# On the spectral expansion of hyperbolic Eisenstein series

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**Abstract**

In this article we determine the spectral expansion, meromorphic continuation, and location of poles with identifiable singularities for the scalar-valued hyperbolic Eisenstein series. Similar to the form-valued hyperbolic Eisenstein series studied in [7], the scalar-valued hyperbolic Eisenstein series is defined for each primitive, hyperbolic conjugacy class within the uniformizing group associated to any finite volume hyperbolic Riemann surface. Going beyond the results in [7] and [11], we establish a precise spectral expansion for the hyperbolic Eisenstein series for any finite volume hyperbolic Riemann surface by first proving that the hyperbolic Eisenstein series is in $L^2$. Our other results, such as meromorphic continuation and determination of singularities, are derived from the spectral expansion.

## 1 Introduction

**1.1. Summary.** Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group of the first kind acting by fractional linear transformations on the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \,|\, z = x + iy,\, y > 0\}$. Let $X := \Gamma\backslash\mathbb{H}$, which is a Riemann surface of finite volume with respect to the natural hyperbolic metric induced from $\mathbb{H}$. Associated to any hyperbolic element $\gamma \in \Gamma$, we define a scalar-valued hyperbolic Eisenstein series $\mathcal{E}_{\mathrm{hyp},\gamma}(z, s)$, which is analogous to the form-valued hyperbolic Eisenstein series defined in [7]; see also [11], section 3. We first prove that the hyperbolic Eisenstein series is in $L^2(X)$. The main result of this article is the determination of the full spectral expansion of $\mathcal{E}_{\mathrm{hyp},\gamma}(z, s)$ based on an explicit computation of the inner product of $\mathcal{E}_{\mathrm{hyp},\gamma}(z, s)$ with any eigenfunction of the hyperbolic Laplacian (see Theorem 4.1). The knowledge of the spectral expansion of $\mathcal{E}_{\mathrm{hyp},\gamma}(z, s)$ enables us to also determine its meromorphic continuation (see Theorem 4.2).

**1.2. Comparison with known results.** As stated, hyperbolic Eisenstein series have been considered elsewhere, most notably in [2], [7], and [11]. In [7], the authors define a form-valued hyperbolic Eisenstein series; their main result, which is an analogue of the classical Kronecker limit formula, is that the constant term in a Laurent expansion at the first pole of their hyperbolic Eisenstein series is the harmonic form dual to the cycle determined by the hyperbolic element $\gamma \in \Gamma$ from which they define their series. In addition, the authors prove the meromorphic continuation and establish the location of singularities when $X$ is compact, though they do not explicitly evaluate the spectral expansion of their form-valued hyperbolic Eisenstein series, nor do they study the case when $X$ is non-compact. In [11], the author proves the meromorphic continuation of the scalar-valued hyperbolic Eisenstein series using perturbation theory, but, again, does not discuss the full spectral expansion. More significantly, the consideration in [11] restricts attention to the case when $X$ is compact, whereas the computations here simply require $X$ to have finite hyperbolic volume.

In this article, we obtain the main results analogous to theorems of [7] by first explicitly computing the spectral expansion of our hyperbolic Eisenstein series, then we extract all other results as corollaries. In the case when $X$ is non-compact, we establish the asymptotic behavior of $\mathcal{E}_{\mathrm{hyp},\gamma}(z, s)$ as $z$ tends to a cusp of $X$, which has not been established elsewhere, so then we can consider both compact and non-compact finite volume Riemann surfaces simultaneously.

We are confident that the techniques developed here will apply to other types of hyperbolic Eisenstein series. For example, in [11], the author studies hyperbolic Eisenstein series which are twisted by modular symbols. In order to apply the ideas from the present paper, we need at our disposal an inner product for functions on $\mathbb{H}$, whose functional equation when acted upon by $\Gamma$, agrees with that of this more general Eisenstein series. In [6], the authors interprete the higher-order parabolic Eisenstein series as components of eigensections of certain unipotent bundles on $X$. We are confident that such an interpretation can be made for hyperbolic Eisenstein series twisted by modular symbols, at which time the techniques of the present article will apply. We will leave this problem for future study.

In a different direction, the article [2] studies the asymptotic behavior of hyperbolic Eisenstein series when considering a degenerating sequence of finite volume hyperbolic Riemann surfaces. In brief, the main result in [2] is that the limit of the (properly scaled) hyperbolic Eisenstein series associated to the pinching geodesic from a degenerating sequence of Riemann surfaces is equal to the parabolic Eisenstein series associated to the newly formed cusp on the limit surface. The method of proof in [2] involves a detailed analysis of the differential equation satisfied by the hyperbolic Eisenstein series. The main results in [2] are reproved in [3] using counting function arguments and Stieltjes integral representations of various Eisenstein series.

# 2 Background material and notation

**2.1. Basic notation.** As mentioned in the introduction, we let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ denote a Fuchsian group of the first kind acting by fractional linear transformations on the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \,|\, z = x + iy,\, y > 0\}$. We let $X := \Gamma\backslash\mathbb{H}$, which is a Riemann surface, and denote by $p : \mathbb{H} \longrightarrow X$ the natural projection. The hyperbolic line element $ds^2_{\mathrm{hyp}}$, resp. the hyperbolic Laplacian $\Delta_{\mathrm{hyp}}$, are given as

$$ds^2_{\mathrm{hyp}} := \frac{dx^2 + dy^2}{y^2}, \quad \text{resp.} \quad \Delta_{\mathrm{hyp}} := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Under the change of coordinates $x := e^\rho \cos(\theta)$ and $y := e^\rho \sin(\theta)$, the hyperbolic line element, resp. the hyperbolic Laplacian, are rewritten as

$$ds^2_{\mathrm{hyp}} = \frac{d\rho^2 + d\theta^2}{\sin^2(\theta)}, \quad \text{resp.} \quad \Delta_{\mathrm{hyp}} = -\sin^2(\theta)\left(\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \theta^2}\right).$$

