THE SINGULARITIES OF THE INVARIANT METRIC ON THE JACOBI LINE BUNDLE

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ABSTRACT. A theorem by Mumford implies that every automorphic line bundle on a pure open Shimura variety, equipped with an invariant smooth metric, can be uniquely extended as a line bundle on a toroidal compactification of the variety, in such a way that the metric acquires only logarithmic singularities. This result is the key of being able to compute arithmetic intersection numbers from these line bundles. Hence, it is natural to ask whether Mumford's result remains valid for line bundles on mixed Shimura varieties.

In this paper we examine the simplest case, namely the Jacobi line bundle on the universal elliptic curve, whose sections are the Jacobi forms. We will show that Mumford's result cannot be extended directly to this case and that a new type of singularity appears.

By using the theory of b-divisors, we show that an analogue of Mumford's extension theorem can be obtained. We also show that this extension is meaningful because it satisfies Chern-Weil theory and a Hilbert-Samuel type of formula.

CONTENTS

1. INTRODUCTION

In [10], [11], Faltings introduced the notion of logarithmically singular metrics on a projective variety defined over a number field and proved that they satisfy a Northcott type property, namely that the set of algebraic points not lying on the singular set of the metric with

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bounded height and degree, is finite. A prominent example of logarithmically singular metric is the Hodge bundle ω on a toroidal compactification of the moduli space of principally polarized abelian varieties of dimension g (with level structure if you do not want to work with stacks) $\overline{\mathscr{A}}_g$ equipped with the Petersson metric.

On the other hand, Mumford [22] introduced the concept of a good metric on a vector bundle, which is a class of singular metrics. He showed that, even being singular, Chern-Weil theory carries over to good metrics. He also proved that the invariant metric on a fully decomposable automorphic vector bundle on a toroidal compactification of the quotient of a hermitian symmetric domain by an arithmetic group is a good metric. This fact allowed him to extend Hirzebruch's proportionality principle to non-compact varieties.

The conclusion of the above facts is that the natural metrics that appear when studying vector bundles on toroidal compactifications of pure Shimura varieties are singular, but the singularities are mild enough so we can use the metrics to study geometric and arithmetic problems.

In [7] and [8], the authors developed a general theory of arithmetic intersections with logarithmically singular metrics that has been extensively used to compute arithmetic intersection numbers [5, 20, 6, 15, 16, 4, 1, 2, 12].

It is natural to ask whether this theory of logarithmically singular metrics can be extended to mixed Shimura varieties, to obtain geometric and arithmetic information of them.

In this paper we examine the first example of a mixed Shimura variety, namely the universal elliptic curve of full level N over the modular curve $E^0(N) \to Y(N)$. On it we consider the Jacobi line bundle, whose sections are the Jacobi forms, equipped with the translation invariant metric.

It turns out that, in this case, a new kind of singularity appears. These singularities are concentrated in codimension two. Therefore, if we remove a set of codimension two, we can extend the Jacobi line bundle to a line bundle with a good hermitian metric on a partial compactification of $E^0(N)$. Since algebraic line bundles can be uniquely extended along codimension two subsets, we obtain a line bundle with a singular metric on a compactification $E(N)$ of the universal elliptic curve.

It turns out that this naive approach is not a good idea. First, it is not functorial. If we consider different toroidal compactifications of $E^0(N)$, then the resulting extensions are not compatible. Second, even if the characteristic forms associated with the metric are locally integrable and define cohomology classes, they fail to satisfy a Chern-Weil theory. The cohomology class of the characteristic form does not agree with the characteristic class of the extended line bundle.

In this paper we propose a different approach to understand the extension of the Jacobi line bundle to a compactification of the universal elliptic curve. The "right" extension is not a line bundle, but a b-$\mathbb{Q}$-Cartier divisor. That is, a limit of different Cartier divisors with rational coefficients on all possible toroidal compactifications of $E^0(N)$. Defined in this way, the extension is obviously functorial because we are taking into account all possible toroidal compactifications. What is remarkable is that, with this interpretation, Chern-Weil theory allows us to interpret intersection products in terms of integrals of singular differential forms (see theorems 5.2 and 5.6). Moreover, there is a Hilbert-Samuel type formula relating the asymptotic of the dimension of the space of Jacobi forms with the self-intersection of the b-divisor (Theorem 5.1).

The non-functoriality of the naive extension is exactly the height jumping introduced by Hain (see [13] and [23]).

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2. The universal elliptic surface

In the whole paper we fix an integer $N \geq 3$. In this section we will revisit the definition of the universal elliptic surface of level N lying over the modular curve of level N. In particular, we will recall the construction of its smooth toroidal compactification. For further details and references the reader is referred to [17].

The modular curve of level N. Let $\mathbb{H}$ denote the upper half-plane given by

$$\mathbb{H} := \{\tau \in \mathbb{C} \,|\, \tau = \xi + i\eta,\, \eta > 0\}$$

and $\mathbb{H}^* := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ the extended upper half-plane. The principal congruence subgroup

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1 \bmod N,\ b \equiv c \equiv 0 \bmod N \right\}$$

of level N acts in the usual way by fractional linear transformations on $\mathbb{H}$; this action naturally extends to $\mathbb{H}^*$. The quotient space $X(N) := \Gamma(N)\backslash\mathbb{H}^*$ is called the modular curve of level N; it is the compactification of $Y(N) := \Gamma(N)\backslash\mathbb{H}$ by adding the so-called cusps.

The modular curve $X(N)$ is a compact Riemann surface of genus

$$g_N = 1 + \frac{N-6}{12} \frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)]}{2N} \,,$$

where the index of $\Gamma(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ is given as

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right).$$

The number p_N of cusps of $X(N)$ is given by

$$p_N = \frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)]}{2N} \,;$$

we denote the cusps by $P_1 := [\infty], P_2, \ldots, P_{p_N}$. We recall that $\Gamma(N)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and that the quotient group $\mathrm{SL}_2(\mathbb{Z})/\Gamma(N)$ acts transitively on the set of cusps (with stabilizers of order N). Therefore, it suffices in the sequel to consider the cusp $P_1 = [\infty]$. Since $N \geq 3$ the group $\Gamma(N)$ is torsion-free. Therefore, $X(N)$ has no elliptic points.

We recall that the modular curve $X(N)$ is the moduli space of elliptic curves with a full level N-structure. A point $[\tau] \in X(N)$ corresponds to the isomorphism class of elliptic curves determined by $\mathbb{C}/(\mathbb{Z}\tau \oplus \mathbb{Z})$ with N-torsion given by $(\mathbb{Z}\frac{\tau}{N} \oplus \mathbb{Z}\frac{1}{N})/(\mathbb{Z}\tau \oplus \mathbb{Z})$.

The universal elliptic surface of level N. We consider the product $\mathbb{H} \times \mathbb{C}$ consisting of elements (τ, z) with $\tau \in \mathbb{H}$ and $z = x + iy \in \mathbb{C}$. On $\mathbb{H} \times \mathbb{C}$ the semi-direct product $\Gamma(N) \ltimes \mathbb{Z}^2$ acts by the assignment

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right] (\tau, z) := \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right),$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(N)$ and $(\lambda, \mu) \in \mathbb{Z}^2$. Since $N \geq 3$, the group $\Gamma(N)$ is torsion-free. Hence, the action of $\Gamma(N) \ltimes \mathbb{Z}^2$ on $\mathbb{H} \times \mathbb{C}$ is free and the quotient space

$$E^0(N) := (\Gamma(N) \ltimes \mathbb{Z}^2)\backslash(\mathbb{H} \times \mathbb{C})$$

is a smooth complex surface together with a smooth surjective morphism

$$\pi^0 \colon E^0(N) \longrightarrow Y(N)$$

with fiber $(\pi^0)^{-1}([\tau]) = \mathbb{C}/(\mathbb{Z}\tau \oplus \mathbb{Z})$.

The surface $E^0(N)$ is known to extend to a compact complex surface $E(N)$ together with a surjective morphism

$$\pi\colon E(N) \longrightarrow X(N),$$

the so-called universal elliptic surface of level N. To describe this extension, it suffices to describe the fibers $\pi^{-1}(P_j)$ above the cusps $P_j \in X(N)$ $(j = 1, \ldots, p_N)$. These are given as N-gons, more precisely as

$$\pi^{-1}(P_j) = \bigcup_{\nu=0}^{N-1} \Theta_{j,\nu},$$

where $\Theta_{j,\nu} \cong \mathbb{P}^1(\mathbb{C})$ is embedded into $E(N)$ with self-intersection number -2, while otherwise

$$\Theta_{j,\nu} \cdot \Theta_{j,\nu'} = \begin{cases} 1 & \nu' = \nu \pm 1, \\ 0 & |\nu - \nu'| \geq 2; \end{cases}$$

here and subsequently, the indices ν, ν' have to be read modulo N.

