HU Berlin Winter Semester 2023/24

Exercises for BMS Basic Course
Commutative Algebra

Prof. Dr. Jürg Kramer

Sheet 5 (Week: 13.11. – 17.11.2023)

Exercise 1

Let A be a commutative ring with 1. Consider the following commutative diagram of A-modules and A-module homomorphisms, where the horizontal rows are exact:

$$\begin{array}{ccccc} M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} \\ N' & \xrightarrow{u'} & N & \xrightarrow{v'} & N'' \end{array}$$

Then, prove the following assertions:

(a) If u', f', and f'' are injective, then f is injective.

(b) If v, f', and f'' are surjective, then f is surjective.

Exercise 2

Let A be a commutative ring with 1 and let P be an A-module. Show that the following statements are equivalent:

(i) P is projective.

(ii) The functor $\operatorname{Hom}_A(P, \,\cdot\,)$ is exact.

(iii) Every short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow P \longrightarrow 0$ of A-modules and A-module homomorphisms is split.

(iv) P is a direct summand of a free A-module, i. e., there exists an A-module Q such that $P \oplus Q$ is a free A-module.

Exercise 3

Denote by $C^0 := C^\infty(\mathbb{R}^n, \mathbb{R})$ the ring of smooth real-valued functions on $\mathbb{R}^n$. A differential form ω on $\mathbb{R}^n$ of degree r is given by

$$\omega = \sum_{\{i_1,\ldots,i_r\} \subseteq \{1,\ldots,n\}} f_{i_1,\ldots,i_r}(x_1,\ldots,x_n)\, \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_r} \qquad (f_{i_1,\ldots,i_r} \in C^0),$$

where the differentials $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ are subject to the relation

$$\mathrm{d}x_j \mathrm{d}x_k = -\mathrm{d}x_k \mathrm{d}x_j \qquad (j, k \in \{1, \dots, n\}).$$

Therefore, ω can be rewritten as

$$\omega = \sum_{1 \le i_1 < \ldots < i_r \le n} g_{i_1,\dots,i_r}(x_1, \dots, x_n)\, \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_r} \qquad (g_{i_1,\dots,i_r} \in C^0).$$

Consider the $\mathbb{R}$-vector spaces

$$C^r := \{\omega \mid \omega \text{ is a differential form of degree } r\}.$$

Furthermore, consider the $\mathbb{R}$-linear map $\mathrm{d}\colon C^r \longrightarrow C^{r+1}$ given by

$$\mathrm{d}\omega := \sum_{1 \le i_1 < \ldots < i_r \le n} \sum_{j=1}^{n} \frac{\partial g_{i_1,\dots,i_r}}{\partial x_j} \mathrm{d}x_j\, \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_r}.$$

(a) Show that

$$\mathbf{C}\colon 0 \longrightarrow C^0 \xrightarrow{\mathrm{d}} C^1 \xrightarrow{\mathrm{d}} C^2 \xrightarrow{\mathrm{d}} \dots$$

is a cochain complex (of $\mathbb{R}$-vector spaces), i. e., $\mathrm{d}^2 = \mathrm{d} \circ \mathrm{d} = 0$.

(b) Compute the cohomology groups $H^r(\mathbf{C})$ in the case $n = 2$ for all $r \in \mathbb{Z}$.

Exercise 4

(a) Give an explicit description of the elements of the sets $\operatorname{Spec}(\mathbb{Z})$, $\operatorname{Spec}(\mathbb{Z}/3\mathbb{Z})$, $\operatorname{Spec}(\mathbb{Z}/6\mathbb{Z})$, $\operatorname{Spec}(\mathbb{C}[X]/(X^2))$, and $\operatorname{Spec}(\mathbb{R}[X]/(X^2+1))$.

(b) Prove that there is a bijection $\operatorname{Max}(\mathbb{C}[X]) \longrightarrow \mathbb{C}$.