In a slight abuse of notation, we will at times identify $X$ with a fundamental domain in $\mathbb{H}$ (say, a Ford domain, bounded by geodesic paths) and identify points on $X$ with their preimages in such a fundamental domain. Given any measurable functions $f$ and $g$ on $X$, their inner product is defined by

$$\langle f, g\rangle := \int\limits_X f(z)\overline{g(z)}\mu_{\mathrm{hyp}}(z)\,,$$

where

$$\mu_{\mathrm{hyp}}(z) := \frac{dx\,dy}{y^2}, \quad \text{or, in other coordinates,} \quad \mu_{\mathrm{hyp}}(z) = \frac{d\theta\,d\rho}{\sin^2(\theta)}\,.$$

Throughout this paper we will assume that $f$ and $g$ have sufficiently many derivatives and moderate growth when $X$ is non-compact, so then we have, by Green's theorem, the identity

$$\langle \Delta_{\mathrm{hyp}} f, g\rangle = \langle f, \Delta_{\mathrm{hyp}} g\rangle\,. \tag{1}$$

We refer to [1] and [4] for precise details as to when (1) is valid.

**2.2. The Γ-function.** The classical Γ-function will play an important role in our computations, so we will summarize here the relevant properties of the Γ-function which we need. Recall that $\Gamma(s)$ is defined for $\mathrm{Re}(s) > 0$ by the integral

$$\Gamma(s) := \int\limits_0^\infty e^{-t} t^s \frac{dt}{t}\,.$$

Integration by parts shows that $\Gamma(s)$ satisfies the recursion formula $\Gamma(s+1) = s\Gamma(s)$, which also provides the meromorphic continuation of $\Gamma(s)$ to all $s \in \mathbb{C}$; its continuation has singularities at the non-positive integers, and each singularity is a simple pole with residue at $s = -n$ equal to $(-1)^n/n!$ ($n \in \mathbb{N}$). From the recursion formula, one can show that the function

$$g(x,s) := \frac{2^{-s}\Gamma(s+1)}{\Gamma\left((s+ix)/2+1\right)\Gamma((s-ix)/2+1)},$$

defined for $x \in \mathbb{R}$ and $s \in \mathbb{C}$, satisfies the relation

$$g(x,s) = \frac{s(s-1)}{s^2+x^2}\, g(x,s-2).$$

Furthermore, $g(x,s)$ is bounded in the vertical strip $a < \mathrm{Re}(s) < b$ for any $a, b \in \mathbb{R}$ satisfying $-1 < a < b$. Similarly, and in fact equivalently, the function

$$h(s) := \frac{\Gamma((s-1/2+ir_j)/2)\Gamma((s-1/2-ir_j)/2)}{\Gamma^2(s/2)}$$

satisfies

$$h(s+2) = \frac{s(s-1)+\lambda_j}{s^2}\, h(s)\,,$$

where $\lambda_j = 1/4 + r_j^2$; it is bounded in the vertical strip $a < \mathrm{Re}(s) < b$ for any $a, b \in \mathbb{R}$ satisfying $0 < a < b$.

Among the many known identities for the $\Gamma$-function, we shall make use of the following fundamental relation

$$\Gamma((s+1)/2)\Gamma(s/2) = \sqrt{\pi}\, 2^{1-s}\, \Gamma(s). \tag{2}$$

In addition to the above identities, we shall make use of Stirling's asymptotic formula for the $\Gamma$-function, which states that

$$\log\Gamma(s) = \frac{1}{2}\log(2\pi) - \frac{1}{2}\log(s) + s\log(s) - s + o(1), \tag{3}$$

which holds when $s \to \infty$ provided $s$ remains in a sector of the form $|\arg(s)| < \pi - \varepsilon$ for some $\varepsilon > 0$. In particular, we have for fixed $\sigma \in \mathbb{R}$ and $t \to \infty$ the asymptotics

$$\log\Gamma(\sigma+it) = \frac{1}{2}\log(2\pi) + \left(\sigma - \frac{1}{2}\right)\log(t) - \frac{\pi t}{2} + it\log(t) - it + o(1) \tag{4}$$

with an implied constant depending on $\sigma$. For both formulas we refer to [4], p. 198.

**2.3. Hyperbolic Eisenstein series.** Let $\gamma$ be a primitive hyperbolic element of $\Gamma$. Hence there is an element $\sigma \in \mathrm{PSL}_2(\mathbb{R})$ such that

$$\sigma^{-1}\gamma\sigma = \begin{pmatrix} e^{\ell_\gamma/2} & 0 \\ 0 & e^{-\ell_\gamma/2} \end{pmatrix}, \tag{5}$$

where $\ell_\gamma$ denotes the hyperbolic length of the closed geodesic $L_\gamma$ on $X$ in the homotopy class determined by $\gamma$. We note that

$$\widetilde{L}_\gamma := p^{-1}\big(L_\gamma\big) = \sigma\widetilde{L},$$

where $\widetilde{L} := \{z \in \mathbb{H} \,|\, x = \mathrm{Re}(z) = 0\}$ is the positive $y$-axis, and that

$$\Gamma_\gamma := \mathrm{Stab}_\Gamma\big(\widetilde{L}_\gamma\big) = \langle\gamma\rangle.$$