In terms of local coordinates the situation above the cusp $P_1 = [\infty]$ can be described as follows: The irreducible fiber $\Theta_\nu := \Theta_{1,\nu} \subset E(N)$ can be covered by two affine charts $W_\nu^0, W_\nu^1 \subset E(N)$, where W_ν^0 contains the point $\Theta_\nu \cap \Theta_{\nu+1}$ and W_ν^1 contains the point $\Theta_\nu \cap \Theta_{\nu-1}$. Since Θ_ν and $\Theta_{\nu+1}$ intersect transversally, we can choose coordinates u_ν, v_ν on the chart W_ν^0 in such a way that $\Theta_\nu|_{W_\nu^0}$ is given by the equation $v_\nu = 0$ and $\Theta_{\nu+1}|_{W_\nu^0}$ by the equation $u_\nu = 0$. Using that $\Theta_\nu \cdot \Theta_\nu = -2$ we obtain that the coordinates of W_ν^1 are given by $u_\nu^{-1}, u_\nu^2 v_\nu$. The open subset $W_{\nu+1}^1$ agrees with W_ν^0. Hence, we deduce

$$u_{\nu+1} = v_\nu^{-1}, \quad v_{\nu+1} = u_\nu v_\nu^2.$$

We finally note the relations

$$u_\nu v_\nu = q_N := \mathrm{e}^{2\pi i\tau/N}, \quad u_\nu^{\nu+1} v_\nu^\nu = \zeta := \mathrm{e}^{2\pi i z}. \tag{2.1}$$

If we want to work with different cusps we will denote by $W_{j,v}^0$ and $W_{j,v}^1$ the analogous affine charts around points over the cusp P_j.

We conclude by introducing the zero section

$$\varepsilon\colon X(N) \longrightarrow E(N)$$

and by recalling that the arithmetic genus of $E(N)$ is given by

$$p_{\mathrm{a},N} = \frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)]}{24} - 1 = \frac{N p_N}{12} - 1.$$

Jacobi forms. Modular forms can be interpreted as global sections of line bundles on the modular curve. The Jacobi forms play a similar role for the universal elliptic curve.

Definition 2.2. Let k, m be non-negative integers. A holomorphic function $f\colon \mathbb{H}\times\mathbb{C}\to\mathbb{C}$ is called *Jacobi form of weight k, index m for* $\Gamma(N)$, if it satisfies the following properties:

(i) The function f satisfies the functional equations

$$f\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right)(c\tau+d)^{-k}\times \tag{2.3}$$

$$\times\exp\left(2\pi i m\left(\lambda^2\tau+2\lambda z-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}\right)\right)=f(\tau,z)$$

for all $\left[\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right),(\lambda,\mu)\right]\in\Gamma(N)\ltimes\mathbb{Z}^2$.

(ii) At the cusp $P_1=[\infty]$, the function f has a Fourier expansion of the form

$$f(\tau,z)=\sum_{\substack{n\in\mathbb{N},\,r\in\mathbb{Z}\\4mn-Nr^2\geq 0}} c(n,r)q_N^n\zeta^r, \tag{2.4}$$

and similar Fourier expansions at the other cusps.

We denote the vector space of Jacobi forms of weight k, index m for $\Gamma(N)$ by $J_{k,m}(\Gamma(N))$.

If condition (ii) on the Fourier expansions is restricted to the summation over $n\in\mathbb{N}_{>0}$ and $r\in\mathbb{Z}$ such that $4mn-Nr^2>0$, the function f is called *Jacobi cusp form of weight k, index m for* $\Gamma(N)$ and the span of these functions is denoted by $J^{\text{cusp}}_{k,m}(\Gamma(N))$.

If condition (ii) on the Fourier expansions is dropped, the function f is called *weak Jacobi form of weight k, index m for* $\Gamma(N)$. The span of these functions is denoted by $J^{\text{weak}}_{k,m}(\Gamma(N))$.

Definition 2.5. Given integers k, m and $N>0$, the condition (2.3) is a cocycle condition that defines a line bundle $L_{k,m,N}$ on $E^0(N)$. This line bundle is called the *Jacobi line bundle*. Its space of global sections, $H^0(E^0(N),L_{k,m,N})$, agrees with the space of weak Jacobi forms of weight k and index m for $\Gamma(N)$.

Riemann theta functions. The Riemann theta function $\theta_{1,1}\colon\mathbb{H}\times\mathbb{C}\to\mathbb{C}$ is defined by the convergent power series

$$\theta_{1,1}(\tau,z):=\sum_{n\in\mathbb{Z}}\exp\left(\pi i\tau\left(n+\frac{1}{2}\right)^2+2\pi i\left(z+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\right) \tag{2.6}$$

and satisfies the functional equation

$$\theta_{1,1}\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right)(c\tau+d)^{-1/2}\times$$

$$\times\exp\left(\pi i\left(\lambda^2\tau+2\lambda z-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}\right)\right)=\chi\begin{pmatrix}a&b\\c&d\end{pmatrix}\theta_{1,1}(\tau,z)$$

for all $\left[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda, \mu)\right] \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ with a character $\chi(\cdot)$, which is an 8-th root of unity. Therefore, $\theta_{1,1}^8$ is a weak Jacobi form of weight 4, index 4 for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. Moreover, from the definition of power series (2.6) it follows that $\theta_{1,1}^8$ is a Jacobi form.

Dimension formulae. We recall the dimension formulae for the space of Jacobi forms. For simplicity, we restrict ourselves to the case $m = k = 4\ell$. We denote by $j \colon E^0(N) \to E(N)$ the open immersion. From [17], we cite the following result.

Proposition 2.7. *There is a distinguished subsheaf $\mathcal{F}_\ell$ of the sheaf $j_* L_{4\ell,4\ell,N}$ such that we have an isomorphism*

$$J_{4\ell,4\ell}^{\mathrm{cusp}}\big(\Gamma(N)\big) \cong H^0\big(E(N), \mathcal{F}_\ell\big).$$

In particular, the dimension of $J_{4\ell,4\ell}^{\mathrm{cusp}}(\Gamma(N))$ is given, when N divides 4ℓ, by

$$\begin{aligned}\dim J_{4\ell,4\ell}^{\mathrm{cusp}}\big(\Gamma(N)\big)& \\ &= p_N\left(\frac{8N\ell^2}{3} - N\ell - \frac{N}{4}Q\big(\frac{16\ell}{N}\big) - \frac{N}{2}\sum_{\substack{\Delta \mid 16\ell/N,\, \Delta<0 \\ 16\ell/(N\Delta)\ \mathrm{squarefree}}} H(\Delta)\right) \\ &= \frac{8Np_N}{3}\ell^2 + o\big(\ell^2\big),\end{aligned}$$

where $Q(n)$ denotes the largest integer whose square divides n and $H(\Delta)$ is the Hurwitz class number.

Proof. The first statement is [17, Theorem 2.6], the second statement is [17, Theorem 3.8], noting that

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] = 2Np_N.$$

To prove the assymptotic estimate one uses that $Q(n)$ is at most $\sqrt{n}$, that the number of divisors of an integer n is $o(n^\varepsilon)$ for any $\varepsilon > 0$ and that, by the Brauer-Siegel theorem, the Hurwitz class number $H(\Delta)$ is $o(|\Delta|^{1/2+\varepsilon})$ for any $\varepsilon > 0$. □

Remark 2.8. The asymptotic growth in ℓ of the dimension of the space of Jacobi forms and of the space of Jacobi cusp forms is the same. To see this we have to estimate the number of conditions on the vanishing of the Fourier coefficients $c(n, r)$ for $4mn - Nr^2 = 0$ with $m = 4\ell$.

Using the transformation given by $a = d = \lambda = 1$ and $b = c = \mu = 0$, we see that the coefficients of the Fourier expansion (2.4) satisfy the periodicity relation

$$c(n + Nr + Nm, r + 2m) = c(n, r). \tag{2.9}$$

Given a Jacobi form of weight k and index m, to impose that it is a Jacobi cusp form is equivalent to impose, at each cusp, the conditions

$$c(n, r) = 0, \text{ for } 4mn - Nr^2 = 0.$$

In view of the periodicity condition (2.9), this is a finite number of conditions that grows linearly with the index. Therefore, the difference $\dim J_{4\ell,4\ell}\big(\Gamma(N)\big) - \dim J^{\mathrm{cusp}}_{4\ell,4\ell}\big(\Gamma(N)\big)$ grows linearly with ℓ and we deduce the asymptotic formula

$$\dim J_{4\ell,4\ell}\big(\Gamma(N)\big) = \frac{8Np_N}{3}\ell^2 + o\big(\ell^2\big).$$

Translation invariant metric. Here we recall the translation invariant metric on the line bundle $L_{k,m,N}$.

Definition 2.10. For $f \in J^{\mathrm{weak}}_{k,m}(\Gamma(N))$, we define

$$\|f(\tau,z)\|^2 := |f(\tau,z)|^2 \exp(-4\pi m y^2/\eta)\eta^k,$$

where we recall that $\eta = \mathrm{Im}(\tau)$ and $y = \mathrm{Im}(z)$.

Lemma 2.11. *For $f \in J^{\mathrm{weak}}_{k,m}(\Gamma(N))$, we have*

$$\left\| f\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right)\right\|^2 = \|f(\tau,z)\|^2$$

for all $\big[\big(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\big), (\lambda,\mu)\big] \in \Gamma(N) \ltimes \mathbb{Z}^2$. In particular, this shows that $\|\cdot\|$ induces a hermitian metric on the line bundle $L_{k,m,N}$.

