Exercises for BMS Basic Course
Commutative Algebra

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Sheet 9 (Week: 11.12. – 15.12.2023)

Exercise 1

Let A be a commutative ring with 1 and $S \subseteq A$ a multiplicatively closed subset. Let M, N be A-modules. Show that there exists a uniquely determined $S^{-1}A$-module isomorphism

$$S^{-1}M \otimes_{S^{-1}A} S^{-1}N \cong S^{-1}(M \otimes_A N),$$

induced by the assignment

$$\frac{m}{s} \otimes \frac{n}{t} \mapsto \frac{m \otimes n}{st} \qquad (m \in M;\, n \in N;\, s, t \in S).$$

Hint: Start with the $S^{-1}A$-bilinear map

$$f \colon S^{-1}M \times S^{-1}N \longrightarrow S^{-1}(M \otimes_A N),$$

induced by the assignment

$$\left(\frac{m}{s}, \frac{n}{t}\right) \mapsto \frac{m \otimes n}{st},$$

and use the universal property of the tensor product to get a uniquely determined $S^{-1}A$-module homomorphism

$$f' \colon S^{-1}M \otimes_{S^{-1}A} S^{-1}N \longrightarrow S^{-1}(M \otimes_A N).$$

Then, verify explicitly that f' is surjective and injective.

Exercise 2

Let A be a commutative noetherian ring with 1.

(a) Let $S \subseteq A$ be a multiplicatively closed subset. Show that $S^{-1}A$ is also a noetherian ring.

(b) Show that the ring $A[[X]]$ of formal power series in the variable X over A is also a noetherian ring.

(c) Let $B \subseteq A$ be a subring. Is B necessarily noetherian?

Exercise 3

Let A be a commutative ring with 1.

(a) Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence of A-modules. Show that M is noetherian if and only if both M' and M'' are noetherian.

(b) Let M_j ($j = 1, \ldots, n$) be noetherian A-modules. Show that their direct sum $\bigoplus_{j=1}^n M_j$ is also a noetherian A-module.

Exercise 4

Let A, B, and C be commutative rings with 1. Let $\varphi\colon A \longrightarrow B$ be a homomorphism of rings and let $\varphi^*\colon \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ be the induced map given by the assignment $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$.

(a) Show that ${\varphi^*}^{-1}(D(f)) = D(\varphi(f))$ for any $f \in A$, and deduce that the map φ^* is continuous.

(b) The ideal $\varphi^{-1}(\mathfrak{b}) \subseteq A$ is called the *contraction* $\mathfrak{b}^{\mathrm{c}}$ of the ideal $\mathfrak{b} \subseteq B$. The ideal generated by $\varphi(\mathfrak{a})$ in B is called the *extension* $\mathfrak{a}^{\mathrm{e}}$ of the ideal $\mathfrak{a} \subseteq A$.

Show that $\mathfrak{a} \subseteq \mathfrak{a}^{\mathrm{ec}}$ and $\mathfrak{b} \supseteq \mathfrak{b}^{\mathrm{ce}}$. Further, show that $\mathfrak{a}^{\mathrm{e}} = \mathfrak{a}^{\mathrm{ece}}$ and $\mathfrak{b}^{\mathrm{c}} = \mathfrak{b}^{\mathrm{cec}}$.

(c) Prove that ${\varphi^*}^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}^{\mathrm{e}})$ for any ideal $\mathfrak{a} \subseteq A$ and that $\overline{\varphi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^{\mathrm{c}})$ for any ideal $\mathfrak{b} \subseteq B$.

(d) Let $\psi\colon B \longrightarrow C$ be a homomorphism of rings. Show that $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

Note that, from (a) and (d), it follows that Spec becomes a contravariant functor from the category of commutative rings with 1 to the category of topological spaces.