gamma z, \partial\mathcal{S}_{1/\varepsilon}\big).$$

Using a counting function similar to (6) with a bound similar to (7), the right-hand side of (25) can be bounded as $O_{\Gamma,\varepsilon}(C_\delta\, k)$ with an implied constant depending solely on Γ and ε, hence solely on Γ.

This completes the proof of the theorem. □

6 Bounds for covers

In this section, we fix a Fuchsian subgroup $\Gamma_0 \subseteq \mathrm{PSL}_2(\mathbb{R})$ of the first kind with quotient space $M_0 := \Gamma_0\backslash\mathbb{H}$. We then consider subgroups $\Gamma \subseteq \Gamma_0$, which are of finite index. The quotient space $M = \Gamma\backslash\mathbb{H}$ then is a finite degree cover of M_0. Our main goal in this section is to give uniform bounds for the quantity $S_{2k}^{\Gamma}(z)$ depending solely on k and Γ_0.

6.1. Theorem. *Let Γ_0 be a fixed Fuchsian subgroup of $\mathrm{PSL}_2(\mathbb{R})$ of the first kind and $\Gamma \subseteq \Gamma_0$ any subgroup of finite index. For any $k \in \mathbb{N}_{>0}$, we then have the bound*

$$\sup_{z \in M} \big(S_{2k}^{\Gamma}(z)\big) = O_{\Gamma_0}(k^{3/2}),$$

where the implied constant depends solely on Γ_0.

Proof. Denote by $\pi : M \longrightarrow M_0$ the covering map and by $\mathcal{C}_0$ the set of cusps of M_0. As before, we choose $\varepsilon > 0$ such that the neighborhoods $U_\varepsilon(p_0)$ of area ε around the cusps $p_0 \in \mathcal{C}_0$ are disjoint. These neighborhoods give rise to the compact subset

$$K_{0,\varepsilon} := M_0 \setminus \bigcup_{p_0 \in \mathcal{C}_0} U_\varepsilon(p_0)$$

of M_0. By means of $K_{0,\varepsilon}$ we obtain the compact subset $K_\varepsilon := \pi^{-1}(K_{0,\varepsilon})$ of M. For z ranging through K_ε, we use Corollary 3.4 to obtain

$$S_{2k}^{\Gamma}(z) \leq \sum_{\gamma \in \Gamma} K_k(t; \rho_{\gamma;z}) \leq \sum_{\gamma \in \Gamma_0} K_k(t; \rho_{\gamma;z}). \tag{26}$$

The proofs of Proposition 4.1 and Theorem 4.2 with Γ and K_ε replaced by Γ_0 and $K_{0,\varepsilon}$, respectively, now show that the right-hand side of inequality (26) can be uniformly bounded as $O_{\Gamma_0}(k)$, keeping in mind that the choice of ε and, hence of the compact subset $K_{0,\varepsilon}$, depend solely on Γ_0.

We are thus left to bound $S^{\Gamma}_{2k}(z)$ in the neighborhoods of the cusps of M obtained by pulling back the neighborhoods $U_\varepsilon(p_0)$ for $p_0 \in \mathcal{C}_0$ to M. In order to do this, we can again assume that p_0 is the cusp at infinity and $U_\varepsilon(p_0)$ is given as the strip

$$\mathcal{S}_{1,1/\varepsilon} := \big\{ z \in \mathbb{H} \,\big|\, 0 \leq x < 1,\, y > 1/\varepsilon \big\}.$$

Furthermore, we may also assume that the cusp $p \in \mathcal{C}$ of M lying over the cusp p_0 is also at infinity of ramification index a, say. The pull-back of the neighborhood $U_\varepsilon(p_0)$ to p via π is then modeled by the strip

$$\mathcal{S}_{a,1/\varepsilon} := \big\{ z \in \mathbb{H} \,\big|\, 0 \leq x < a,\, y > 1/\varepsilon \big\},$$

which contains the strip

$$\mathcal{S}_{a,a/\varepsilon} := \big\{ z \in \mathbb{H} \,\big|\, 0 \leq x < a,\, y > a/\varepsilon \big\}$$

of area ε. As in the proof of Theorem 5.2, we distinguish two cases.

