Exercises for BMS Basic Course
Commutative Algebra

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Sheet 4 (Week: 06.11. – 10.11.2023)

Exercise 1

Let K be a field and $K[W, X, Y, Z]$ be the polynomial ring in the four variables W, X, Y, Z over K. Let $\mathfrak{a} \subseteq K[W, X, Y, Z]$ be the ideal generated by the (2×2)-minors of the matrix

$$\begin{pmatrix} X & Y & Z \\ Y & Z & W \end{pmatrix},$$

i. e., $\mathfrak{a} = (WY - Z^2, WX - YZ, XZ - Y^2)$, and put $A := K[W, X, Y, Z]/\mathfrak{a}$; we denote by $\overline{W}, \overline{X}, \overline{Y}, \overline{Z}$ the images of W, X, Y, Z in the ring A, respectively.

(a) Show that A is a finitely generated module over $B := K[\overline{W}, \overline{X}]$.

(b) Show that A is in fact a free B-module by constructing a basis for A over B.

(c) Show that A is not finitely generated as a $K[\overline{X}, \overline{Y}]$-module.

Exercise 2

Let A be a commutative ring with 1.

(a) Show that the following are equivalent:

(i) A sequence of A-modules

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

is exact.

(ii) For all A-modules N, the induced sequence

$$0 \longrightarrow \mathrm{Hom}_A(M'', N) \xrightarrow{\overline{v}} \mathrm{Hom}_A(M, N) \xrightarrow{\overline{u}} \mathrm{Hom}_A(M', N)$$

is exact.

(b) Give a counterexample to the analogous claim of part (a) if, in addition, u is assumed to be injective and $\overline{u}$ is assumed to be surjective.

Exercise 3

Consider the commutative diagram of short exact sequences of A-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & N' & \xrightarrow{u'} & N & \xrightarrow{v'} & N'' & \longrightarrow & 0.
\end{array}$$

Then the induced sequence

$$0 \longrightarrow \ker(f') \xrightarrow{\overline{u}} \ker(f) \xrightarrow{\overline{v}} \ker(f'') \xrightarrow{d} \operatorname{coker}(f') \xrightarrow{\overline{u}'} \operatorname{coker}(f) \xrightarrow{\overline{v}'} \operatorname{coker}(f'') \longrightarrow 0$$

with $\overline{u}, \overline{v}, \overline{u}', \overline{v}'$, and d defined as in the course, is an exact sequence of A-modules with well-defined A-module homomorphisms.

The exactness at $\ker(f')$, $\ker(f)$, $\operatorname{coker}(f)$, and $\operatorname{coker}(f'')$ have been proven in the course. Finish the proof by showing the exactness at $\ker(f'')$ and $\operatorname{coker}(f')$.

Exercise 4

Let A be a commutative ring with 1. Prove the following assertions:

(a) The set $\{\mathfrak{p}\}$ is closed in $\operatorname{Spec}(A)$ if and only if $\mathfrak{p}$ is a maximal ideal.

(b) $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$.

(c) $\mathfrak{q} \in \overline{\{\mathfrak{p}\}} \iff \mathfrak{q} \supseteq \mathfrak{p}$.

Answer the following questions and prove your answers:

(d) Is $\operatorname{Spec}(A)$ a T_0- or *Kolmogorov*-space? This means that if $\mathfrak{p}$ and $\mathfrak{q}$ are distinct points of $\operatorname{Spec}(A)$, then either there is a neighborhood of $\mathfrak{p}$ which does not contain $\mathfrak{q}$, or else there is a neighborhood of $\mathfrak{q}$ which does not contain $\mathfrak{p}$.

(e) Is $\operatorname{Spec}(A)$ a T_1- or *Hausdorff*-space? This means that if $\mathfrak{p}$ and $\mathfrak{q}$ are distinct points of $\operatorname{Spec}(A)$, then there is a neighborhood U of $\mathfrak{p}$ and a neighborhood V of $\mathfrak{q}$ such that $U \cap V = \emptyset$.