Proof. This is a straightforward calculation. □

Lemma 2.12. *Locally, in the affine chart W^0_ν over the cusp $P_1 = [\infty]$, the hermitian metric $\|\cdot\|$ is described by the formula*

$$\begin{aligned}\log\big(\|f(\tau,z)\|^2\big)\Big|_{W^0_\nu} &= \log\big(|f(\tau,z)|^2\big)\Big|_{W^0_\nu} \\ &+ \frac{m}{N}\left((\nu+1)^2\log(u_\nu\bar{u}_\nu) + \nu^2\log(v_\nu\bar{v}_\nu) - \frac{\log(u_\nu\bar{u}_\nu)\log(v_\nu\bar{v}_\nu)}{\log(u_\nu\bar{u}_\nu)+\log(v_\nu\bar{v}_\nu)}\right) \\ &+ k\log\left(-\frac{N}{4\pi}\big(\log(u_\nu\bar{u}_\nu)+\log(v_\nu\bar{v}_\nu)\big)\right).\end{aligned}$$

Proof. Taking absolute values, we derive from (2.1)

$$\begin{aligned}\eta &= -\frac{N}{2\pi}\log|q_N| = -\frac{N}{2\pi}\log|u_\nu v_\nu| \\ &= -\frac{N}{4\pi}\big(\log(u_\nu\bar{u}_\nu)+\log(v_\nu\bar{v}_\nu)\big),\end{aligned}$$

$$\begin{aligned}y &= -\frac{1}{2\pi}\log|\zeta| = -\frac{1}{2\pi}\log|u_\nu^{\nu+1}v_\nu^{\nu}| \\ &= -\frac{1}{4\pi}\big((\nu+1)\log(u_\nu\bar{u}_\nu)+\nu\log(v_\nu\bar{v}_\nu)\big).\end{aligned}$$

With these formulae we compute

$$-\frac{4\pi m y^2}{\eta} = \frac{m}{N}\frac{\big((\nu+1)\log(u_\nu\bar{u}_\nu)+\nu\log(v_\nu\bar{v}_\nu)\big)^2}{\log(u_\nu\bar{u}_\nu)+\log(v_\nu\bar{v}_\nu)} =$$

$$\frac{m}{N}\left((\nu+1)^2\log(u_\nu\bar{u}_\nu)+\nu^2\log(v_\nu\bar{v}_\nu)-\frac{\log(u_\nu\bar{u}_\nu)\log(v_\nu\bar{v}_\nu)}{\log(u_\nu\bar{u}_\nu)+\log(v_\nu\bar{v}_\nu)}\right).$$

From this the proof follows immediately from the definition of the hermitian metric $\|\cdot\|$. □

3. Mumford-Lear extensions and b-divisors

In this section we will introduce Mumford-Lear extensions of a line bundle and relate them with b-divisors. We first recall the different notions of growth for metrics and differential forms that will be useful in the sequel.

Notations. Let X be a complex algebraic manifold of dimension d and D a normal crossing divisor of X. Write $U = X \setminus D$, and let $j\colon U \longrightarrow X$ be the inclusion. We will denote by $\mathscr{E}_X^*$ the sheaf of algebras of smooth complex differential forms on X and by $\mathscr{E}_U^*$ the restriction of this sheaf to U.

Let V be an open coordinate subset of X with coordinates $z_1, \dots, z_d$; we put $r_i = |z_i|$. We say that V *is adapted to* D, if the divisor D is locally given by the equation $z_1 \cdots z_k = 0$. We assume that the coordinate neighborhood V is small enough; more precisely, we will assume that all the coordinates satisfy $r_i < 1/e^e$, which implies that $\log 1/r_i > e$ and $\log(\log 1/r_i) > 1$.

If f and g are two functions with non-negative real values, we will write $f \prec g$, if there exists a constant $C > 0$ such that $f(x) \leq C \cdot g(x)$ for all x in the domain of definition under consideration.

log-log growth forms.

Definition 3.1. We say that a smooth complex function f on $X \setminus D$ has *log-log growth along* D, if we have

$$|f(z_1, \dots, z_d)| \prec \prod_{i=1}^{k} \log(\log(1/r_i))^M \tag{3.2}$$

for every coordinate subset V adapted to D and some positive integer M. The *sheaf of differential forms on X with log-log growth along D* is the subalgebra of $j_*\mathscr{E}_U^*$ generated, in each coordinate neighborhood V adapted to D, by the functions with log-log growth along D and the differentials

$$\frac{\mathrm{d}\, z_i}{z_i\log(1/r_i)},\ \frac{\mathrm{d}\, \bar{z}_i}{\bar{z}_i\log(1/r_i)}, \qquad \text{for } i = 1, \dots, k,$$
$$\mathrm{d}\, z_i,\ \mathrm{d}\, \bar{z}_i, \qquad \text{for } i = k+1, \dots, d.$$

If D is clear from the context, a differential form with log-log growth along D will be called a *log-log growth form*.

Note that, to check that a function or a form has log-log growth it is enough to check the defining condition at the elements of an open covering of X made of coordinate subsets adapted to D.

Dolbeault algebra of pre-log-log forms. Clearly, the forms with log-log growth form an algebra but not a differential algebra. To remedy this we impose conditions on the derivatives as well.

Definition 3.3. A log-log growth form ω such that $\partial\omega$, $\bar{\partial}\omega$ and $\partial\bar{\partial}\omega$ are also log-log growth forms is called a *pre-log-log form (along D)*. The sheaf of pre-log-log forms is the subalgebra of $j_*\mathscr{E}_U^*$ generated by the pre-log-log forms. We will denote this complex by $\mathscr{E}_X^*\langle\langle D\rangle\rangle_{\mathrm{pre}}$. The pre-log-log forms of degree zero are called *pre-log-log* functions.

The sheaf $\mathscr{E}_X^*\langle\langle D\rangle\rangle_{\mathrm{pre}}$, together with its real structure, its bigrading, and the usual differential operators ∂, $\bar{\partial}$ is easily shown to be a sheaf of Dolbeault algebras. Moreover, it is the maximal subsheaf of Dolbeault algebras of the sheaf of differential forms with log-log growth.

Metrics with logarithmic growth and pre-log metrics. Let L be a line bundle on X and let $\|\cdot\|$ be a smooth hermitian metric on $L|_U$.

Definition 3.4. We will say that the metric $\|\cdot\|$ has *logarithmic growth (along D)* if, for every point $x \in X$, there is a coordinate neighbourhood V of x adapted to D, a nowhere zero regular section s of L on V, and an integer $M \geq 0$ such that

$$\prod_{i=1}^{k} \log(1/r_i)^{-M} \prec \|s(z_1,\ldots,z_d)\| \prec \prod_{i=1}^{k} \log(1/r_i)^{M} \tag{3.5}$$

Definition 3.6. We will say that the metric $\|\cdot\|$ is a *pre-log metric (along D)*, if it has logarithmic growth and, for every rational section s of L, the function $\log\|s\|$ is a pre-log-log form along $D \setminus \mathrm{div}(s)$ on $X \setminus \mathrm{div}(s)$.

Mumford-Lear extensions. We are now able to define Mumford-Lear extensions. For the remainder of the section we fix a complex algebraic manifold X of dimension d, D and U as before, and we also fix a hermitian line bundle $\overline{L} = (L, \|\cdot\|)$ on U.

Definition 3.7. We say that $\overline{L}$ *admits a Mumford-Lear extension to X*, if there is an integer $e \geq 1$, a line bundle $\mathcal{L}$ on X, an algebraic subset $S \subset X$ of codimension at least 2 that is contained in D, a smooth hermitian metric $\|\cdot\|$ on $\mathcal{L}|_U$ that has logarithmic growth along $D \setminus S$ and an isometry $\alpha\colon (L, \|\cdot\|)^{\otimes e} \to (\mathcal{L}|_U, \|\cdot\|)$. The 5-tuple $(e, \mathcal{L}, S, \|\cdot\|, \alpha)$

is called a *Mumford-Lear extension* of $\overline{L}$. When the isomorphism α, the metric and the set S can be deduced by the context, we will denote the Mumford-Lear extension by $(e, \mathcal{L})$. If $e = 1$, we will denote it by the line bundle $\mathcal{L}$.

Remark 3.8. The name Mumford-Lear extension arises because they generalize (in the case of line bundles) the extensions of hermitian vector bundles considered by Mumford in [22] and the extensions of line bundles considered by Lear in his thesis [21].

The Mumford-Lear extensions satisfy the following unicity property.

Proposition 3.9. *Assume that $\overline{L}$ admits a Mumford-Lear extension to X. Let $(e_1, \mathcal{L}_1, S_1, \|\cdot\|_1, \alpha_1)$ and $(e_2, \mathcal{L}_2, S_2, \|\cdot\|_2, \alpha_2)$ be two Mumford-Lear extensions of $\overline{L}$ to X. Then, there is a unique isomorphism*

$$\beta\colon \mathcal{L}_1^{\otimes e_2} \to \mathcal{L}_2^{\otimes e_1}$$

such that the diagram

$$\begin{array}{ccc} & & \mathcal{L}_1^{\otimes e_2}|_U \\ & \overset{\alpha_1^{\otimes e_2}}{\nearrow} & \\ L^{\otimes e_1 e_2} & & \Big\downarrow \beta|_U \\ & \underset{\alpha_2^{\otimes e_1}}{\searrow} & \\ & & \mathcal{L}_2^{\otimes e_1}|_U \end{array}$$

is commutative.

Proof. The composition $\alpha_2^{\otimes e_1} \circ (\alpha_1^{-1})^{\otimes e_2}$ defines an isomophism between the line bundles $\mathcal{L}_1^{\otimes e_2}|_U$ and $\mathcal{L}_2^{\otimes e_1}|_U$ that is the only one that makes the diagram in the theorem commutative. Put $S = S_1 \cup S_2$. The proof of [22, Proposition 1.3] shows that this isomorphism extends uniquely to an isomorphism $\beta_1\colon \mathcal{L}_1^{\otimes e_2}|_{X\setminus S} \to \mathcal{L}_2^{\otimes e_1}|_{X\setminus S}$. Since X is smooth and S has codimension 2, the isomorphism β_1 extends to a unique isomorphism $\beta\colon \mathcal{L}_1^{\otimes e_2} \to \mathcal{L}_2^{\otimes e_1}$ satisfying the condition of the proposition. □

The next result is an immediate consequence of Proposition 3.9.

