Exercises for BMS Basic Course
Commutative Algebra

Prof. Dr. Jürg Kramer

Sheet 10 (Week: 18.12. – 22.12.2023)

Exercise 1

(a) Let A be a commutative ring with 1 and assume that the polynomial ring $A[X]$ in one variable is noetherian. Show that the ring A is also noetherian.

(b) Let A be a commutative noetherian ring with 1. Show that the polynomial ring $A[X_1, X_2, X_3, \ldots]$ in infinitely many variables is not noetherian.

(c) Show that a field K, which is finitely generated as a $\mathbb{Z}$-algebra, necessarily has to be a finite field.

Exercise 2

(a) Let A be a commutative ring with 1, $\mathfrak{a} \subseteq A$ an ideal, and $b \in A$. Consider the ideal $\mathfrak{b} := (\mathfrak{a}, 1 - bX)$ in the polynomial ring $A[X]$. Show that $b \in \mathfrak{r}(\mathfrak{a})$ if and only if $1 \in \mathfrak{b}$.

(b) Let k be an algebraically closed field. Give an alternative proof of the Strong Nullstellensatz, i.e., the implication

$$\mathfrak{a} \subseteq k[X_1, \ldots, X_n], \text{ ideal} \quad \Longrightarrow \quad I(V(\mathfrak{a})) = \mathfrak{r}(\mathfrak{a}).$$

Hint: Add a new variable X to the polynomial ring $k[X_1, \ldots, X_n]$ and use part (a); this is the so-called *"trick" of Rabinowitch*. In addition, use the Weak Nullstellensatz.

(c) Let k be an algebraically closed field and $\mathfrak{a} = (X_1^2 + X_2^2 - 1, X_2 - 1) \subseteq k[X_1, X_2]$ an ideal.
Determine the affine algebraic set $V(\mathfrak{a})$ and its vanishing ideal $I(V(\mathfrak{a}))$. Do we have the equality $\mathfrak{a} = I(V(\mathfrak{a}))$? Prove or disprove this.

Exercise 3

(a) Show that an ideal $\mathfrak{q} \subseteq \mathbb{Z}$ is primary if and only if $\mathfrak{q} = (0)$ or $\mathfrak{q} = (p^n)$ for some prime number p and some $n \in \mathbb{N}_{>0}$.

(b) Let k be a field, $A = k[X, Y, Z]/(XY - Z^2)$, and $\mathfrak{p} = (\bar{X}, \bar{Z})$, where $\bar{X}, \bar{Z}$ denote the classes of X, Z in A, respectively. Show that $\mathfrak{p}^2$ is not a primary ideal of A.

(c) Let A be a commutative ring with 1 and $\mathfrak{p} \subsetneq A$ a prime ideal. Show that a $\mathfrak{p}$-primary ideal $\mathfrak{q} \subseteq A$ need not be equal to some power of $\mathfrak{p}$. Reciprocally, show that a power $\mathfrak{p}^n$ with $n \in \mathbb{N}_{>0}$ need not be primary in general.

Exercise 4

(a) Let A be a commutative ring with 1. Show that $A[X] \otimes_A A[Y] \cong A[X, Y]$.

(b) Consider the natural embedding $i \colon \mathbb{C}[X] \longrightarrow \mathbb{C}[X, Y]$. Describe the induced map $i^* \colon \mathrm{Max}(\mathbb{C}[X, Y]) \longrightarrow \mathrm{Max}(\mathbb{C}[X])$ of the maximal ideals and show that it is well-defined.

(c) Does the Zariski topology on $\mathrm{Spec}(\mathbb{C}[X, Y])$ coincide with the product of the Zariski topologies on $\mathrm{Spec}(\mathbb{C}[X])$ and $\mathrm{Spec}(\mathbb{C}[Y])$?