HU Berlin Winter Semester 2023/24

Exercises for BMS Basic Course
Commutative Algebra

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Sheet 1 (Week: 16.10. - 20.10.2023)

Exercise 1

Let A be a commutative ring with 1. Prove the following assertions:

(a) The ideal $\mathfrak{p} \subseteq A$ is a prime ideal if and only if $A/\mathfrak{p}$ is an integral domain.

(b) The ideal $\mathfrak{m} \subseteq A$ is a maximal ideal if and only if $A/\mathfrak{m}$ is a field.

(c) Let $A \neq \{0\}$. Then, there exists at least one maximal ideal $\mathfrak{m} \subsetneq A$.

Exercise 2

Let $f\colon A \longrightarrow B$ be a ring homomorphism. Prove the following assertions:

(a) For every prime ideal $\mathfrak{q} \subsetneq B$, the ideal $f^{-1}(\mathfrak{q}) \subsetneq A$ is also prime.
Is this statement also true for maximal ideals? Prove or disprove this claim.

(b) If f is surjective, then the converse statement of part (a) is also true:
For every prime ideal $\mathfrak{p} \subsetneq A$ with $\ker(f) \subseteq \mathfrak{p}$, the ideal $f(\mathfrak{p}) \subsetneq B$ is also prime.

Exercise 3

Let A be a commutative ring with 1 and let $\mathfrak{a} \subseteq A$ be an ideal.

(a) Prove that the set

$$\mathfrak{N}_A := \big\{a \in A \,\big|\, \exists\, n \in \mathbb{N}_{>0} :\ a^n = 0\big\}$$

of nilpotent elements of A is an ideal of A, the so-called *nilradical of* A.

(b) Prove that the set

$$\mathfrak{r}(\mathfrak{a}) := \big\{a \in A \,\big|\, \exists\, n \in \mathbb{N}_{>0} : a^n \in \mathfrak{a}\big\}$$

is an ideal of A, the so-called *radical of* $\mathfrak{a}$.

(c) Let $\pi\colon A \longrightarrow A/\mathfrak{a}$ denote the canonical projection. Show that

$$\mathfrak{r}(\mathfrak{a}) = \pi^{-1}\big(\mathfrak{N}_{A/\mathfrak{a}}\big).$$

Exercise 4

Let A be a commutative ring with 1. Let $\mathfrak{a} \subseteq A$ be an ideal and let $\mathfrak{r}(\mathfrak{a})$ denote its radical. Prove that the following properties hold:

(a) $\mathfrak{r}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(\mathfrak{a})$.

(b) $\mathfrak{r}(\mathfrak{a} \cdot \mathfrak{b}) = \mathfrak{r}(\mathfrak{a} \cap \mathfrak{b}) = \mathfrak{r}(\mathfrak{a}) \cap \mathfrak{r}(\mathfrak{b})$.

(c) $\mathfrak{r}(\mathfrak{a}) = (1) \iff \mathfrak{a} = (1)$.

(d) $\mathfrak{r}(\mathfrak{a} + \mathfrak{b}) = \mathfrak{r}(\mathfrak{r}(\mathfrak{a}) + \mathfrak{r}(\mathfrak{b}))$.

(e) For $n \in \mathbb{N}_{>0}$ and every prime ideal $\mathfrak{p} \subsetneq A$, we have $\mathfrak{r}(\mathfrak{p}^n) = \mathfrak{p}$.