Corollary 3.10. *Assume the hypothesis of the previous proposition. Let s be a rational section of L, so $\alpha_1(s^{\otimes e_1})^{\otimes e_2}$ and $\alpha_2(s^{\otimes e_2})^{\otimes e_1}$ are rational sections of $\mathcal{L}_1^{\otimes e_2}$ and $\mathcal{L}_2^{\otimes e_1}$, respectively. Then,*

$$\operatorname{div}(\alpha_1(s^{\otimes e_1})^{\otimes e_2}) = \operatorname{div}(\alpha_2(s^{\otimes e_2})^{\otimes e_1})$$

as Cartier divisors on X. □

This result allows us to associate to each rational section of L a $\mathbb{Q}$-Cartier divisor on X. We will denote by $\mathbb{Q}$-$\mathrm{Ca}(X)$ the group of $\mathbb{Q}$-Cartier divisors of X.

Definition 3.11. Assume that $\overline{L}$ admits a Mumford-Lear extension to X and let $(e, \mathcal{L}, S, \|\cdot\|, \alpha)$ be one such extension. Let s be a rational section of L. Then, we define *the divisor of s on X* as the $\mathbb{Q}$-Cartier divisor

$$\operatorname{div}_X(s) = \frac{1}{e} \operatorname{div}(\alpha(s^{\otimes e})) \in \mathbb{Q}\text{-}\mathrm{Ca}(X),$$

where $\operatorname{div}(\alpha(s^{\otimes e}))$ is the divisor of $\alpha(s^{\otimes e})$ viewed as a rational section of $\mathcal{L}$ on the whole of X.

Mumford-Lear extensions and birational morphisms. We now consider Mumford-Lear extensions on different birational models of X.

Notation 3.12. Let $\mathcal{C}$ be the category whose objects are pairs (Y, φ_Y), where Y is a smooth complex variety and $\varphi_Y : Y \to X$ is a proper birational morphism, and whose arrows are morphisms $\varphi : Y \to Z$ such that $\varphi_Z \circ \varphi = \varphi_Y$. We denote by $\mathcal{BIR}(X)$ the set of isomorphism classes in $\mathcal{C}$. Since the set of morphisms between two objects of $\mathcal{C}$ is either empty or contains a single element, the set $\mathcal{BIR}(X)$ is itself a small category equivalent to $\mathcal{C}$. In fact, $\mathcal{BIR}(X)$ is a directed set. As a shorthand, an element (Y, φ_Y) of $\mathcal{BIR}(X)$ will be denoted by the variety Y, the morphism φ_Y being implicit. For an object Y of $\mathcal{BIR}(X)$ we will denote $U_Y = \varphi_Y^{-1}(U)$ and $D_Y = \varphi_Y^{-1}(D)$. By abuse of notation we will denote also by φ_Y the induced proper birational morphism from U_Y to U. Finally, we will denote by $\mathcal{BIR}'(X)$ the subset consisting of the elements Y with D_Y a normal crossing divisor. This is a cofinal subset.

Definition 3.13. We say that $\overline{L}$ *admits all Mumford-Lear extensions over* X if, for every object Y of $\mathcal{BIR}'(X)$, the hermitian line bundle $\varphi_Y^* \overline{L}$ on U_Y admits a Mumford-Lear extension to Y.

Definition 3.14. Assume that $\overline{L}$ admits all Mumford-Lear extensions over X. For every $Y \in \mathcal{BIR}'(X)$, choose a Mumford-Lear extension $(e', \mathcal{L}', S', \|\cdot\|', \alpha')$ of $\varphi_Y^* \overline{L}$ to Y. Then, the *divisor of s on Y* is defined as

$$\operatorname{div}_Y(s) = \frac{1}{e'} \operatorname{div}(\alpha'(s^{\otimes e'})) \in \mathbb{Q}\text{-}\mathrm{Ca}(Y).$$

The $\mathbb{Q}$-Cartier divisors of Definition 3.14 do not need to be compatible with inverse images. As we will see in concrete examples, it may happen that there are morphisms $\varphi : Y \to Z$ in $\mathcal{BIR}'(X)$ such that

$$\varphi^* \operatorname{div}_Z(s) \neq \operatorname{div}_Y(s).$$

This lack of compatibility with inverse images is related with the phenomenon of height jumping (see [13] and [23] for a discussion of height jumping).

In contrast, the divisors associated with Mumford-Lear extensions are compatible with direct images.

Proposition 3.15. *Assume that $\overline{L}$ admits all Mumford-Lear extensions over X. Let $\varphi\colon Y \to Z$ be a morphism in $\mathcal{BIR}'(X)$ and s a section of L. Then,*

$$\varphi_* \operatorname{div}_Y(s) = \operatorname{div}_Z(s).$$

Proof. Let T be the subset of Z, where φ^{-1} is not defined. Since Z is smooth, hence normal, by Zariski's main theorem, T has codimension at least 2. Write $W = Z \setminus T$ and let $U' = U_Z \cap W$. Then, $\overline{L}$ induces a line bundle on U' that admits a Mumford-Lear extension to W.

Since T has codimension 2, the restriction map

$$\mathbb{Q}\text{-Ca}(Z) \to \mathbb{Q}\text{-Ca}(W)$$

is an isomorphism. Moreover, using the definition is easy to see that

$$\operatorname{div}_Y(s)|_W = \operatorname{div}_W(s) = \operatorname{div}_Z(s)|_W.$$

Thus, the proposition follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{Q}\text{-Ca}(Y) & \longrightarrow & \mathbb{Q}\text{-Ca}(W) \\ \downarrow{\scriptstyle \varphi_*} & & \| \\ \mathbb{Q}\text{-Ca}(Z) & \xrightarrow{\simeq} & \mathbb{Q}\text{-Ca}(W) \end{array}.$$

□

B-Divisors. Recall that the Zariski-Riemann space of X is the projective limit

$$\mathfrak{X} = \varprojlim_{\mathcal{BIR}(X)} Y.$$

We are not going to use the structure of this space, which is introduced here merely in order to make later definitions more suggestive.

For the definition of b-divisors, we will follow the point of view of [3]. The groups $\mathbb{Q}\text{-Ca}(Y)$, $Y \in \mathcal{BIR}(X)$, form a projective system with respect to the push-forward morphisms and an inductive system with respect to the pull-back morphisms. We define the group of $\mathbb{Q}$-Cartier divisor on $\mathfrak{X}$ as the inductive limit

$$\mathbb{Q}\text{-Ca}(\mathfrak{X}) = \varinjlim_{\mathcal{BIR}(X)^{\text{op}}} \mathbb{Q}\text{-Ca}(Y)$$

and the group of $\mathbb{Q}$-Weil divisors on $\mathfrak{X}$ as the projective limit

$$\mathbb{Q}\text{-We}(\mathfrak{X}) = \varprojlim_{\mathcal{BIR}(X)} \mathbb{Q}\text{-Ca}(Y).$$

Since, for any morphism φ in $\mathcal{BIR}(X)$ the composition $\varphi_* \circ \varphi^*$ is the identity, it is easy to see that there is a map $\mathbb{Q}\text{-Ca}(\mathfrak{X}) \to \mathbb{Q}\text{-We}(\mathfrak{X})$. Note also that, since $\mathcal{BIR}'(X)$ is cofinal, the above projective and inductive limit can be taken over $\mathcal{BIR}'(X)$.

The group of $\mathbb{Q}$-Weil divisors of $\mathfrak{X}$ is closely related to the group of b-divisors of X defined in [24]. Thus, a $\mathbb{Q}$-Weil divisors of $\mathfrak{X}$ will be called a b-divisors of X.

The following definition makes sense thanks to Proposition 3.9.

Definition 3.16. Assume that $\overline{L}$ admits all Mumford-Lear extensions over X. Let s be rational section of L. Then, the *b-divisor associated* to s is

$$\text{b-div}(s) = (\text{div}_Y(s))_{Y\in\mathcal{BIR}'(X)} \in \mathbb{Q}\text{-We}(\mathfrak{X}).$$

When it is needed to specify with respect to which metric we are compactifying the divisor, we will write b-div$(s, \|\cdot\|)$.

Integrable b-divisors. From now on we restrict ourselves to the case when X is a surface. We want to extend the intersection product of divisors as much as possible to b-divisors.

It is clear that there is an intersection pairing

$$\mathbb{Q}\text{-Ca}(\mathfrak{X}) \times \mathbb{Q}\text{-We}(\mathfrak{X}) \to \mathbb{Q}$$

defined as follows. Let $C \in \mathbb{Q}\text{-Ca}(\mathfrak{X})$ and $E \in \mathbb{Q}\text{-We}(\mathfrak{X})$. Then, there is an object $Y \in \mathcal{BIR}'(X)$ and a divisor $C_Y \in \mathbb{Q}\text{-Ca}(Y)$ such that C is the image of C_Y. Let E_Y be the component of E on Y. Then, by the projection formula, the intersection product $C_Y \cdot E_Y$ does not depend on the choice of Y. Thus, we define

$$C \cdot E = C_Y \cdot E_Y.$$

But, in general, we can not define the intersection product of two elements of $\mathbb{Q}$-We$(\mathfrak{X})$. The following definition is the analogue for b-divisors of the concept of an L^2-function. Recall that, since $\mathcal{BIR}'(X)$ is a directed set, it is in particular a net.

Definition 3.17. A divisor $C = (C_Y)_{Y\in\mathcal{BIR}'(X)} \in \mathbb{Q}\text{-We}(\mathfrak{X})$ is called integrable if the limit

$$\varinjlim_{\mathcal{BIR}'(X)} C_Y \cdot C_Y$$

exists and is finite.

