HU Berlin Winter Semester 2023/24

Exercises for BMS Basic Course
Commutative Algebra

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Sheet 8 (Week: 04.12. – 08.12.2023)

Exercise 1

Let A be a commutative ring with 1. An A-module N is said to be *flat*, if the functor $T_N = - \otimes_A N$ is exact in the category of A-modules.

(a) Show that the following statements are equivalent for every A-module N:

- (i) N is flat.
- (ii) $\mathrm{Tor}_n^A(M, N) = 0$ for $n \in \mathbb{N}_{>0}$ and all A-modules M.
- (iii) $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules M.

(b) Let N, N', N'' be A-modules. Show that if $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ is exact, and both N and N'' are flat, then N' is also flat.

Exercise 2

Let A be a commutative ring with 1 and $S \subseteq A$ a multiplicatively closed subset.

(a) Show that addition and multiplication in $S^{-1}A$ defined by

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st} \quad \text{and} \quad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st} \qquad (a, b \in A;\ s, t \in S)$$

are independent of the choice of representatives (a, s) and (b, t) in $A \times S$.

(b) Give an example where the natural ring homomorphism $f\colon A \longrightarrow S^{-1}A$ given by the assignment $a \mapsto \frac{a}{1}$ is not injective.

(c) Show that the prime ideals of $S^{-1}A$ are in bijective correspondence with the prime ideals of A, which are disjoint to S.

Exercise 3

Let A be a commutative ring with 1 and $S \subseteq A$ a multiplicatively closed subset. Furthermore, let M be an A-module and let M', M'' be A-submodules of M. Then, prove the following assertions:

(a) Localization at S commutes with sums, i. e., we have the equality

$$S^{-1}(M' + M'') = S^{-1}M' + S^{-1}M''.$$

(b) Localization at S commutes with intersections, i. e., we have the equality

$$S^{-1}(M' \cap M'') = S^{-1}M' \cap S^{-1}M''.$$

(c) Localization at S commutes with quotients, i. e., we have the $S^{-1}A$-module isomorphism

$$S^{-1}(M/M') \cong (S^{-1}M)/(S^{-1}M').$$

(d) Localization at S is an exact functor from the category of A-modules to the category of $S^{-1}A$-modules, i. e., if

$$M' \xrightarrow{\varphi} M \xrightarrow{\psi} M''$$

is an exact sequence of A-modules, then the sequence of $S^{-1}A$-modules

$$S^{-1}M' \xrightarrow{S^{-1}\varphi} S^{-1}M \xrightarrow{S^{-1}\psi} S^{-1}M''$$

is also exact.

Exercise 4

Let X be a non-empty topological space.

(a) Let $Y \subseteq X$ be a topological subspace which is irreducible. Show that the closure $\overline{Y}$ of Y in X is also irreducible.

(b) Show that every irreducible topological subspace Y of X is contained in a maximal irreducible topological subspace.

(c) Show that the maximal irreducible topological subspaces of X are closed and cover X. They are called the *irreducible components of* X.

(d) Let A be a commutative ring with 1. Show that the irreducible components of $\mathrm{Spec}(A)$ are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A.