(i) If $0 < \varepsilon < 2\pi/k$, i.e., $a/\varepsilon > ak/(2\pi)$, we show as in Proposition 5.1 that

$$\sup_{z \in \mathcal{S}_{a,a/\varepsilon}} \big(S^{\Gamma}_{2k}(z) \big) = \sup_{z \in \partial\mathcal{S}_{a,a/\varepsilon}} \big(S^{\Gamma}_{2k}(z) \big),$$

and we are reduced to bound $S^{\Gamma}_{2k}(z)$ in the annulus $\mathcal{S}_{a,1/\varepsilon} \setminus \mathcal{S}_{a,a/\varepsilon}$, which will be done below.

(ii) If $\varepsilon \geq 2\pi/k$, i.e., $a/\varepsilon \leq ak/(2\pi)$, we proceed as in the corresponding part of the proof of Theorem 5.2 to find

$$\sup_{z \in \mathcal{S}_{a,ak/(2\pi)}} \big(S^{\Gamma}_{2k}(z) \big) = \sup_{z \in \partial\mathcal{S}_{a,ak/(2\pi)}} \big(S^{\Gamma}_{2k}(z) \big),$$

where $\mathcal{S}_{a,ak/(2\pi)}$ is the strip

$$\mathcal{S}_{a,ak/(2\pi)} := \big\{ z \in \mathbb{H} \,\big|\, 0 \leq x < a,\, y > ak/(2\pi) \big\},$$

which reduces the problem to bound $S^{\Gamma}_{2k}(z)$ to the region $\mathcal{S}_{a,a/\varepsilon} \setminus \mathcal{S}_{a,ak/(2\pi)}$. As in the proof of Theorem 5.2, we next use inequality (23), observing that we now have

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & an \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}.$$

The first two summands in (23) can be bounded by an obvious adaption as $O(k^{3/2})$ as z ranges through the set $\mathcal{S}_{a,a/\varepsilon} \setminus \mathcal{S}_{a,ak/(2\pi)}$, where we use in particular that $y \leq ak/(2\pi)$. Furthermore, by increasing the range of summation in the sums (24) and (25) by replacing $\Gamma \setminus \Gamma_\infty$ by $\Gamma_0 \setminus \Gamma_{0,\infty}$, the argument given in the proof of Theorem 5.2 shows that the third and fourth summand in (23) can both be bounded as $O_{\Gamma_0}(k)$. All in all, we obtain in case (ii)

$$\sup_{z \in \mathcal{S}_{a,a/\varepsilon}} \big(S^{\Gamma}_{2k}(z) \big) = O_{\Gamma_0}(k^{3/2}),$$

and we are also in this case reduced to bound $S^{\Gamma}_{2k}(z)$ in the annulus $\mathcal{S}_{a,1/\varepsilon} \setminus \mathcal{S}_{a,a/\varepsilon}$, which we do next.

To this end, we make again use of the estimate (23) with z ranging through $\mathcal{S}_{a,1/\varepsilon} \setminus \mathcal{S}_{a,a/\varepsilon}$. By estimating the third and the fourth summand in (23) as in (24) and (25) with $\Gamma \setminus \Gamma_\infty$ replaced by $\Gamma_0 \setminus \Gamma_{0,\infty}$, respectively, these two summands can be bounded as $O_{\Gamma_0}(k)$. By proceeding as in the proof of Theorem 5.2, the first and the second summand in (23) can be estimated as $O(k^{1/2}/\varepsilon)$ using that $y \leq a/\varepsilon$.

By adding up all the above estimates, the proof of the theorem is complete. $\square$

6.2. Remark. We note that, if in addition to the hypotheses of Theorem 6.1, the fixed Fuchsian subgroup Γ_0 of $\mathrm{PSL}_2(\mathbb{R})$ of the first kind is cocompact and, hence the subgroup $\Gamma \subseteq \Gamma_0$ of finite index is also cocompact, then the proof of Theorem 6.1 in combination with Corollary 4.3 shows that for any $k \in \mathbb{N}_{>0}$, we then have the bound

$$\sup_{z\in M}\big(S_{2k}^{\Gamma}(z)\big) = O_{\Gamma_0}(k),$$

where the implied constant depends solely on Γ_0.

7 Optimality of the bounds

In this section we show that the bounds obtained in Corollary 4.3 and Theorem 5.2 are optimal, at least in certain cases.

