# Exercises for BMS Basic Course
# Commutative Algebra

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**Sheet 3 (Week: 30.10. – 03.11.2023)**

### Exercise 1

Let $A \coloneqq C([0,1],\mathbb{R})$ denote the ring of continuous, real-valued functions on the interval $[0,1]$.

(a) Let $x \in [0,1]$. Show that the set

$$\mathfrak{m}_x \coloneqq \{f \in A \,|\, f(x) = 0\}$$

is a maximal ideal in $A$.

(b) For an integer $n > 1$, let $f_1, \ldots, f_n \in A$ be continuous functions without any common zero. Show that $f_1^2 + \ldots + f_n^2$ is a unit in $A$.

(c) Let $\mathfrak{a} \subsetneq A$ be a proper ideal and let

$$V(\mathfrak{a}) \coloneqq \{a \in [0,1] \,|\, f(a) = 0 \text{ for every } f \in \mathfrak{a}\}$$

be the set of common zeros of all functions in $\mathfrak{a}$. Prove that $V(\mathfrak{a}) \neq \emptyset$.

(d) Use (c) to deduce that $\mathrm{Max}(A) = \{\mathfrak{m}_x \,|\, x \in [0,1]\}$ and show that the map

$$[0,1] \longrightarrow \mathrm{Max}(A),$$

induced by the assignment $x \mapsto \mathfrak{m}_x$, is a bijection.

### Exercise 2

(a) Use the classification of finitely generated abelian groups to derive a classification of finitely generated $\mathbb{Z}$-modules up to isomorphism.

(b) Show that $\mathbb{Q}$ is not $\mathbb{Z}$-free. Further, prove that $\mathbb{Z}/n\mathbb{Z}$ is not $\mathbb{Z}$-free, but $\mathbb{Z}/n\mathbb{Z}$-free.

(c) Let $A$ be a commutative ring with 1. Show that every finitely generated $A$-module is $A$-free if and only if $A$ is a field.

(d) Let $A$ be a commutative ring with 1 and $M$ an $A$-module. Fix an $A$-endomorphism $f \in \mathrm{End}_A(M)$. Then, we can define a scalar multiplication by elements of the polynomial ring $A[T]$ on $M$ by setting

$$(p(T), m) \mapsto p(f)(m) \qquad (p \in A[T],\, m \in M).$$

Show that this endows $M$ with the structure of an $A[T]$-module.

**Exercise 3**

(a) Let $A$ be a commutative ring with 1 and $\mathfrak{a} \subseteq A$ an ideal such that for all finitely generated $A$-modules $M$, the equality $\mathfrak{a}M = M$ implies $M = 0$. Show that $\mathfrak{a}$ is contained in the Jacobson radical $\mathfrak{R}_A$ of $A$.

(b) Let $A$ be a commutative ring with 1 and $\mathfrak{a} \subseteq A$ an ideal. Let $M$ be an $A$-module such that $\mathfrak{a}M = M$. If $\mathfrak{a}$ is nilpotent, i. e., $\mathfrak{a}^n = 0$ for some $n \in \mathbb{N}_{>0}$, show that $M = 0$.

(c) Let $A$ be a local ring with maximal ideal $\mathfrak{m}$. Let $M$ be an $A$-module and $N$ a finitely generated $A$-module together with an $A$-module homomorphism $f\colon M \longrightarrow N$. If the induced $A$-module homomorphism $f_{\mathfrak{m}}\colon M/\mathfrak{m}M \longrightarrow N/\mathfrak{m}N$ is surjective, show that $f$ is also surjective.

(d) Let $A$ be a commutative ring with 1 and $M$ a finitely generated $A$-module. Show that every surjective $A$-module endomorphism $f\colon M \longrightarrow M$ is also injective.
*Hint:* View $M$ as an $A[T]$-module as in Exercise 2, part (d).

**Exercise 4**

Let $A$ be a commutative ring with 1. For $f \in A$, we define the *distinguished* or *basic set*

$$D(f) := \operatorname{Spec}(A) \setminus V(f)$$

to be the complement of $V(f)$ in $\operatorname{Spec}(A)$. Show that the distinguished sets $D(f)$ $(f \in A)$ are open and that they form a basis of open sets for the Zariski topology of $\operatorname{Spec}(A)$.

Furthermore, prove that for $f, g \in A$ we have:

(a) $D(f) \cap D(g) = D(f \cdot g)$.

(b) $D(f) = \emptyset \iff f$ is nilpotent.

(c) $D(f) = \operatorname{Spec}(A) \iff f \in A^{\times}$.

(d) $D(f) = D(g) \iff \mathfrak{r}(f) = \mathfrak{r}(g)$.
Here, $\mathfrak{r}(f)$ denotes the radical of the principal ideal $(f)$.