Using the coordinates $\rho = \rho(z)$ and $\theta = \theta(z)$ introduced in subsection 2.1, the *hyperbolic Eisenstein series* $\mathcal{E}_{\mathrm{hyp},\gamma}(z,s)$ *associated to* $\gamma \in \Gamma$ is defined by

$$\mathcal{E}_{\mathrm{hyp},\gamma}(z,s) := \sum_{\eta \in \Gamma_\gamma \backslash \Gamma} \big(\sin(\theta(\sigma^{-1}\eta z))\big)^s . \tag{6}$$

Recalling that the hyperbolic distance $d_{\mathrm{hyp}}(z,\widetilde{L})$ from $z$ to the geodesic line $\widetilde{L}$ is characterized by the formula

$$\sin(\theta(z)) \cdot \cosh(d_{\mathrm{hyp}}(z,\widetilde{L})) = 1 ,$$

we can rewrite the hyperbolic Eisenstein series (6) as

$$\mathcal{E}_{\mathrm{hyp},\gamma}(z,s) = \sum_{\eta \in \Gamma_\gamma \backslash \Gamma} \big(\cosh(d_{\mathrm{hyp}}(\eta z,\widetilde{L}_\gamma))\big)^{-s} .$$

Referring to [2], [3], [10], or [11], where detailed proofs are provided, we recall that the series (6) converges absolutely and locally uniformly for any $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, and that it is invariant with respect to $\Gamma$. A straightforward computation shows that the series (6) satisfies the differential equation

$$\big(\Delta_{\mathrm{hyp}} - s(1-s)\big)\mathcal{E}_{\mathrm{hyp},\gamma}(z,s) = s^2\,\mathcal{E}_{\mathrm{hyp},\gamma}(z,s+2) . \tag{7}$$

**2.4. Spectral expansions.** Under the hypotheses made in subsection 2.1, there is a spectral expansion in terms of the eigenfunctions $\psi_j$ associated to the discrete eigenvalues $\lambda_j$ of the hyperbolic Laplacian $\Delta_{\mathrm{hyp}}$ and the (parabolic) Eisenstein series $\mathcal{E}_{\mathrm{par},P}$ associated to the cusps $P$ of $X$. For any function $f$ on $X$, for which $f$ and $\Delta_{\mathrm{hyp}} f$ are bounded, the spectral expansion is given by the identity

$$f(z) = \sum_{j=0}^{\infty} \langle f,\psi_j\rangle\,\psi_j(z) + \frac{1}{4\pi}\sum_{P \text{ cusp}} \int_{-\infty}^{\infty} \langle f,\mathcal{E}_{\mathrm{par},P}\rangle\,\mathcal{E}_{\mathrm{par},P}(z,1/2+ir)\,dr . \tag{8}$$

We refer to [4] for all aspects of these results, in particular to Theorem 7.3 on p. 103.

**2.5. Counting functions.** Using the notations of subsection 2.3, we define the *hyperbolic counting function* $N_{\mathrm{hyp},\gamma}(T;z)$ as

$$N_{\mathrm{hyp},\gamma}(T;z) := \mathrm{card}\big\{\eta \in \Gamma_\gamma\backslash\Gamma \mid d_{\mathrm{hyp}}(\eta z,\widetilde{L}_\gamma) < T\big\}$$

Equivalently, the function $N_{\mathrm{hyp},\gamma}(T;z)$ counts the number of geodesic paths from $z \in X$ to the closed geodesic $L_\gamma$ on $X$ of length less than $T$. Using the counting function $N_{\mathrm{hyp},\gamma}(T;z)$ we can express the hyperbolic Eisenstein series (6) as a Stieltjes integral, namely we have

$$\mathcal{E}_{\mathrm{hyp},\gamma}(z,s) = \int_0^\infty \big(\cosh(u)\big)^{-s}\,dN_{\mathrm{hyp},\gamma}(u;z) . \tag{9}$$

This representation of $\mathcal{E}_{\mathrm{hyp},\gamma}(z,s)$ plays an important role in [3].

# 3 Preliminary inner product computations

**3.1. Lemma.** *For any* $x \in \mathbb{R}$ *and any* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > -1$*, we have*

$$\int_0^\pi \big(\sin(u)\big)^s e^{-xu}\,du = \pi e^{-\pi x/2}\frac{2^{-s}\Gamma(s+1)}{\Gamma((s+ix)/2+1)\Gamma((s-ix)/2+1)} .$$

*Proof.* We set

$$f(x,s) := \int_0^\pi \big(\sin(u)\big)^s e^{-xu} du\,.$$

Using integration by parts, we arrive at the relation, as long as $\mathrm{Re}(s) > 1$,

$$f(x,s) = \frac{s(s-1)}{s^2+x^2}\, f(x,s-2).$$

As discussed in subsection 2.2, the function

$$g(x,s) = \frac{2^{-s}\Gamma(s+1)}{\Gamma((s+ix)/2+1)\Gamma((s-ix)/2+1)}$$

satisfies the relation

$$g(x,s) = \frac{s(s-1)}{s^2+x^2}\, g(x,s-2).$$

Obviously, both $f(x,s)$ and $g(x,s)$ are bounded and holomorphic, and $g(x,s) \neq 0$ in the vertical strip $1 < \mathrm{Re}(s) < 4$. Therefore, the function $h(x,s) = f(x,s)/g(x,s)$ satisfies $h(x,s) = h(x,s-2)$ and is bounded and holomorphic for all $s \in \mathbb{C}$, and hence is constant in $s$, meaning $h(x,s) = C(x)$. To evaluate $C(x)$, let us take $s = 0$. For this, we have

$$f(x,0) = \int_0^\pi e^{-xu} du = \frac{1}{x}(1-e^{-\pi x}).$$

Also,

$$g(x,0) = \frac{1}{\Gamma(ix/2+1)\Gamma(-ix/2+1)}\,.$$

Taking $w = ix/2$ and using the well-known identity

$$\Gamma(w)\Gamma(1-w) = \frac{\pi}{\sin(\pi w)}\,,$$

we find

$$\Gamma(ix/2+1)\Gamma(-ix/2+1) = \frac{ix}{2}\Gamma(ix/2)\Gamma(1-ix/2) = \frac{ix}{2}\frac{\pi}{\sin(\pi ix/2)}\,.$$