Proposition 3.18. *Let $C_1, C_2 \in \mathbb{Q}\text{-We}(\mathfrak{X})$. If C_1 and C_2 are integrable, then*

$$\varinjlim_{\mathcal{BIR}'(X)} C_{1,Y} \cdot C_{2,Y}$$

exists and is finite.

Proof. Let $C = (C_Y) \in \mathbb{Q}\text{-We}(\mathfrak{X})$ and let $\varphi\colon Y \to Z$ be an arrow in $\mathcal{BIR}'(X)$. Since $\varphi_* C_Y = C_Z$, we deduce that

$$C_Y = \varphi^* C_Z + E,$$

where E is an exceptional divisor for the morphism φ. Hence,

$$C_Y \cdot C_Y = (\varphi^* C_Z + E) \cdot (\varphi^* C_Z + E) = C_Z \cdot C_Z + E \cdot E.$$

Thus, by the Hodge index theorem,

$$C_Y \cdot C_Y - C_Z \cdot C_Z = E \cdot E \leq 0.$$

Hence,

$$\begin{aligned}0 &\geq (C_{1,Y} \pm C_{2,Y})^2 - (C_{1,Z} \pm C_{2,Z})^2\\ &= C_{1,Y}^2 - C_{1,Z}^2 + C_{2,Y}^2 - C_{2,Z}^2 \pm 2(C_{1,Y} \cdot C_{2,Y} - C_{1,Z} \cdot C_{2,Z}).\end{aligned}$$

Therefore,

$$\begin{aligned}C_{1,Y} \cdot C_{2,Y} - C_{1,Z} \cdot C_{2,Z} &\leq -\frac{1}{2}(C_{1,Y}^2 - C_{1,Z}^2 + C_{2,Y}^2 - C_{2,Z}^2)\\ &= \frac{1}{2}(|C_{1,Y}^2 - C_{1,Z}^2| + |C_{2,Y}^2 - C_{2,Z}^2|)\end{aligned}$$

and

$$\begin{aligned}C_{1,Y} \cdot C_{2,Y} - C_{1,Z} \cdot C_{2,Z} &\geq \frac{1}{2}(C_{1,Y}^2 - C_{1,Z}^2 + C_{2,Y}^2 - C_{2,Z}^2)\\ &= -\frac{1}{2}(|C_{1,Y}^2 - C_{1,Z}^2| + |C_{2,Y}^2 - C_{2,Z}^2|).\end{aligned}$$

Thus,

$$|C_{1,Y} \cdot C_{2,Y} - C_{1,Z} \cdot C_{2,Z}| \leq \frac{1}{2}(|C_{1,Y}^2 - C_{1,Z}^2| + |C_{2,Y}^2 - C_{2,Z}^2|).$$

Thus, the convergence of $(C_{1,Y}^2)_Y$ and $(C_{2,Y}^2)_Y$ implies the convergence of $(C_{1,Y} \cdot C_{2,Y})_Y$. □

4. The Mumford-Lear extension of the Jacobi line bundle

In this section we will study the Mumford-Lear extensions of the Jacobi line bundle.

The functions $f_{n,m}$. We first study a family of functions that will be useful latter. Let (n, m) be a pair of coprime positive integers. Let u, v be coordinates of $\mathbb{C}^2$ and denote $U \subset \mathbb{C}^2$ the open subset defined by $|uv| < 1$. Let $D \subset U$ be the normal crossing divisor of equation $uv = 0$.

Proposition 4.1. *Let $f_{n,m}$ be the function on U given by*

$$f_{n,m}(u, v) = \frac{1}{nm} \frac{\log(u\overline{u}) \log(v\overline{v})}{n \log(u\overline{u}) + m \log(v\overline{v})}$$

This function satisfies the following properties.

(i) The function $f_{n,m}$ is a pre-log-log function along $D \setminus \{(0, 0)\}$.

(ii) The equality $\partial\overline{\partial} f_{n,m} \wedge \partial\overline{\partial} f_{n,m} = 0$ holds. The differential forms $f_{n,m}$, $\partial f_{n,m}$, $\overline{\partial} f_{n,m}$, and $\partial\overline{\partial} f_{n,m}$ and all the products between them are locally integrable. Moreover, any product of $\partial\overline{\partial} f_{n,m}$ with a pre-log-log form along D is also locally integrable.

(iii) Let $\varphi\colon U \to U$ be the map given by $(s,t) \mapsto (st,t)$. Note that this map identifies U with a chart of the blow-up of U along $(0,0)$. Then,

$$\varphi^* f_{n,m}(s,t) = \frac{1}{nm(n+m)} \log(t\overline{t}) + f_{n,n+m}(s,t).$$

Proof. Put

$$P_{n,m}(u,v) = n\log(u\overline{u}) + m\log(v\overline{v})$$
$$a = \log(v\overline{v})\frac{\mathrm{d}\,u}{u}, \qquad b = \log(u\overline{u})\frac{\mathrm{d}\,v}{v}.$$

With these notations, we have

$$\partial f_{n,m} = \frac{1}{nmP_{n,m}^2}(n\log(u\overline{u})b + m\log(v\overline{v})a), \tag{4.2}$$

$$\partial\overline{\partial} f_{n,m} = \frac{2}{P_{n,m}^3}(b-a)\wedge(\overline{a}-\overline{b}), \tag{4.3}$$

$$\partial f_{n,m} \wedge \partial\overline{\partial} f_{n,m} = \frac{2}{nmP_{n,m}^4} a\wedge b\wedge(\overline{a}-\overline{b}). \tag{4.4}$$

From equation (4.3), it follows that $\partial\overline{\partial} f_{n,m}\wedge\partial\overline{\partial} f_{n,m} = 0$.

We now prove (i). Let $p = (0,v_0) \in D \setminus \{(0,0)\}$. Let V be a neighborhood of p such that $|\log(v\overline{v})| \leq K$, $u\overline{u} < 1$ and

$$n|\log(u\overline{u})| \geq 2mK \geq 2m|\log(v\overline{v})|,$$

for some positive constant K. Therefore, on all the points of V, the estimate

$$|P_{n,m}| \geq \frac{n}{2}|\log(u\overline{u})|$$

holds. Then, for $(u,v) \in V$,

$$|f_{n,m}(u,v)| \leq \frac{2}{n^2m}\frac{|\log(u\overline{u})\log(v\overline{v})|}{|\log(u\overline{u})|} \leq \frac{2K}{n^2m}. \tag{4.5}$$

Similarly, if t_1 and t_2 are smooth tangent vectors on V with bounded coefficients, from (4.2) and (4.3), we derive

$$|\partial f_{n,m}(t_1)| \leq \frac{C_1}{|\log(u\overline{u})|^2|u|} \tag{4.6}$$

$$|\partial\overline{\partial} f_{n,m}(t_1,t_2)| \leq \frac{C_2}{|\log(u\overline{u})|^3|u|^2} \tag{4.7}$$

for suitable positive constants C_1 and C_2. The estimates (4.5), (4.6) and (4.7) show that $f_{n,m}$ is a pre-log-log function. Thus, we have proved (i).

Since pre-log-log forms are always locally integrable (cf. [8, Proposition 7.6]), in order to check (ii), it is only necessary to study a neighborhood of the point $(0,0)$. Thus, we restrict ourselves to the open W defined by $|u| < 1/e$ and $|v| < 1/e$.

We show the local integrability of a form of the type $\partial\overline{\partial} f_{n,m} \wedge \psi$ for a pre-log-log form ψ, being the other cases analogous.

By the definition of pre-log-log forms, equation (4.3) shows that $\partial\overline{\partial} f_{n,m} \wedge \psi = g(u,v)\, \mathrm{d}\, u \wedge \mathrm{d}\, \overline{u} \wedge \mathrm{d}\, v \wedge \mathrm{d}\, \overline{v}$, with g a function satisfying

$$|g(u,v)| \leq C_1 \frac{|\log(\log(u\overline{u}))\log(\log(v\overline{v}))|^M}{|P^3_{n,m} u\overline{u}v\overline{v}|}.$$

for certain positive constants C_1 and M. Using the geometric vs. arithmetic mean inequality, and the fact that the logarithm grows slower than any polynomial, we see that g can be bounded as

$$|g(u,v)| \leq \frac{C_2}{|\log(u\overline{u})\log(v\overline{v})|^{1+\varepsilon} u\overline{u}v\overline{v}},$$

with C_2 and ε positive. Since the differential form

$$\frac{\mathrm{d}\, u \wedge \mathrm{d}\, \overline{u} \wedge \mathrm{d}\, v \wedge \mathrm{d}\, \overline{v}}{|\log(u\overline{u})\log(v\overline{v})|^{1+\varepsilon} u\overline{u}v\overline{v}}$$

is locally integrable, we deduce that $\partial\overline{\partial} f_{n,m} \wedge \psi$ is locally integrable.

Every product between a smooth form and any of the forms $f_{n,m}$, $\partial f_{n,m}$, $\overline{\partial} f_{n,m}$, and $\partial\overline{\partial} f_{n,m}$, will satisfy growth estimates not worse than the one satisfied by $\partial\overline{\partial} f_{n,m} \wedge \psi$, except the product $\partial\overline{\partial} f_{n,m} \wedge \partial\overline{\partial} f_{n,m}$. Since this last product is zero we conclude (ii).

The statement (iii) follows from a direct computation. □

THE MUMFORD-LEAR EXTENSION OF THE JACOBI LINE BUNDLE TO $E(N)$. We now denote by $D = E(N) \setminus E^0(N)$. This is a normal crossings divisor. Let Σ be the set of double points of D and put $D^0 = D \setminus \Sigma$ for the smooth part of D. Let H be the divisor of $E(N)$ defined as the image of the zero section $X(N) \to E(N)$.

