HU Berlin Winter Semester 2023/24

Exercises for BMS Basic Course
Commutative Algebra
Prof. Dr. Jürg Kramer

Sheet 13 (Week: 22.01. – 26.01.2024)

Exercise 1

(a) Let $A \subseteq B$ be commutative rings with 1. Show that the integral closure of A in B is a subring of B containing A.

(b) Let $A \subseteq B \subseteq C$ be commutative rings with 1. Let C be integral over B and B integral over A. Prove that C is integral over A.

Exercise 2

Let $A \subseteq B$ be commutative rings with 1 and assume that B is integral over A. Then, prove the following assertions:

(a) If $\mathfrak{b} \subseteq B$ is an ideal and $\mathfrak{a} := \mathfrak{b}^{\mathrm{c}} = A \cap \mathfrak{b}$, then $B/\mathfrak{b}$ is integral over $A/\mathfrak{a}$.

(b) If $S \subseteq A$ is a multiplicatively closed subset, then $S^{-1}B$ is integral over $S^{-1}A$.

Exercise 3

Let $d \in \mathbb{Z}$ be a square-free integer and consider the quadratic number field $K = \mathbb{Q}(\sqrt{d})$. Show that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{d})$ is given by

$$\mathfrak{O}_K = \begin{cases} \mathbb{Z}[\sqrt{d}], & \text{if } d \equiv 2,3 \mod 4, \\ \mathbb{Z}[\frac{1+\sqrt{d}}{2}], & \text{if } d \equiv 1 \mod 4. \end{cases}$$

Exercise 4

A presheaf $\mathcal{F}$ on a topological space X is called a *sheaf*, if the following condition holds:

Let $U \subseteq X$ be an open subset and $U = \bigcup_{i \in I} U_i$ an open covering of U. Let $\{s_i\}_{i \in I}$ with $s_i \in \mathcal{F}(U_i)$ $(i \in I)$ be a collection of compatible sections, i. e., $\rho_{U_i, U_i \bigcap U_j}(s_i) = \rho_{U_j, U_i \bigcap U_j}(s_j)$ for all $i, j \in I$. Then, there exists a unique $s \in \mathcal{F}(U)$ such that $\rho_{U, U_i}(s) = s_i$ for all $i \in I$.

For $X = \mathbb{R}$, prove the following assertions:

(a) The presheaf $\mathcal{C}$ given by the assignment

$$\mathcal{C}(U) \coloneqq \{f \colon U \longrightarrow \mathbb{R} \mid f \text{ is continuous on } U\}$$

for every open subset $U \subseteq X$, is a sheaf.

(b) The presheaf $\mathcal{F}$ given by the assignment

$$\mathcal{F}(U) \coloneqq \{f \colon U \longrightarrow \mathbb{R} \mid f \text{ is bounded on } U\}$$

for every open subset $U \subseteq X$, is not a sheaf.