Writing $\sin(\pi ix/2) = i\sinh(\pi x/2)$, we then have

$$g(x,0) = \frac{1}{\Gamma(ix/2+1)\Gamma(-ix/2+1)} = \frac{2\sinh(\pi x/2)}{\pi x}\,.$$

Therefore,

$$C(x) = \frac{f(x,0)}{g(x,0)} = \frac{(1-e^{-\pi x})/x}{2\sinh(\pi x/2)/(\pi x)} = \pi e^{-\pi x/2}.$$

Recalling that $f(x,s) = C(x)g(x,s)$, the stated assertion now follows. □

**3.2. Lemma.** *For any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the hyperbolic Eisenstein series $\mathcal{E}_{\mathrm{hyp},\gamma}(z,s)$ is bounded as a function of $z \in X$. If $X$ is non-compact and $P \in X$ is a cusp satisfying $P = \tau(i\infty)$ for suitable $\tau \in \mathrm{PSL}_2(\mathbb{R})$, we have the estimate*

$$\big|\mathcal{E}_{\mathrm{hyp},\gamma}(z,s)\big| = O\big(\mathrm{Im}(\tau^{-1}z)^{-\mathrm{Re}(s)}\big)$$

*as $z \to P$.*

*Proof.* If $X$ is compact, the boundedness of the hyperbolic Eisenstein series (6) follows from the discussion in subsection 2.3, or the analysis given in [3], section 4.1. It remains to determine the asymptotic behavior of $\mathcal{E}_{\mathrm{hyp},\gamma}(z,s)$, when $X$ is non-compact and $z$ approaches a cusp of $X$.

Without loss of generality, we may assume that the cusp $P$ of $X$ corresponds to the cusp $i\infty$ of a fundamental domain $\mathcal{F} \subseteq \mathbb{H}$ of $X$, which we identify with $X$ and fix for this proof. We then choose $y_0$ sufficiently large such that every point on the geodesic $\widetilde{L}_\gamma \subseteq \mathbb{H}$ has imaginary part less than $y_0$. Let now $z = x + iy \in X$ be such that $y > y_0$, and let $L_0$ denote the horocycle about $z$ at height $y_0$; the hyperbolic distance $d := d_{\mathrm{hyp}}(z, L_0)$ from $z$ to $L_0$ equals $d = \log(y/y_0)$. We consider the counting function

$$N'_{\mathrm{hyp},\gamma}(T; L_0) := \mathrm{card}\{\eta \in \Gamma_\gamma\backslash\Gamma \mid d_{\mathrm{hyp}}(\eta\, L_0, \widetilde{L}_\gamma) < T\}\,. \tag{10}$$

Now, every element of the set

$$\big\{\eta \in \Gamma_\gamma\backslash\Gamma \mid d_{\mathrm{hyp}}(\eta z, \widetilde{L}_\gamma) < T\big\}$$

corresponds to a geodesic path $L$ on $X$ from $z$ to $L_\gamma$ of length less than $T$, which necessarily intersects the horocycle $L_0$ on $X$. Let $d_1$ be the length of the portion of $L$ from $z$ to $L_0$, and let $d_2$ be the length of the portion of $L$ from $L_0$ to $L_\gamma$. Trivially, we have that $d_1 + d_2$ is the length of $L$ and that $d_1 \geq d$. Therefore, we find $d_1 + d_2 < T$, and hence $d_2 < T - d$. This analysis proves the inclusion of sets

$$\big\{\eta \in \Gamma_\gamma\backslash\Gamma \mid d_{\mathrm{hyp}}(\eta z, \widetilde{L}_\gamma) < T\big\} \subseteq \{\eta \in \Gamma_\gamma\backslash\Gamma \mid d_{\mathrm{hyp}}(\eta\, L_0, \widetilde{L}_\gamma) < T - d\}$$

which implies the inequality

$$N_{\mathrm{hyp},\gamma}(T; z) \leq N'_{\mathrm{hyp},\gamma}(T - d; L_0) \tag{11}$$

for $T > d$. Trivially, we also have that $N_{\mathrm{hyp},\gamma}(T; z) = 0$ for $T < d$. Recalling the representation (9) of the hyperbolic Eisenstein series, we have

$$\big|\mathcal{E}_{\mathrm{hyp},\gamma}(z,s)\big| \leq \int_d^\infty \big(\cosh(u)\big)^{-\mathrm{Re}(s)}\, dN_{\mathrm{hyp},\gamma}(u; z) \leq \int_d^\infty \big(\cosh(u)\big)^{-\mathrm{Re}(s)}\, dN'_{\mathrm{hyp},\gamma}(u - d; L_0)\,.$$

Using the elementary bound $\cosh(u) \geq e^u/2$ and letting $v = u - d$, we get the estimate

$$\big|\mathcal{E}_{\mathrm{hyp},\gamma}(z,s)\big| \leq \left(\frac{y}{2y_0}\right)^{-\mathrm{Re}(s)} \int_0^\infty e^{-v\mathrm{Re}(s)}\, dN'_{\mathrm{hyp},\gamma}(v; L_0)\,. \tag{12}$$