Consider the divisor on $E(N)$ given by

$$C = 8H + \sum_{j=1}^{p_N} \sum_{\nu=0}^{N-1} \left(N - 4\nu + \frac{4\nu^2}{N} \right) \Theta_{j,\nu}, \tag{4.8}$$

Choose a smooth hermitian metric $\|\cdot\|'$ on $\mathcal{O}(C)$ and let s be a section of $\mathcal{O}(C)$ with $\operatorname{div} s = C$.

Proposition 4.9. *The hermitian line bundle* $\overline{L} = (L_{4,4,N}, \|\cdot\|)$ *satisfies the following properties.*

(i) The restriction of the metric $\|\cdot\|$ *to* $E^0(N)$ *is smooth. Moreover, the divisor of the restriction of* $\theta^8_{1,1}$ *to* $E^0(N)$ *is* $8H$*. Therefore, there is a unique isomorphism* $\alpha\colon L_{4,4,N} \to \mathcal{O}(C)\,|_{E^0(N)}$ *that sends* $\theta^8_{1,1}$ *to* s*.*

(ii) Each point p belonging to only one component $\Theta_{j,\nu}$ has a neighborhood V on which

$$\log\|\theta_{1,1}^8\|^2 = \log\|s\|'^2 + \psi_1,$$

where ψ_1 is a pre-log-log along D^0.

(iii) On the affine coordinate chart $W_{j,\nu}^0$ defined on Section 2, we can write

$$\log\|\theta_{1,1}^8\|^2 = \log\|s\|'^2 + \psi_2 - \frac{4}{N}\frac{\log(u_\nu\bar{u}_\nu)\log(v_\nu\bar{v}_\nu)}{\log(u_\nu\bar{u}_\nu)+\log(v_\nu\bar{v}_\nu)},$$

where ψ_2 is pre-log-log along D.

Consequently, if we denote also by $\|\cdot\|$ the singular metric on $\mathcal{O}(C)$ induced by α and $\|\cdot\|$, then the 5-tuple $(1,\mathcal{O}(C),\Sigma,\|\cdot\|,\alpha)$ is a Mumford-Lear extension of the hermitian line bundle $\overline{L}$ to $E(N)$ and the divisor of $\theta_{1,1}^8$ on the universal elliptic surface $E(N)$ is given by

$$\operatorname{div}_{E(N)}(\theta_{1,1}^8) = C.$$

Proof. The metric $\|\cdot\|$ on $L_{4,4,N}$ over the open subset $E^0(N)$ is induced by a smooth metric on the trivial line bundle over $\mathbb{H}\times\mathbb{C}$. Since $N\geq 3$, the map $\mathbb{H}\times\mathbb{C}\to E^0(N)$ is étale. Hence, the metric $\|\cdot\|$ is smooth on $E^0(N)$. Therefore, the components of $\operatorname{div}_{E(N)}(\theta_{1,1}^8)$ that meet the open subset $E^0(N)$ come from the theta function. Is well known that, for fixed $\tau\in\mathbb{H}$, the zeros of the Riemann theta function $\theta_{1,1}(\tau,z)$ are located at $z\in\mathbb{Z}\tau\oplus\mathbb{Z}$; all the zeros are simple. This proves that the restriction of $\operatorname{div}_{E(N)}(\theta_{1,1}^8)$ to $E^0(N)$ is given by $8H$. This finishes the proof of (i).

By the normality of the group $\Gamma(N)$ in $\mathrm{SL}_2(\mathbb{Z})$, in order to prove that $\overline{L}$ admits a Mumford-Lear extension and compute the divisor $\operatorname{div}_{E(N)}(\theta_{1,1}^8)$, it suffices to work over the cusp $P_1=[\infty]$.

Consider the open affine chart $W=W_{1,\nu}^0$. By Lemma 2.12,

$$\begin{aligned}\log\left(\|\theta_{1,1}^8(\tau,z)\|^2\right)\Big|_{W_\nu^0} &= \log\left(|\theta_{1,1}^8(\tau,z)|^2\right)\Big|_W\\ &+\frac{4}{N}\left((\nu+1)^2\log(u_\nu\bar{u}_\nu)+\nu^2\log(v_\nu\bar{v}_\nu)-\frac{\log(u_\nu\bar{u}_\nu)\log(v_\nu\bar{v}_\nu)}{\log(u_\nu\bar{u}_\nu)+\log(v_\nu\bar{v}_\nu)}\right)\\ &+4\log\left(-\frac{N}{4\pi}\big(\log(u_\nu\bar{u}_\nu)+\log(v_\nu\bar{v}_\nu)\big)\right).\end{aligned}$$

We first study the term $\log|\theta_{1,1}^8|^2$. For this, we rewrite expression (2.6) defining $\theta_{1,1}$ in terms of the local coordinates u_ν, v_ν. Using formulas (2.1), we obtain

$$\begin{aligned}\theta_{1,1}(\tau,z) &= \sum_{n\in\mathbb{Z}} \mathrm{e}^{\pi i(n+1/2)} q_N^{N/2(n+1/2)^2}\zeta^{n+1/2)}\\ &= \sum_{n\in\mathbb{Z}} \mathrm{e}^{\pi(n+1/2)} u_\nu^{N/2(n+1/2)^2+(\nu+1)(n+1/2)} v_\nu^{N/2(n+1/2)^2+\nu(n+1/2)}.\end{aligned}$$

Since the vertical component $\Theta_{1,\nu}$ is characterized by the equation $v_\nu = 0$, the multiplicity of $\theta_{1,1}$ along $\Theta_{1,\nu}$ is given by

$$\min_{n\in\mathbb{Z}}\left(\frac{N}{2}n^2+\left(\frac{N}{2}+\nu\right)n+\frac{N}{8}+\frac{\nu}{2}\right).$$

For a real number x we write $\lfloor x\rfloor$ for the bigger integer smaller or equal to x and $\epsilon(x)=x-\lfloor x\rfloor$. Then, one easily checks that

$$\min_{n\in\mathbb{Z}}\left(\frac{N}{2}n^2+\left(\frac{N}{2}+\nu\right)n+\frac{N}{8}+\frac{\nu}{2}\right) \\ =\frac{N}{2}\left(\epsilon^2\left(-\frac{\nu}{N}\right)-\epsilon\left(-\frac{\nu}{N}\right)\right)+\frac{N}{8}-\frac{\nu^2}{2N}.$$

Note that this quantity depends on the value of ν and not just on the residue class of ν modulo N. This is because $\theta_{1,1}(\tau,z)$ is a multi-valued function on $E^0(N)$.

Similarly, the multiplicity of $\theta_{1,1}$ along $\Theta_{1,\nu+1}$ is given by

$$\frac{N}{2}\left(\epsilon^2\left(-\frac{\nu+1}{N}\right)-\epsilon\left(-\frac{\nu+1}{N}\right)\right)+\frac{N}{8}-\frac{(\nu+1)^2}{2N}.$$

Therefore, on $W\setminus H$, we can write

$$\log|\theta_{1,1}^8|^2= \\ \left(4N\left(\epsilon^2\left(-\frac{\nu+1}{N}\right)-\epsilon\left(-\frac{\nu+1}{N}\right)\right)+N-\frac{4(\nu+1)^2}{N}\right)\log u_\nu\overline{u}_\nu+ \\ \left(4N\left(\epsilon^2\left(-\frac{\nu}{N}\right)-\epsilon\left(-\frac{\nu}{N}\right)\right)+N-\frac{4\nu^2}{N}\right)\log v_\nu\overline{v}_\nu+\psi_3,$$

where ψ_3 is a smooth function.

We next consider the remaining terms of the expresion of $\log\|\theta_{1,1}^8\|^2$. The term

$$4\log\Big(-\frac{N}{4\pi}\big(\log(u_\nu\bar{u}_\nu)+\log(v_\nu\bar{v}_\nu)\big)\Big)$$

is pre-log-log along D.

The terms $\frac{4\nu^2}{N}\log(v_\nu\bar{v}_\nu)$ and $\frac{4(\nu+1)^2}{N}\log(u_\nu\bar{u}_\nu)$ add $4\nu^2/N$ and $4(\nu+1)^2/N$ to the multiplicity of the components $\Theta_{1,\nu}$ and $\Theta_{1,\nu+1}$ respectively. Summing up, we obtain that

$$\log\left(\|\theta_{1,1}^8(\tau,z)\|^2\right)\Big|_{W\setminus H}= \\ \left(4N\left(\epsilon^2\left(-\frac{\nu}{N}\right)-\epsilon\left(-\frac{\nu}{N}\right)\right)+N\right)\log v_\nu\overline{v}_\nu+ \\ \left(4N\left(\epsilon^2\left(-\frac{\nu+1}{N}\right)-\epsilon\left(-\frac{\nu+1}{N}\right)\right)+N\right)\log u_\nu\overline{u}_\nu- \\ \frac{4}{N}\left(\frac{\log(u_\nu\bar{u}_\nu)\log(v_\nu\bar{v}_\nu)}{\log(u_\nu\bar{u}_\nu)+\log(v_\nu\bar{v}_\nu)}\right)+\psi_2,$$

where ψ_2 is pre-log-log along D.

In order to finish the proof of (iii), it only remains to observe that, for $\nu = 0, \dots, N$,

$$4N\left(\epsilon^2\left(-\frac{\nu}{N}\right) - \epsilon\left(-\frac{\nu}{N}\right)\right) = \frac{4\nu^2}{N} - 4\nu.$$

Statement (ii) follows from (iii) and the fact that, by Proposition 4.1 (i), the term

$$\frac{4}{N}\left(- \frac{\log(u_\nu \bar{u}_\nu)\log(v_\nu \bar{v}_\nu)}{\log(u_\nu \bar{u}_\nu) + \log(v_\nu \bar{v}_\nu)}\right)$$

is pre-log-log along D^0. □

The self-intersection of C. By Proposition 4.9, the Mumford-Lear extension of $\overline{L}$ to $E(N)$ is isomorphic to $\mathcal{O}(C)$. We next compute the self-intersection $C \cdot C$.

