Exercises for BMS Basic Course
Commutative Algebra

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Sheet 2 (Week: 23.10. – 27.10.2023)

Exercise 1

(a) Determine the nilradical $\mathfrak{N}_{\mathbb{Z}}$ and the Jacobson radical $\mathfrak{R}_{\mathbb{Z}}$ of the ring of integers $\mathbb{Z}$.

(b) Is the ring of integers $\mathbb{Z}$ a local ring? Prove or disprove this statement.
For which $n \in \mathbb{N}$ is the quotient ring $\mathbb{Z}/(n)$ a local ring? Prove your answer.

(c) Let A be a commutative ring with 1 and $\mathfrak{N}_A$ its nilradical. Show that the following are equivalent:

(i) A has exactly one prime ideal.

(ii) Every element of A is either a unit or nilpotent.

(iii) $A/\mathfrak{N}_A$ is a field.

Exercise 2

Let A be a commutative ring with 1.

(a) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ ($n \in \mathbb{N}$ with $n > 1$) be prime ideals of A and let $\mathfrak{a}$ be an ideal contained in the union $\bigcup_{j=1}^n \mathfrak{p}_j$. Prove that $\mathfrak{a} \subseteq \mathfrak{p}_j$ for at least one $j \in \{1, \ldots, n\}$.

(b) Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ ($n \in \mathbb{N}$ with $n > 1$) be ideals of A and let $\mathfrak{p}$ be a prime ideal containing the intersection $\bigcap_{j=1}^n \mathfrak{a}_j$. Prove that $\mathfrak{a}_j \subseteq \mathfrak{p}$ for at least one $j \in \{1, \ldots, n\}$.
Moreover, show: If $\mathfrak{p} = \bigcap_{j=1}^n \mathfrak{a}_j$, then $\mathfrak{p} = \mathfrak{a}_j$ for some $j \in \{1, \ldots, n\}$.

Exercise 3

Let A be a commutative ring with 1. We consider the set

$$\mathrm{Spec}(A) \coloneqq \{\mathfrak{p} \subsetneq A \,|\, \mathfrak{p} \text{ prime ideal}\}$$

of all prime ideals of A. For a subset $S \subseteq A$, let

$$V(S) \coloneqq \{\mathfrak{p} \in \mathrm{Spec}(A) \,|\, \mathfrak{p} \supseteq S\}$$

denote the set of all prime ideals of A containing S.

(a) Prove that $V(S) = V(\mathfrak{a}) = V(\mathfrak{r}(\mathfrak{a}))$, where $\mathfrak{a}$ denotes the ideal generated by the set S in A.

(b) Show that the sets $V(S)$, where S ranges over the subsets of A, satisfy the axioms for closed sets of a topology in $\mathrm{Spec}(A)$.

The resulting topology on $\mathrm{Spec}(A)$ is called the *Zariski topology*; the topological space $\mathrm{Spec}(A)$ is called the *prime spectrum of* A.