HU Berlin Winter Semester 2023/24

Exercises for BMS Basic Course
Commutative Algebra

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Sheet 7 (Week: 27.11. – 01.12.2023)

Exercise 1

Let p be a prime number. Show that there are exactly p equivalence classes of extensions of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$ in the category of $\mathbb{Z}$-modules, namely, the split extension

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{\iota} \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0,$$

where ι and π are the canonical injection and projection, respectively, and the extensions

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{m_p} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{m_j} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0 \qquad (j = 1, 2, \ldots, p-1),$$

where m_p and m_j denote the multiplications by p and j, respectively.

Exercise 2

Let A be a commutative ring with 1 and let M, N, and P be A-modules. Prove the following assertions using the universal property of the tensor product:

(a) $M \otimes_A N \cong N \otimes_A M$.

(b) $(M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P) \cong M \otimes_A N \otimes_A P$.

(c) $(M \oplus N) \otimes_A P \cong (M \otimes_A P) \oplus (N \otimes_A P)$.

(d) $A \otimes_A M \cong M$.

Exercise 3

Let A be a commutative ring with 1, M an A-module, and $\mathfrak{a}, \mathfrak{b} \subseteq A$ ideals. Prove the following assertions:

(a) $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

(b) $M \otimes_A A/\mathfrak{a} \cong M/\mathfrak{a}M$.

(c) $A/\mathfrak{a} \otimes_A A/\mathfrak{b} \cong A/(\mathfrak{a} + \mathfrak{b})$.

(d) $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m,n)\mathbb{Z}$, where (m,n) denotes the greatest common divisor of m and n.

Exercise 4

Let X be a non-empty topological space. Then, X is called *irreducible*, if for any closed subsets X_1 and X_2 of X the equality $X = X_1 \cup X_2$ implies $X = X_1$ or $X = X_2$.

(a) Show that X is irreducible if and only if every pair of non-empty open sets in X has a non-empty intersection. Further, show that X is irreducible if and only if every non-empty open subset of X is dense in X.

(b) Let A be a commutative ring with 1. Show that $\operatorname{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.