Proposition 4.10. *The self-intersection product $C \cdot C$ is given by*

$$C \cdot C = \frac{16(N^2+1)p_N}{3N}.$$

In particular, for $N = 4$, we have $p_4 = 6$, hence $C \cdot C = 136$.

Proof. Using the adjunction formula (see for instance [17] proof of Proposition 3.2), we obtain

$$H \cdot H = -\frac{Np_N}{12}.$$

Moreover,

$$H \cdot \Theta_{j,\nu} = \begin{cases} 1, & \text{if } \nu = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\Theta_{j,\nu} \cdot \Theta_{j',\nu'} = \begin{cases} -2, & \text{if } j = j',\ \nu = \nu' \\ 1, & \text{if } j = j',\ \nu \equiv \nu' \pm 1 \mod N \\ 0, & \text{otherwise.} \end{cases}$$

From these intersection products and the explicit description of C in (4.8), we derive the result. □

The b-divisor of the Jacobi line bundle.

Theorem 4.11. *The line bundle $\overline{L} = (L_{4,4,N}, \|\cdot\|)$ admits all Mumford-Lear extensions over $E(N)$. Moreover, the associated b-divisor is integrable and the equality*

$$\text{b-div}(\theta_{1,1}^8) \cdot \text{b-div}(\theta_{1,1}^8) = \frac{16Np_N}{3}$$

holds.

Proof. Recall that $\Sigma \subset D$ denotes the set of double points of D. By Proposition 4.9 (ii), the restriction $\overline{L}|_{E(N)\setminus\Sigma}$ has a pre-log metric along $D \setminus \Sigma$. Therefore, if $p \notin \Sigma$ and $\varphi \colon X \to E(N)$ is the blow-up of $E(N)$ at p, we deduce that $\varphi^*\mathcal{O}(C)$ is a Mumford-Lear extension of $\overline{L}$ and that

$$\operatorname{div}_X(\theta_{1,1}^8) = \varphi^* \operatorname{div}_{E(N)}(\theta_{1,1}^8).$$

Assume now that $p \in \Sigma$ and $\varphi \colon X \to E(N)$ is the blow-up of $E(N)$ at p. Write Σ_X for the set of double points of the total transform of D. Then, $\#\Sigma_X = \#\Sigma + 1$, because we can write $\Sigma_X = (\Sigma \setminus \{p\}) \cup \{p_1, p_2\}$, where $\{p_1, p_2\}$ is the intersection of the exceptional divisor E of the blow-up with the strict transform of D.

Proposition 4.1 (iii) and Proposition 4.9 (iii) imply that

$$(N, \varphi^*\mathcal{O}(NC) \otimes \mathcal{O}(-2E))$$

is a Mumford-Lear extension of $\overline{L}$ to X (in this case the 2-codimensional set is Σ_X, and the isomorphism and the metric are the ones induced by α and $\|\cdot\|$). Moreover,

$$\operatorname{div}_X(\theta_{1,1}^8) = \varphi^* \operatorname{div}_{E(N)}(\theta_{1,1}^8) - \frac{2}{N}E.$$

A similar phenomenon occurs on any smooth surface birational to $E(N)$. To describe it we need a little of terminology. Let $\varphi \colon X \to E(N)$ be a proper birational morphism with X smooth. A point $p \in X$ will be called mild if the metric of $\varphi^*\overline{L}$ is smooth or pre-log in a neighborhood of p. Put $\Sigma_X \subset X$ for the set of non-mild points. Let n, m be positive integers with $\gcd(n, m) = 1$. We will say that a point p has type (n, m) and multiplicity μ if there is a coordinate neighborhood centered at p, with coordinates (u, v) such that

$$\log \|\theta_{1,1}^8\| = \log \|s\|' + \psi - \frac{\mu}{nm} \frac{\log(u\bar{u}) \log(v\bar{v})}{n \log(u\bar{u}) + m \log(v\bar{v})},$$

where ψ is a smooth function on the pre-image of $E^0(N)$ with log-log growth along the pre-image of $E(N) \setminus E^0(N)$. Observe that $E(N)$ has Np_N non-mild points, all of type $(1, 1)$ and multiplicity $4/N$.

Assume that Σ_X is finite and that $\overline{L}$ admits a Mumford-Lear extension $(e_X, \mathcal{O}(C_X), \Sigma_X, \|\cdot\|, \alpha)$ to X. Let D_X be the total transform of D to X. If $\varphi \colon X' \to X$ is the blow-up at a mild point $p \notin \Sigma_X$, then $\Sigma_{X'} = \varphi^{-1}\Sigma_X$ is finite and $(e_X, \varphi^*\mathcal{O}(C_X))$ is a Mumford-Lear extension of $\overline{L}$ to X'. In particular,

$$\operatorname{div}_{X'}(\theta_{1,1}^8) = \varphi'^* \operatorname{div}_X(\theta_{1,1}^8).$$

Let now $\varphi \colon X' \to X$ be the blow-up of X at a point $p \in \Sigma_X$, with type (n, m) and multiplicity a/b, with a, b integers. Then, by Proposition 4.1(iii), $\Sigma_{X'} = (\Sigma_X \setminus \{p\}) \cup \{p_1, p_2\}$, where $\{p_1, p_2\}$ is the intersection of the exceptional divisor $E_{X'}$ of the blow-up with the strict transform of D_X. Moreover,

$$(bnm(n+m)e_X, \varphi^*\mathcal{O}(bnm(n+m)C_X) \otimes (-aE_{X'}))$$

is a Mumford-Lear extension of $\overline{L}$ to X'. Hence,

$$\operatorname{div}_{X'}(\theta^8_{1,1}) = \varphi^* \operatorname{div}_X(\theta^8_{1,1}) - \frac{a}{bnm(n+m)} E_{X'}.$$

Note also that the singular point p gives rise to two points in $\Sigma_{X'}$, both of multiplicity a/b, one of type $(n+m, m)$ and the other of type $(n, n+m)$. Since the self-intersection of the exceptional divisor $E_{X'}$ is -1, we deduce that

$$\operatorname{div}_{X'}(\theta^8_{1,1})^2 = \varphi'^* \operatorname{div}_X(\theta^8_{1,1})^2 - \frac{a^2}{b^2n^2m^2(n+m)^2}. \tag{4.12}$$

Since the elements of $\mathcal{BIR}'(E(N))$ can be obtained by successive blow-ups at points, for all $X \in \mathcal{BIR}'(E(N))$, the set Σ_X is finite and $\overline{L}$ admits a Mumford-Lear extensions to X. Hence, $\overline{L}$ admits all Mumford-Lear extensions over $E(N)$.

From the previous discussion, it is clear that, to study the b-divisor b-div$(\theta^8_{1,1})$, we can forget the blow-ups at mild points and concentrate on blow-ups along non-mild points.

Consider the labeled binary tree with root labeled by $(1,1)$ and such that, if a node is labeled (n,m), the two child nodes are labeled $(n+m, m)$ and $(n, n+m)$. Then, the labels of the tree are in bijection with the set of ordered pairs of coprime positive integers. This tree also describes the type of the non-mild points that appear by successive blow-ups starting with a point of type $(1,1)$.

By equation (4.12) and this description of the singular points that appear in the tower of blow-ups, we deduce that the b-divisor b-div$(\theta^8_{1,1})$ is integrable if and only if the series

$$\sum_{\substack{n>0,\, m>0 \\ (n,m)=1}} \frac{1}{n^2m^2(n+m)^2}$$

is absolutely convergent. Since this is the case, we conclude that the b-divisor b-div$(\theta^8_{1,1})$ is integrable. Moreover, since $X(N)$ has p_N cusps and over each cusp $E(N)$ has N points of type $(1,1)$ and multiplicity $4/N$, we deduce from equation (4.12)

$$\text{b-div}(\theta^8_{1,1})^2 = C \cdot C - \frac{4^2Np_N}{N^2} \sum_{\substack{n>0,\, m>0 \\ (n,m)=1}} \frac{1}{n^2m^2(n+m)^2}.$$

Now we compute

$$\sum_{\substack{n>0,\, m>0\\ (n,m)=1}} \frac{1}{n^2m^2(n+m)^2} = \frac{\sum_{n>0,\, m>0} \frac{1}{n^2m^2(n+m)^2}}{\sum_{k>0} \frac{1}{k^6}}$$

$$= \frac{\zeta(2,2;2)}{\zeta(6)} = \frac{\frac{1}{3}\zeta(6)}{\zeta(6)} = \frac{1}{3},$$

where $\zeta(2,2;2)$ is the special value of the Tornheim zeta function that is computed in [25].

Therefore,

$$\text{b-div}(\theta_{1,1}^8)^2 = C \cdot C - \frac{16p_N}{3N} = \frac{16(N^2+1)}{3N}p_N - \frac{16}{3N}p_N = \frac{16Np_N}{3}$$

concluding the proof of the theorem. □

Remark 4.13. We can rewrite the formula in Theorem 4.11 as

$$\text{b-div}(\theta_{1,1}^8)^2 = 4 \cdot 4 \cdot \frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)]}{2} \frac{\zeta(2,2;2)}{\zeta(6)}.$$

Thus, this degree can be interpreted as the product of the weight of the Jacobi form, its index, the degree of the map $X(N) \to X(1)$ and a quotient of zeta values.