7.1. Optimality in the cocompact setting. In order to address optimality in case that the Fuchsian subgroup Γ of the first kind under consideration is cocompact, we assume in addition that Γ does not contain elliptic elements. We then let ω denote the Hodge bundle on M. For k large enough, we then have by the Riemann-Roch theorem that

$$d = \dim_{\mathbb{C}}\big(\mathcal{S}_{2k}^{\Gamma}\big) = \dim_{\mathbb{C}}\big(H^0(M,\omega^{\otimes 2k})\big) = 2k\deg(\omega) + 1 - g = 2k\,\frac{\mathrm{vol}_{\mathrm{hyp}}(M)}{4\pi} + 1 - g.$$

From this we derive for k large enough

$$\sup_{z\in M}\big(S_{2k}^{\Gamma}(z)\big)\,\mathrm{vol}_{\mathrm{hyp}}(M) \geq \int\limits_M S_{2k}^{\Gamma}(z)\,\mu_{\mathrm{hyp}}(z) = d = 2k\,\frac{\mathrm{vol}_{\mathrm{hyp}}(M)}{4\pi} + 1 - g.$$

Dividing by $\mathrm{vol}_{\mathrm{hyp}}(M) = 4\pi(g-1)$, yields

$$\sup_{z\in M}\big(S_{2k}^{\Gamma}(z)\big) \geq \frac{2k-1}{4\pi}\,,$$

which shows that the bound obtained in Corollary 4.3 is optimal for k being large enough.

7.2. Optimality in the cofinite setting. In this subsection we will show that the bound obtained in Theorem 5.2 in the cofinite setting is optimal in case that $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. For this, let $f \in \mathcal{S}_{2k}^{\Gamma}$ be an L^2-normalized, primitive, Hecke eigenform with Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n)\, e^{2\pi i n z}.$$

In [22], letting $\varepsilon > 0$, the author proves as the main result the bounds

$$k^{1/2-\varepsilon} \ll_{\varepsilon} \sup_{z\in M}\big(|f(z)|^2\, y^{2k}\big) \ll_{\varepsilon} k^{1/2+\varepsilon},$$

with an implied constant depending only on ε. The lower bound, which is of interest for this subsection, is obtained as follows. For fixed $y > 0$, we compute

$$\int\limits_0^1 |f(x+iy)|^2\, y^{2k}\,\mathrm{d}x = \sum_{n=1}^{\infty} |\lambda_f(n)|^2\, y^{2k}\, e^{-4\pi n y} \geq |\lambda_f(1)|^2\, y^{2k}\, e^{-4\pi y}. \tag{27}$$

From [22], we then recall the formula

$$|\lambda_f(1)|^2 = \frac{\pi}{2}\,\frac{(4\pi)^{2k}}{\Gamma(2k)}\,\frac{1}{L\big(\mathrm{Sym}^2(f),1\big)}\,,$$

where $L(\mathrm{Sym}^2(f), s)$ ($s \in \mathbb{C}$) denotes the symmetric square L-function associated to the primitive Hecke eigenform f, which can be bounded as

$$k^{-\varepsilon} \ll_\varepsilon L\big(\mathrm{Sym}^2(f), 1\big) \ll_\varepsilon k^\varepsilon$$

for any $\varepsilon > 0$. Using Stirling's formula, we arrive at the estimate

$$|\lambda_f(1)|^2 \gg_\varepsilon (2k)^{1/2-\varepsilon} \left(\frac{4\pi e}{2k}\right)^{2k}. \tag{28}$$

Using (28), we derive from (27) the lower bound

$$\int_0^1 |f(x+iy)|^2 \, y^{2k} \, \mathrm{d}x \gg_\varepsilon (2k)^{1/2-\varepsilon} \left(\frac{2\pi e}{k}\right)^{2k} \frac{y^{2k}}{e^{4\pi y}}. \tag{29}$$

Evaluating (29) at $y = k/(2\pi)$, we thus obtain the claimed lower bound

$$\sup_{z \in M} \big(|f(z)|^2 \, y^{2k}\big) \geq \int_0^1 |f(x+iy)|^2 \, y^{2k} \, \mathrm{d}x \gg_\varepsilon k^{1/2-\varepsilon}$$

for k large enough with an implied constant depending on the choice of $\varepsilon > 0$.
Let now $\{f_1, \ldots, f_d\}$ be an orthonormal basis of $\mathcal{S}^\Gamma_{2k}$ consisting of primitive Hecke eigenforms. Since $d \gg k$, we arrive with $y = k/(2\pi)$ at

$$\sup_{z \in M} \big(S^\Gamma_{2k}(z)\big) \geq \sum_{j=1}^d \int_0^1 \big|f_j(x+iy)\big|^2 \, y^{2k} \, \mathrm{d}x \gg_\varepsilon k^{3/2-\varepsilon}$$

for k large enough with an implied constant depending on the choice of $\varepsilon > 0$.

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