HU Berlin Winter Semester 2023/24

Exercises for BMS Basic Course
Commutative Algebra

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Sheet 11 (Week: 08.01. – 12.01.2024)

Exercise 1

Let k be a field, $A = k[X,Y]$, $\mathfrak{q} = (X, Y^2)$, and $\mathfrak{m} = (X,Y)$. Show that:

(a) The ideal $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal.

(b) There is a chain of strict inclusions $\mathfrak{m}^2 \subsetneq \mathfrak{q} \subsetneq \mathfrak{m}$.

(c) Deduce from part (b) that $\mathfrak{q}$ is not a power of a prime ideal of A.

Exercise 2

(a) Find two different minimal primary decompositions of the ideal $\mathfrak{a} = (X^2, XY)$ in the polynomial ring $k[X,Y]$ over the field k. Illustrate on the basis of this example the first and the second uniqueness theorems for minimal primary decompositions.

(b) Let $\mathfrak{p}_1 = (X,Y)$, $\mathfrak{p}_2 = (X,Z)$, and $\mathfrak{m} = (X,Y,Z)$ be three ideals in the polynomial ring $k[X,Y,Z]$ over the field k. Letting $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2$, show that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of $\mathfrak{a}$ and determine the isolated and embedded prime ideals.

Exercise 3

Let A be a commutative ring with 1 and $\mathfrak{p} \subseteq A$ be a prime ideal. For any $n \in \mathbb{N}_{>0}$, the n-th symbolic prime power $\mathfrak{p}^{(n)}$ of $\mathfrak{p}$ is defined to be the contraction of the ideal $\mathfrak{p}_\mathfrak{p}^n \subseteq A_\mathfrak{p}$ under the localization homomorphism $A \longrightarrow A_\mathfrak{p}$. Show that:

(a) The n-th symbolic prime power $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary ideal.

(b) If $\mathfrak{p}^n$ has a primary decomposition, then $\mathfrak{p}^{(n)}$ is its $\mathfrak{p}$-primary component.

(c) We have $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ if and only if $\mathfrak{p}^n$ is $\mathfrak{p}$-primary.

Exercise 4

Let A be a commutative ring with 1 and $S \subseteq A$ a multiplicatively closed subset. Let $f \in A$ and $\mathfrak{p} \in \operatorname{Spec}(A)$.

(a) Show that the natural ring homomorphism $A \longrightarrow S^{-1}A$ induces a homeomorphism

$$\operatorname{Spec}(S^{-1}A) \cong \{\mathfrak{q} \in \operatorname{Spec}(A) \mid S \cap \mathfrak{q} = \emptyset\} \subseteq \operatorname{Spec}(A).$$

(b) Show that the natural ring homomorphism $A \longrightarrow A_f$ induces a homeomorphism

$$\operatorname{Spec}(A_f) \cong D(f) \subseteq \operatorname{Spec}(A).$$

(c) Show that the natural ring homomorphism $A \longrightarrow A_{\mathfrak{p}}$ induces a homeomorphism

$$\operatorname{Spec}(A_{\mathfrak{p}}) \cong \{\mathfrak{q} \in \operatorname{Spec}(A) \mid \mathfrak{q} \subseteq \mathfrak{p}\} \subseteq \operatorname{Spec}(A).$$