Exercises for BMS Basic Course
Commutative Algebra
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Sheet 14 (Week: 29.01. – 02.02.2024)

Exercise 1

(a) Let A be a noetherian integral domain of dimension 1. Show that any two different, non-zero prime ideals $\mathfrak{p}_1, \mathfrak{p}_2 \subsetneq A$ are coprime, i. e., $\mathfrak{p}_1 + \mathfrak{p}_2 = (1)$.

(b) Let $(A, \mathfrak{m})$ be a local integral domain which is not a field. Suppose that $\mathfrak{m}$ is principal and satisfies $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$. Prove that A is a discrete valuation ring.
Hint: Let $\mathfrak{m} = (x)$. Show that every non-zero element $a \in A$ can be uniquely written as $a = ux^k$, where u is a unit in A and $k \in \mathbb{N}$. Use this to define a discrete valuation on the field of fractions of A having A as its valuation ring.

Exercise 2

Let $K = \mathbb{Q}(\sqrt{-5})$. Consider the ideals

$$\mathfrak{p}_1 = (2, 1 + \sqrt{-5}), \quad \mathfrak{p}_2 = (3, 1 + \sqrt{-5}), \quad \mathfrak{p}_3 = (3, 1 - \sqrt{-5}).$$

in the ring of integers $\mathfrak{O}_K = \mathbb{Z}[\sqrt{-5}]$. Prove the following five statements:

(a) The ideals $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are prime.

(b) $(2) = \mathfrak{p}_1^2$.

(c) $(3) = \mathfrak{p}_2\mathfrak{p}_3$.

(d) $(1 + \sqrt{-5}) = \mathfrak{p}_1\mathfrak{p}_2$.

(e) $(1 - \sqrt{-5}) = \mathfrak{p}_1\mathfrak{p}_3$.

Conclude that $(6) = \mathfrak{p}_1^2\mathfrak{p}_2\mathfrak{p}_3$ and observe that the two distinct factorizations of the number 6, namely

$$\begin{aligned} 6 &= 2 \cdot 3 \\ &= (1 + \sqrt{-5})(1 - \sqrt{-5}) \end{aligned}$$

into irreducible elements in $\mathfrak{O}_K$, arise from different groupings of the unique factorization of the principal ideal (6) into prime ideals.

Definitions for Exercise 3

(1) A *directed set* is a partially ordered set $(I, \leq)$ such that for any $i, j \in I$ there exists a $k \in I$ such that $i \leq k$ and $j \leq k$.

(2) Let $\mathcal{C}$ be a category and $(I, \leq)$ a directed set. A *direct system in $\mathcal{C}$ indexed by I* consists of a family $\{M_i\}_{i\in I}$ of objects of $\mathcal{C}$ and a family of morphisms $\{\varphi_{ij} \colon M_i \longrightarrow M_j\}_{i\leq j\in I}$ satisfying

(S1) $\varphi_{ii} = \mathrm{id}_{M_i}$ for each $i \in I$, where $\mathrm{id}_{M_i} \colon M_i \longrightarrow M_i$ is the identity morphism;

(S2) $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$, whenever $i \leq j \leq k$.

(3) Let $\mathcal{C}$ be a category, $(I, \leq)$ a directed set, and $\big(\{M_i\}_{i\in I}, \{\varphi_{ij} \colon M_i \longrightarrow M_j\}_{i\leq j\in I}\big)$ a direct system in $\mathcal{C}$ indexed by I. The *direct limit* of such a direct system, if it exists, is an object $\varinjlim_{i\in I} M_i$ of $\mathcal{C}$ together with morphisms $\mu_i \colon M_i \longrightarrow \varinjlim_{i\in I} M_i$ satisfying the following universal property:

(L1) $\mu_i = \mu_j \circ \varphi_{ij}$, whenever $i \leq j$;

(L2) Let N be an object of $\mathcal{C}$ together with morphisms $\nu_i \colon M_i \longrightarrow N$ for all $i \in I$ such that $\nu_i = \nu_j \circ \varphi_{ij}$, whenever $i \leq j$. Then, there exists a unique morphism $\psi \colon \varinjlim_{i\in I} M_i \longrightarrow N$ such that $\nu_i = \psi \circ \mu_i$ for all $i \in I$.

Exercise 3

Let A be a commutative ring with 1 and let $\mathfrak{M}_A$ denote the category of A-modules and A-module homomorphisms.

(a) Let $(I, \leq)$ be a directed set, $\{M_i\}_{i\in I}$ a family of A-modules, and $\{\varphi_{ij} \colon M_i \longrightarrow M_j\}_{i\leq j\in I}$ a family of A-module homomorphisms constituting a direct system in $\mathfrak{M}_A$ indexed by I. Construct a direct limit of this direct system by defining

$$\varinjlim_{i\in I} M_i := \coprod_{i\in I} M_i / \sim ,$$

where we put

$$x_i \sim x_j \quad \Longleftrightarrow \quad \exists k \in I \colon \varphi_{ik}(x_i) = \varphi_{jk}(x_j) \qquad (x_i \in M_i, x_j \in M_j).$$

(b) Show that if $\mu_i(x_i) = 0$, then there exists $i \leq j$ such that $\varphi_{ij}(x_i) = 0$ in M_j.

(c) Let $A = \mathbb{Z}$ and let I be the set of positive integers equipped with the partial order given by $m \leq n$ if and only if m divides n. Set $M_n = \mathbb{Z}/n\mathbb{Z}$ and for $m \leq n$ define $\varphi_{m,n} \colon \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$ to be multiplication by n/m.
Prove that the direct limit $\varinjlim_{n\in I} M_n$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$.

Exercise 4

Let $\mathcal{F}$ be a presheaf of rings on a topological space X and let $x \in X$. The *stalk* $\mathcal{F}_x$ *of* $\mathcal{F}$ *at* x is defined as

$$\mathcal{F}_x := \varinjlim_{\substack{U \subseteq X, \text{open} \\ x \in U}} \mathcal{F}(U),$$

where the direct limit is taken over all open neighborhoods U of x via the restriction maps of the presheaf $\mathcal{F}$.

Now, consider the presheaf $\mathcal{F}$ of real-valued differentiable functions on the open unit disc $\{z \in \mathbb{C} \,|\, |z| < 1\}$ (in the classical topology). Show that the stalk of $\mathcal{F}$ at the origin is a local ring.