HU Berlin Winter Semester 2023/24

Exercises for BMS Basic Course
Commutative Algebra
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Sheet 6 (Week: 20.11. – 24.11.2023)

Exercise 1

Let A be a commutative ring with 1 and let $\mathbf{f}, \mathbf{g} \colon (\mathbf{C}, \partial^{\mathbf{C}}) \longrightarrow (\mathbf{D}, \partial^{\mathbf{D}})$ be two morphisms between chain complexes of A-modules. We say that $\mathbf{f}$ and $\mathbf{g}$ are *chain homotopic*, if for each $n \in \mathbb{Z}$, there is an A-module homomorphism

$$s_n \colon C_n \longrightarrow D_{n+1}$$

such that $f_n - g_n = \partial^{\mathbf{D}}_{n+1} \circ s_n + s_{n-1} \circ \partial^{\mathbf{C}}_n$.
Assuming that $\mathbf{f}$ and $\mathbf{g}$ are chain homotopic, show that for each $n \in \mathbb{Z}$, the induced A-module homomorphism on the level of homology

$$f_{n*} - g_{n*} \colon H_n(\mathbf{C}) \longrightarrow H_n(\mathbf{D})$$

is the zero A-module homomorphism, so that the morphisms $\mathbf{f}$ and $\mathbf{g}$ induce the same A-module homomorphism from $H_n(\mathbf{C})$ to $H_n(\mathbf{D})$.

Exercise 2

Let A be a commutative ring with 1. Prove the following assertions:

(a) Let M and N be A-modules, let $f \colon M \longrightarrow N$ be an A-module homomorphism, and let

$$\mathbf{P} \colon \ \ldots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

and

$$\mathbf{Q} \colon \ \ldots \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \ldots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow N \longrightarrow 0$$

be projective resolutions of M and N, respectively. Further, let $\mathbf{f}, \mathbf{g} \colon \mathbf{P} \longrightarrow \mathbf{Q}$ be morphisms of chain complexes extending f, i. e., we have the following commutative diagram:

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \ldots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & {\scriptstyle g_n}\downarrow\downarrow{\scriptstyle f_n} & & {\scriptstyle g_{n-1}}\downarrow\downarrow{\scriptstyle f_{n-1}} & & & & {\scriptstyle g_1}\downarrow\downarrow{\scriptstyle f_1} & & {\scriptstyle g_0}\downarrow\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
\ldots & \longrightarrow & Q_n & \longrightarrow & Q_{n-1} & \longrightarrow & \ldots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

Show that $\mathbf{f}$ and $\mathbf{g}$ are chain homotopic.

(b) Let $\mathfrak{M}_A$ be the category of A-modules, let $T\colon \mathfrak{M}_A \longrightarrow \mathfrak{M}_A$ be a covariant functor, and let M be an A-module. Then using part (a), show that the definition of the left-derived functor $LT(M)$ of T is independent of the choice of a projective resolution of M.

Exercise 3

Show that for any $n \in \mathbb{N}_{>0}$ and any abelian group G, the following assertions hold:

(a) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, G) \cong \{g \in G \mid n \cdot g = 0\}$.

(b) $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$.

Exercise 4

Let A be a commutative ring with 1, and let M, N be A-modules. Let $\mathcal{E}_A(M, N)$ denote the set of equivalence classes of extensions of M by N. Then, show that there is a bijection of sets

$$\mathcal{E}_A(M, N) \approx \mathrm{Ext}^1_A(M, N).$$

Hint: Use the map $\mathcal{E}_A(M, N) \longrightarrow \mathrm{Ext}^1_A(M, N)$ defined in the course.