The result now follows from elementary counting arguments which imply that the integral in (12) converges for $\mathrm{Re}(s) > 1$ (see [5] and [8]). $\square$

**3.3. Lemma.** *For any smooth, bounded, real-valued function $\phi$ on $X$, we have for sufficiently small $\varepsilon > 0$ the estimate*

$$\langle \mathcal{E}_{\mathrm{hyp},\gamma}, \phi\rangle = \frac{2^{-s}\pi\Gamma(s+1)}{\Gamma^2\left(s/2+1\right)} \cdot \int_{L_\gamma} \phi(z)\, ds_{\mathrm{hyp}}(z) + O\big(\varepsilon/\sqrt{s}\big)$$

*as $s \to \infty$, where the implied constant depends on $\phi$ and $\varepsilon$.*

*Proof.* Without loss of generality it suffices to prove the lemma in the case when $\sigma$ in (5) is the identity matrix. Then, using the series expansion for the hyperbolic Eisenstein series (6), we can unfold the integral in question, resulting in the expression

$$\langle \mathcal{E}_{\mathrm{hyp},\gamma}, \phi\rangle = \int_0^{\ell_\gamma} \int_0^\pi \phi\big(e^\rho e^{i\theta}\big)\big(\sin(\theta)\big)^s \frac{d\theta\, d\rho}{\sin^2(\theta)}\,.$$

For sufficiently small $\varepsilon > 0$, let us write

$$\int_0^{\ell_\gamma}\int_0^{\pi} \dots = \int_0^{\ell_\gamma}\int_0^{\pi/2-\varepsilon} \dots + \int_0^{\ell_\gamma}\int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} \dots + \int_0^{\ell_\gamma}\int_{\pi/2+\varepsilon}^{\pi} \dots$$

Given $\varepsilon > 0$, there is a constant $a_\varepsilon$ with $0 < a_\varepsilon < 1$ and $|\sin(\theta)| \le a_\varepsilon$, whenever $\theta \in [0, \pi/2 - \varepsilon] \cup [\pi/2 + \varepsilon, \pi]$. Then, we have the bound

$$\int_0^{\ell_\gamma}\int_0^{\pi/2-\varepsilon} \phi\big(e^\rho e^{i\theta}\big)\big(\sin(\theta)\big)^{s-2} d\theta\, d\rho + \int_0^{\ell_\gamma}\int_{\pi/2+\varepsilon}^{\pi} \phi\big(e^\rho e^{i\theta}\big)\big(\sin(\theta)\big)^{s-2} d\theta\, d\rho = O\big(a_\varepsilon^{s-2}\big) \tag{13}$$

as $s \to \infty$. Hence we can write

$$\langle \mathcal{E}_{\mathrm{hyp},\gamma}, \phi \rangle = \int_0^{\ell_\gamma}\int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} \phi\big(e^\rho e^{i\theta}\big)\big(\sin(\theta)\big)^{s-2} d\theta\, d\rho + O\big(a_\varepsilon^{s-2}\big) \tag{14}$$

as $s \to \infty$. Now, using a Taylor series expansion for the function $\phi$ with respect to the variable $\theta$ about the point $\theta = \pi/2$, we have that

$$\int_0^{\ell_\gamma} \phi\big(e^\rho e^{i\theta}\big) d\rho = \int_{L_\gamma} \phi(z)\, ds_{\mathrm{hyp}}(z) + O\big((\theta - \pi/2)\big) \tag{15}$$

for $|\theta - \pi/2| < \varepsilon$. Observe that

$$\int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} (\pi/2 - \theta)\big(\sin(\theta)\big)^{s-2} d\theta = \int_{-\varepsilon}^{\varepsilon} u\big(\cos(u)\big)^{s-2} du\,,$$

which we use for simplicity of exposition. Now, on the interval $[-\varepsilon, \varepsilon]$, we consider the estimate

$$u\big(\cos(u)\big)^{s-2} \le g(u),$$

where

$$g(u) := u\big(1 - b_\varepsilon u^2\big)^{s-2} \quad \text{with} \quad b_\varepsilon := \frac{1}{2} - \frac{\varepsilon^2}{24}\,.$$

Since the function $g(u)$ assumes its extrema on $[-\varepsilon, \varepsilon]$ for $u = \pm b_\varepsilon^{-1/2}(2s-3)^{-1/2}$, we get the bound

$$u\big(\cos(u)\big)^{s-2} \le \frac{1}{\sqrt{b_\varepsilon}\sqrt{2s-3}}\left(1 - \frac{1}{2s-3}\right)^{s-2} = O\big(1/\sqrt{s}\big),$$

from which we derive

$$\int_{-\varepsilon}^{\varepsilon} u\big(\cos(u)\big)^{s-2} du = O\big(\varepsilon/\sqrt{s}\big) \tag{16}$$

as $s \to \infty$. By combining (14), (15), and (16), we arrive at

$$\langle \mathcal{E}_{\mathrm{hyp},\gamma}, \phi \rangle = \int_{L_\gamma} \phi(z)\, ds_{\mathrm{hyp}}(z) \cdot \int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} \big(\sin(\theta)\big)^{s-2} d\theta + O\big(\varepsilon/\sqrt{s}\big) \tag{17}$$

as $s \to \infty$. Noting that

$$\int\limits_{0}^{\pi/2-\varepsilon} \big(\sin(\theta)\big)^{s-2} d\theta + \int\limits_{\pi/2+\varepsilon}^{\pi} \big(\sin(\theta)\big)^{s-2} d\theta = O\big(a_{\varepsilon}^{s-2}\big)$$

