Exercises for BMS Basic Course
Commutative Algebra

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Sheet 12 (Week: 15.01. – 19.01.2024)

Exercise 1

Let $\mathfrak{a} = (X^2Y, XY^2)$ be an ideal in the polynomial ring $k[X,Y]$ over the field k.

(a) Compute the ideal quotients $(\mathfrak{a} : XY)$, $(\mathfrak{a} : (X^2, XY))$, and $(\mathfrak{a} : (XY, Y^2))$.

(b) Find a minimal primary decomposition of the ideal $\mathfrak{a}$.

(c) Determine the isolated and the embedded prime ideals associated to $\mathfrak{a}$, and find the uniquely determined primary components.

Exercise 2

Let $A = C([0,1], \mathbb{R})$ denote the ring of continuous, real-valued functions on the interval $[0,1]$. Show that the zero ideal in A is not decomposable.

Hint: Recall from Sheet 3, Exercise 1, that every maximal ideal in A is of the form $\mathfrak{m}_x = \{f \in A \mid f(x) = 0\}$ for some $x \in [0,1]$. Assuming now that there is a primary decomposition $(0) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$, construct a non-zero element $f \in \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ using this exercise.

Exercise 3

Let k be an algebraically closed field. For a subset $S \subseteq k[X_1, \ldots, X_n]$, recall that

$$V(S) := \big\{(x_1, \ldots, x_n) \in k^n \,\big|\, f(x_1, \ldots, x_n) = 0,\, \forall f \in S\big\}$$

is called an *affine algebraic set* in k^n.

(a) Show that by defining the affine algebraic sets of k^n to be closed, one obtains a topology on k^n. This topology is called the *Zariski topology* on k^n.

(b) Show that there is a bijection between the affine algebraic sets of k^n and the ideals of $k[X_1, \ldots, X_n]$, which coincide with their radicals, the so-called *radical ideals*.

(c) Show that there is a bijection between the irreducible affine algebraic sets of k^n, the so-called *affine algebraic varieties*, and the prime ideals of $k[X_1, \ldots, X_n]$.

(d) Let $\mathfrak{a} = (X^2Y, XY^2) \subseteq k[X,Y]$. Use Exercise 1 to find a decomposition of the affine algebraic set $V(\mathfrak{a})$ into irreducible affine algebraic sets of k^2.

Exercise 4

Let X be a topological space. A *presheaf* $\mathcal{F}$ *of abelian groups (resp. rings) on* X consists of the data:

(1) for every open subset $U \subseteq X$, an abelian group (resp. a ring) $\mathcal{F}(U)$,

(2) for every inclusion $V \subseteq U$ of open subsets of X, a homomorphism of abelian groups (resp. rings) $\varrho_{UV} \colon \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$,

subject to the conditions:

(i) $\mathcal{F}(\emptyset) = \{0\}$,

(ii) if U is an open subset of X, then $\varrho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$,

(iii) if $W \subseteq V \subseteq U$ are three open subsets of X, then $\varrho_{UW} = \varrho_{VW} \circ \varrho_{UV}$.

Prove the following assertions:

(a) Consider the topological space $X = \mathbb{R}^n$ equipped with the euclidean topology. Show that the assignment

$$\mathcal{C}(U) := \{f \colon U \longrightarrow \mathbb{R} \mid f \text{ is smooth on } U\}$$

for every open subset $U \subseteq X$ gives rise to a presheaf $\mathcal{C}$ of rings on $\mathbb{R}^n$.

(b) Let $x \in X$ be a fixed point in a topological space X and let G be an abelian group. Show that the assignment

$$\mathcal{G}(U) := \begin{cases} \{0\}, & \text{if } x \notin U, \\ G, & \text{if } x \in U, \end{cases}$$

for every open subset $U \subseteq X$ gives rise to a presheaf $\mathcal{G}$ of abelian groups on X.

The presheaf $\mathcal{G}$ is called the *skyscraper presheaf at* x *with stalk* G.