HU Berlin Winter Semester 2023/24

Commutative Algebra

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Test Exam

Problem 1 (10 points)

Let A be a commutative ring with 1.

(a) Let $S \subseteq A$ be a multiplicatively closed subset. State the universal property of the localization $S^{-1}A$ of A at S.

(b) If A is a local ring with $\mathfrak{m}$ its unique maximal ideal, show that $A_{\mathfrak{m}} = A$ using the universal property of localization.

(c) Let $S \subseteq A$ be a multiplicatively closed subset and $\mathfrak{N}(A)$ be the nilradical of A. Show that $\mathfrak{N}(S^{-1}A) = S^{-1}(\mathfrak{N}(A))$.

Problem 2 (10 points)

Let A be a commutative ring with 1.

(a) Provide the three equivalent definitions of a noetherian ring A and state Hilbert's Basis Theorem for A.

(b) Let M', M, and M'' be A-modules fitting into the short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

Show that if M' and M'' are finitely generated, then M is finitely generated.

Problem 3 (10 points)

(a) Let A be a commutative ring with 1 and let M, N be A-modules. Provide the definition of an extension of M by N.

(b) Compute the $\mathbb{Z}$-module

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$$

and give an interpretation of its elements as extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$.

Problem 4 (10 points)

(a) State the weak and the strong geometric versions of Hilbert's Nullstellensatz.

(b) Give a counter-example to the weak version of Hilbert's Nullstellensatz, when k is not algebraically closed.

(c) Recall, that the closed sets of the Zariski topology on k^n are given by the affine algebraic sets

$$V(S) = \{(x_1, \dots, x_n) \in k^n \mid f(x_1, \dots, x_n) = 0, \forall f \in S\},$$

where $S \subseteq k[X_1, \dots, X_n]$. Show that, in this topology, the closure $\overline{X}$ of an arbitrary subset $X \subseteq k^n$ is given by $\overline{X} = V(I(X))$, where

$$I(X) = \{f \in k[X_1, \dots, X_n] \mid f(x_1, \dots, x_n) = 0, \forall (x_1, \dots, x_n) \in X\}$$

is the vanishing ideal of X.

Problem 5 (10 points)

(a) Let A be a commutative ring with 1. Provide the definition of a primary ideal in A.

(b) Let $\mathfrak{a} := (X^3, X^2Y, XY^2)$ be an ideal in the polynomial ring $k[X, Y]$ over the field k. Find a minimal primary decomposition for the ideal $\mathfrak{a}$.

(c) Determine the isolated and the embedded associated prime ideals of $\mathfrak{a}$, and find the uniquely determined primary components.

Problem 6 (10 points)

(a) Let $A \subseteq B$ be integral domains and let B be integral over A. Show that A is a field if and only if B is a field.

(b) Let A be a local noetherian ring with its unique maximal ideal $\mathfrak{m}$. Show that $\mathfrak{m}/\mathfrak{m}^2 = 0$ if and only if A is a field.