as $s \to \infty$, combined with (17), we arrive at the estimate

$$\langle \mathcal{E}_{\mathrm{hyp},\gamma}, \phi \rangle = \int\limits_{L_\gamma} \phi(z)\, ds_{\mathrm{hyp}}(z) \cdot \int\limits_{0}^{\pi} \big(\sin(\theta)\big)^{s-2} d\theta + O\big(\varepsilon/\sqrt{s}\big)$$

as $s \to \infty$. To finish, we use Lemma 3.1 with $x = 0$ to give

$$\langle \mathcal{E}_{\mathrm{hyp},\gamma}, \phi \rangle = \frac{2^{-s}\pi\Gamma(s+1)}{\Gamma^2(s/2+1)} \cdot \int\limits_{L_\gamma} \phi(z)\, ds_{\mathrm{hyp}}(z) + O\big(\varepsilon/\sqrt{s}\big)$$

as $s \to \infty$. This completes the proof of the lemma. □

# 4 Spectral expansion and meromorphic continuation

We are now in position to state and prove the main result of this paper.

**4.1. Theorem.** *For any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the hyperbolic Eisenstein series $\mathcal{E}_{\mathrm{hyp},\gamma}(z,s)$ associated to $\gamma \in \Gamma$ admits the spectral expansion*

$$\mathcal{E}_{\mathrm{hyp},\gamma}(z,s) = \sum_{j=0}^{\infty} a_{j,\gamma}(s)\, \psi_j(z) + \frac{1}{4\pi} \sum_{P \text{ cusp}} \int\limits_{-\infty}^{\infty} a_{1/2+ir,\gamma,P}(s)\, \mathcal{E}_{\mathrm{par},P}(z, 1/2+ir)\, dr\,. \tag{18}$$

*The coefficient $a_{j,\gamma}(s)$ is given by the formula*

$$a_{j,\gamma}(s) = \sqrt{\pi} \cdot \frac{\Gamma((s-1/2+ir_j)/2)\Gamma((s-1/2-ir_j)/2)}{\Gamma^2(s/2)} \cdot \int\limits_{L_\gamma} \psi_j(z)\, ds_{\mathrm{hyp}}(z); \tag{19}$$

*here we have written the eigenvalue $\lambda_j$ of the eigenfunction $\psi_j$ in the form $\lambda_j = 1/4 + r_j^2$. An analogous formula holds for the coefficient $a_{1/2+ir,\gamma,P}(s)$; it is given at the end of the proof below.*

*Proof.* The hyperbolic Eisenstein series $\mathcal{E}_{\mathrm{hyp},\gamma}(z,s)$ is a smooth function on $X$, which is bounded by Lemma 3.2. The differential equation (7) allows us to conclude that $\Delta_{\mathrm{hyp}}\, \mathcal{E}_{\mathrm{hyp},\gamma}(z,s)$ is also smooth and bounded on $X$. The existence of the spectral expansion (18) now follows from [4], Theorem 7.3.

The coefficient $a_{j,\gamma}(s)$ is given by the inner product $\langle \mathcal{E}_{\mathrm{hyp},\gamma}, \psi_j \rangle$, which converges by the asymptotic bound proved in Lemma 3.2, and known asymptotic bounds for eigenfunctions of the hyperbolic Laplacian. Using the differential equation (7) and integration by parts, which is justified again using Lemma 3.2 and [4], Theorem 3.1, we have the relation

$$\begin{aligned} &\lambda_j a_{j,\gamma}(s) = \lambda_j \langle \mathcal{E}_{\mathrm{hyp},\gamma}, \psi_j \rangle = \langle \mathcal{E}_{\mathrm{hyp},\gamma}, \Delta_{\mathrm{hyp}}\, \psi_j \rangle = \\ &\langle \Delta_{\mathrm{hyp}}\, \mathcal{E}_{\mathrm{hyp},\gamma}, \psi_j \rangle = s(1-s) a_{j,\gamma}(s) + s^2 a_{j,\gamma}(s+2)\,, \end{aligned}$$

which implies

$$a_{j,\gamma}(s+2) = \frac{s(s-1)+\lambda_j}{s^2}\, a_{j,\gamma}(s). \tag{20}$$

From subsection 2.2, we recall the function

$$h(s) = \frac{\Gamma((s-1/2+ir_j)/2)\Gamma((s-1/2-ir_j)/2)}{\Gamma^2(s/2)}\,,$$

which satisfies the recursion formula

$$h(s+2) = \frac{s(s-1)+\lambda_j}{s^2}\,h(s). \tag{21}$$

From (20) and (21), we conclude that the quotient $a_{j,\gamma}(s)/h(s)$ is invariant under $s \mapsto s+2$; furthermore, it is bounded in a vertical strip, say $2 < \mathrm{Re}(s) < 5$. Therefore, the quotient $a_{j,\gamma}(s)/h(s)$ is constant. In other words, we have

$$a_{j,\gamma}(s) = b_{j,\gamma} \cdot \frac{\Gamma((s-1/2+ir_j)/2)\Gamma((s-1/2-ir_j)/2)}{\Gamma^2(s/2)} \tag{22}$$

for some constant $b_{j,\gamma}$, which is independent of $s$, but possibly depends on $j$ and $\gamma$.

We are left to determine the constant $b_{j,\gamma}$, which we will do now. Using Stirling's formula (3) for real $s$ tending to infinity, we get

$$\log\left(\frac{\Gamma((s-1/2+ir_j)/2)\Gamma((s-1/2-ir_j)/2)}{\Gamma^2((s-1/2)/2)}\right) = o(1) \tag{23}$$

as $s \to \infty$. Using Stirling's formula (3) a second time for real $s$ tending to infinity, we find

$$\log\left(\frac{\Gamma((s-1/2)/2)}{\Gamma(s/2)}\right) = -\frac{1}{4}\log((s-1/2)/2) + o(1) \tag{24}$$

as $s \to \infty$. Combining the asymptotics (23), (24) with (22), we obtain

$$\log\left(\langle \mathcal{E}_{\mathrm{hyp},\gamma}, \psi_j\rangle\right) = \log(b_{j,\gamma}) - \frac{1}{2}\log(s-1/2) + \frac{1}{2}\log(2) + o(1) \tag{25}$$

as $s \to \infty$. Now, recall Lemma 3.3 with $\phi = \psi_j$, namely the formula

$$\langle \mathcal{E}_{\mathrm{hyp},\gamma}, \psi_j\rangle = \frac{2^{-s}\pi\Gamma(s+1)}{\Gamma^2\left(s/2+1\right)} \cdot \int\limits_{L_\gamma} \psi_j(z)\,ds_{\mathrm{hyp}}(z) + O\big(c_\varepsilon^{s-2}\big) \tag{26}$$

as $s \to \infty$. Using (2), we can rewrite the $\Gamma$-factor as

$$\frac{2^{-s}\pi\Gamma(s+1)}{\Gamma^2(s/2+1)} = \sqrt{\pi}\,\frac{\Gamma((s+1)/2)}{\Gamma(s/2+1)}\,.$$

Using Stirling's formula (3) a third time for real $s$ tending to infinity, we get the asymptotics

$$\log\left(\frac{\Gamma((s+1)/2)}{\Gamma(s/2+1)}\right) = -\frac{1}{2}\log(s-1/2) + \frac{1}{2}\log(2) + o(1) \tag{27}$$

as $s \to \infty$. Combining (27) with (26), yields the formula

$$\log\left(\langle \mathcal{E}_{\mathrm{hyp},\gamma}, \psi_j\rangle\right) = \log\left(\int\limits_{L_\gamma} \psi_j(z)\,ds_{\mathrm{hyp}}(z)\right) + \log(\sqrt{\pi}) - \frac{1}{2}\log(s-1/2) + \frac{1}{2}\log(2) + o(1) \tag{28}$$

as $s \to \infty$. Finally, by comparing (25) with (28), we find

$$b_{j,\gamma} = \sqrt{\pi} \cdot \int\limits_{L_\gamma} \psi_j(z)\,ds_{\mathrm{hyp}}(z)\,,$$

as claimed.

Proceeding as in the discrete case, we obtain for the coefficient $a_{1/2+ir,\gamma,P}(s)$ the formula

$$a_{1/2+ir,\gamma,P}(s) = \sqrt{\pi} \cdot \frac{\Gamma((s-1/2+ir)/2)\Gamma((s-1/2-ir)/2)}{\Gamma^2(s/2)} \cdot \int\limits_{L_\gamma} \mathcal{E}_{\mathrm{par},P}(z,1/2+ir)\,ds_{\mathrm{hyp}}(z).$$

This completes the proof of the theorem. □

**4.2. Theorem.** *The hyperbolic Eisenstein series $\mathcal{E}_{\mathrm{hyp},\gamma}(z,s)$ admits a meromorphic continuation to all $s\in\mathbb{C}$. The singularities of the function $\Gamma^2(s/2)\,\mathcal{E}_{\mathrm{hyp},\gamma}(z,s)$ are located at the points*

*(a) $s=1/2\pm ir_j-2n$, where $n\in\mathbb{N}$ and $\lambda_j=1/4+r_j^2$ is the eigenvalue of the $L^2$-eigenfunction $\psi_j$ on $X$, with residues*

$$\mathrm{Res}_{s=1/2\pm ir_j-2n}\big[\Gamma^2(s/2)\,\mathcal{E}_{\mathrm{hyp},\gamma}(z,s)\big]=$$
$$\frac{(-1)^n\sqrt{\pi}\,\Gamma(\pm ir_j-n)}{n!}\cdot\psi_j(z)\cdot\int\limits_{L_\gamma}\psi_j(z)\,ds_{\mathrm{hyp}}(z)\,.$$

*(b) $s=1-\rho-2n$ with $n\in\mathbb{N}_{>0}$, or $s=\rho-2n$ with $n\in\mathbb{N}$, where $w=\rho$ is a pole of the Eisenstein series $\mathcal{E}_{\mathrm{par},P}(z,w)$ satisfying $0<\mathrm{Re}(\rho)<1/2$, with residues*

$$\mathrm{Res}_{s=1-\rho-2n}\big[\Gamma^2(s/2)\,\mathcal{E}_{\mathrm{hyp},\gamma}(z,s)\big]=\frac{(-1)^n\sqrt{\pi}\,\Gamma(1/2-\rho-n)}{2n!}\times$$
$$\times\sum_{P\text{ cusp}}\Bigg[\mathrm{CT}_{w=\rho}\,\mathcal{E}_{\mathrm{par},P}(z,w)\cdot\int\limits_{L_\gamma}\mathrm{Res}_{w=\rho}\,\mathcal{E}_{\mathrm{par},P}(z,w)\,ds_{\mathrm{hyp}}(z)+$$
$$+\mathrm{Res}_{w=\rho}\,\mathcal{E}_{\mathrm{par},P}(z,w)\cdot\int\limits_{L_\gamma}\mathrm{CT}_{w=\rho}\,\mathcal{E}_{\mathrm{par},P}(z,w)\,ds_{\mathrm{hyp}}(z)\Bigg]\,.$$

*In case $s=\rho-2n$, the $\Gamma$-factor in the above formula has to be replaced by $\Gamma(-1/2+\rho-n)$.*

*Proof.* In order to derive the meromorphic continuation of $\mathcal{E}_{\mathrm{hyp},\gamma}(z,s)$ we use the spectral expansion (18). We start by giving the meromorphic continuation for the series in (18) arising from the discrete spectrum. The explicit formula (19) in terms of $\Gamma$-functions proves the meromorphic continuation for the coefficients $a_{j,\gamma}(s)$ to all $s\in\mathbb{C}$. Now, using the well-known sup-norm bound

$$\sup_{z\in X}|\psi_j(z)|=O\big(\sqrt{r_j}\big)$$

for the eigenfunctions together with Stirling's formula (4), we find

$$a_{j,\gamma}(s)\,\psi_j(z)=O\big(r_j^{\mathrm{Re}(s)}e^{-\pi r_j/2}\big)\,,$$

which proves that the series in (18) arising from the discrete spectrum is locally absolutely and uniformly convergent as a function of $s\in\mathbb{C}$ away from the poles. The location of the poles calculation and the determination of the residues arising from this part is straightforward referring to the corresponding facts for the $\Gamma$-function recalled in subsection 2.2.

We now turn to give the meromorphic continuation of the integral in (18) arising from the continuous spectrum. Assuming $1/2<\mathrm{Re}(s)<5/2$ and using the residue theorem we can rewrite the integral in question as

$$\frac{\sqrt{\pi}}{4\pi i}\int\limits_{\mathrm{Re}(w)=1/2}\Gamma((s-1+w)/2)\Gamma((s-w)/2)\,\mathcal{E}_{\mathrm{par},P}(z,w)\int\limits_{L_\gamma}\mathcal{E}_{\mathrm{par},P}(z,w)\,ds_{\mathrm{hyp}}(z)\,dw=$$
$$\frac{\sqrt{\pi}}{4\pi i}\int\limits_{\mathrm{Re}(w)=-1/2}\Gamma((s-1+w)/2)\Gamma((s-w)/2)\,\mathcal{E}_{\mathrm{par},P}(z,w)\int\limits_{L_\gamma}\mathcal{E}_{\mathrm{par},P}(z,w)\,ds_{\mathrm{hyp}}(z)\,dw+$$
$$+\frac{\sqrt{\pi}}{2}\sum_{\substack{\rho\text{ pole of }\mathcal{E}_{\mathrm{par},P}(z,w)\\0<\mathrm{Re}(\rho)<1/2}}\Gamma((s-1+\rho)/2)\Gamma((s-\rho)/2)\times$$

$$\times \left[ \mathrm{CT}_{w=\rho}\, \mathcal{E}_{\mathrm{par},P}(z,w) \cdot \int\limits_{L_\gamma} \mathrm{Res}_{w=\rho}\, \mathcal{E}_{\mathrm{par},P}(z,w)\, ds_{\mathrm{hyp}}(z) + \right.$$

$$\left. \mathrm{Res}_{w=\rho}\, \mathcal{E}_{\mathrm{par},P}(z,w) \cdot \int\limits_{L_\gamma} \mathrm{CT}_{w=\rho}\, \mathcal{E}_{\mathrm{par},P}(z,w)\, ds_{\mathrm{hyp}}(z) \right]. \tag{29}$$

While the left-hand side integral in (29) is holomorphic for $1/2 < \mathrm{Re}(s) < 5/2$, the integral on the right-hand side is holomorphic for $-1/2 < \mathrm{Re}(s) < 3/2$. Since the sum in (29) is meromorphic for all $s \in \mathbb{C}$, formula (29) establishes the meromorphic continuation of the term

$$\frac{1}{4\pi} \sum_{P\,\mathrm{cusp}} \int\limits_{-\infty}^{\infty} a_{1/2+ir,\gamma,P}(s)\, \mathcal{E}_{\mathrm{par},P}(z, 1/2+ir)\, dr \tag{30}$$

in the spectral expansion (18) to the half-plane $\mathrm{Re}(s) > -1/2$. Now, moving the integral along $\mathrm{Re}(w) = -1/2$ in (29) to the vertical line $\mathrm{Re}(w) = -3/2$ using Cauchy's theorem, we obtain the meromorphic continuation of (30) to the half-plane $\mathrm{Re}(s) > -3/2$; note that this time no further residues occur, since $\mathcal{E}_{\mathrm{par},P}(z,w)$ has no poles in the strip $-3/2 < \mathrm{Re}(w) < -1/2$. Continuing in this way, we obtain the meromorphic continuation of (30) to all $s \in \mathbb{C}$.

The location of the poles and their residues can finally be easily read off from the sum over the poles of the Eisenstein series in (29) along the same lines as it was done for the discrete part. □

**4.3. Remark.** As in [7], one can consider an analogue of the Kronecker limit formula, which amounts to understanding the second order term in the Laurent expansion of $\mathcal{E}_{\mathrm{hyp},\gamma}(z,s)$ at a pole. From the spectral expansion given in Theorem 4.1, one easily obtains the spectral expansion of the function which appears in the next order term of the Laurent expansion at a pole.

**4.4. Remark.** As stated in the introduction, hyperbolic Eisenstein series twisted by modular symbols were defined in [11], and their meromorphic continuation was determined using perturbation theory. In the language of [9], one can refer to such series as higher order, non-holomorphic, hyperbolic Eisenstein series. In [6], the authors study higher order, non-holomorphic, *parabolic* Eisenstein series using the framework of unipotent vector bundles. Beginning with the linear algebra associated to vector bundles, one can define an inner product for smooth sections, which would point toward an inner product for the higher order, non-holomorphic, *hyperbolic* Eisenstein series defined in [11]. With this, the methods of the present paper can be applied after establishing a spectral theorem associated to unipotent vector bundles.

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