5. Interpretation and open questions

In the previous section we have seen that, when taking into account the invariant metric, the natural way to extend the Cartier divisor $\text{div}(\theta_{1,1}^8)$ associated to a section of the Jacobi line bundle, from the universal family of elliptic curves to a compactification of it, is not as a Cartier divisor, but as a $\mathbb{Q}$-b-divisor. In particular, this implies that we can not restrict ourselves to a single toroidal compactification, but we have to consider the whole tower of toroidal compactifications. Considering purely the arithmetic definition of Jacobi forms, this fact was already observed by the third author [19, Remark 2.19].

In this section we will give further evidence that $\text{b-div}(\theta_{1,1}^8)$ is the natural extension of $\text{div}(\theta_{1,1}^8)$ by showing that if satisfies direct generalizations of classical theorems on hermitial line bundles. We will also state some open problems and future lines of research.

A Hilbert-Samuel formula. First, we observe that $\text{b-div}(\theta_{1,1}^8)^2$ satisfies a Hilbert-Samuel type formula.

Theorem 5.1. *For each $N \geq 3$, the equality*

$$\text{b-div}(\theta_{1,1}^8)^2 = \lim_{\ell\to\infty} \frac{\dim J_{4\ell,4\ell}(\Gamma(N))}{\ell^2/2!}$$

holds.

Proof. By Remark 2.8 and Theorem 4.11 we have

$$\lim_{\ell\to\infty} \frac{\dim J_{4\ell,4\ell}\big(\Gamma(N)\big)}{\ell^2/2!} = \lim_{\ell\to\infty} \frac{\frac{8Np_N}{3}\ell^2 + o\big(\ell^2\big)}{\ell^2/2!} = \frac{16Np_N}{3} = \text{b-div}(\theta_{1,1}^8)^2.$$

□

Chern-Weil theory. The second task is to show that the self-intersection product in the sense of b-divisors is compatible with Chern-Weil theory. Let X be a proper complex variety, $U \subset X$ a dense open subset, L a line bundle on U, and $\|\cdot\|$ a smooth metric on L. Then, we set

$$c_1(L, \|\cdot\|) = \frac{1}{2\pi i}\partial\bar{\partial}\log\|s\|^2,$$

where $s \neq 0$ is a rational section of L. Note that $c_1(L, \|\cdot\|)$ does not depend on the section s. Moreover, it is a smooth form on U, but does not necessarily extend to the whole of X. Assume that L is the restriction to U of a line bundle L' on X such that the metric $\|\cdot\|$ is singular along $X \setminus U$. Due to the presence of singularities, even if the form $c_1(L, \|\cdot\|)$ is locally integrable on X, the cohomolgy class of the current $[c_1(L, \|\cdot\|)]$ does not need to represent the class of the line bundle L'.

In particular, let us write

$$c_1(L_{4,4,N}, \|\cdot\|) = \frac{1}{2\pi i}\partial\bar{\partial}\log\|\theta_{1,1}^8\|^2.$$

The objective of this paragraph is to make precise the idea that the form $c_1(L_{4,4,N}, \|\cdot\|)$ does not represent the first Chern class of a particular Mumford-Lear extension of $L_{4,4,n}$, but rather the class of the b-divisor $\text{b-div}(\theta_{1,1}^8)$.

Theorem 5.2. *For each $N \geq 3$, the equality*

$$\text{b-div}(\theta_{1,1}^8)^2 = \int_{E^0(N)} c_1(L_{4,4,N}, \|\cdot\|)^{\wedge 2} \tag{5.3}$$

holds.

Proof. First, observe that the integral on the right-hand side is an improper integral, but, by propositions 4.1 and 4.9, we know that it is well defined and finite.

We now give two proofs that both terms of equation (5.3) agree. Both methods consist in computing the integral on the right-hand side. In the first method, we replace the singular metric with another singular metric for which classical Chern-Weil theory applies and hence the corresponding integral is given by intersection theory and then we use the theory of residues to compare both integrals. The second method is based on computing the integral on the right-hand side explicitly on a fundamental domain.

Let C be the divisor on Proposition 4.9. Then, $\mathcal{O}(C)$ is a Mumford-Lear extension of $L_{4,4,N}$ to $E(X)$. By Proposition 4.9 (iii) there exists a pre-log metric $\|\cdot\|'$ on the line bundle $\mathcal{O}(C)$ such that each double point $p_{j,\nu} = \Theta_{j,\nu} \cap \Theta_{j,\nu+1} \in W^0_{j,\nu}$ has a neighborhood in which

$$\log \|\theta^8_{1,1}\|^2 = \log \|\theta^8_{1,1}\|'^2 - \frac{4}{N} \frac{\log(u_\nu \bar{u}_\nu) \log(v_\nu \bar{v}_\nu)}{\log(u_\nu \bar{u}_\nu) + \log(v_\nu \bar{v}_\nu)}. \tag{5.4}$$

Write $\omega = c_1(L_{4,4,N}, \|\cdot\|)$, $\omega' = c_1(\mathcal{O}(C), \|\cdot\|')$ and

$$f = \log \|\theta^8_{1,1}\|^2 - \log \|\theta^8_{1,1}\|'^2.$$

Thus,

$$\omega = \omega' + \frac{1}{2\pi i} \partial \bar{\partial} f.$$

Since Chern-Weil theory can be extended to pre-log singularities ([22], [7]), the equality

$$\int_{E(N)} \omega'^{\wedge 2} = C \cdot C$$

holds. Since

$$\int_{E(N)} \omega^{\wedge 2} = \int_{E(N)} \omega'^{\wedge 2} - \int_{E(N)} \mathrm{d} \left(\frac{2}{2\pi i} \partial f \wedge \omega' + \frac{1}{(2\pi i)^2} \partial f \wedge \partial \bar{\partial} f \right),$$

we are led to compute the second integral of the right-hand side of the previous equation. Note that the minus sign in the above formula comes from the fact that $\mathrm{d}\, \partial = -\partial \bar{\partial}$. Since pre-log-log forms have no residues, in order to compute this integral we can focus on the double points $p_{j,\nu}$, $j = 1, \dots, p_N$, $\nu = 0, \dots, N-1$ of D. For each point $p_{j,\nu}$ and $0 < \varepsilon < 1/e$, let $V_{j,\nu,\varepsilon}$ be the polycylinder

$$V_{j,\nu,\varepsilon} = \{(u_\nu, v_\nu) \in W^0_{j,\nu} \mid |u_\nu| \leq \varepsilon,\ |v_\nu| \leq \varepsilon\}.$$

Then, by Stokes theorem,

$$- \int_{E(N)} \mathrm{d} \left(\frac{2}{2\pi i} \partial f \wedge \omega' + \frac{1}{(2\pi i)^2} \partial f \wedge \partial \bar{\partial} f \right) = \sum_{j=1}^{p_N} \sum_{\nu=0}^{N-1} \lim_{\varepsilon \to 0} \int_{\partial V_{j,\nu,\varepsilon}} \frac{2}{2\pi i} \partial f \wedge \omega' + \frac{1}{(2\pi i)^2} \partial f \wedge \partial \bar{\partial} f.$$

Using that ω' is a pre-log-log form and equation (4.2), it is easy to see that

$$\lim_{\varepsilon \to 0} \int_{\partial V_{j,\nu,\varepsilon}} \frac{2}{2\pi i} \partial f \wedge \omega' = 0.$$

As a shorthand, write (u, v) for the coordinates (u_ν, v_ν) of $W^0_{j,\nu}$. We decompose the boundary of $V_{j,\nu,\varepsilon}$ as $\partial V_{j,\nu,\varepsilon} = A_\varepsilon \cup B_\varepsilon$, where

$$A_\varepsilon = \{(u, v) \in W^0_{j,\nu} \mid |u| \leq \varepsilon,\ |v| = \varepsilon\},$$
$$B_\varepsilon = \{(u, v) \in W^0_{j,\nu} \mid |u| = \varepsilon,\ |v| \leq \varepsilon\}.$$

Using equations (5.4) and (4.4) and taking care of the canonical orientation of a complex manifold, we see that

$$\int_{A_\varepsilon} \frac{1}{(2\pi i)^2}\partial f \wedge \partial\bar{\partial} f = \frac{16}{N^2}\int_0^\varepsilon \frac{2(\log(\varepsilon^2))^2\log(r^2)2r\,\mathrm{d}\,r}{(\log(r^2)+\log(\varepsilon^2))^4 r^2} = -\frac{16}{6N^2}.$$

Similarly,

$$\int_{B_\varepsilon} \frac{1}{(2\pi i)^2}\partial f \wedge \partial\bar{\partial} f = \frac{16}{N^2}\int_0^\varepsilon \frac{2(\log(\varepsilon^2))^2\log(r^2)2r\,\mathrm{d}\,r}{(\log(r^2)+\log(\varepsilon^2))^4 r^2} = -\frac{16}{6N^2}.$$

Hence,

$$\lim_{\varepsilon\to 0}\int_{\partial V_{j,\nu,\varepsilon}} \left(\frac{2}{2\pi i}\partial f\wedge\omega' + \frac{1}{(2\pi i)^2}\partial f \wedge \partial\bar{\partial} f\right) = -\frac{16}{3N^2}.$$

Therefore,

$$\int_{E^0(N)} c_1(L_{4,4,N}, \|\cdot\|)^{\wedge 2} = C\cdot C - \frac{16p_N}{3N} = \text{b-div}(\theta^8_{1,1})^2.$$

We next give a second proof of this equality. Let (τ, z) be the usual coordinate of $\mathbb{H}\times\mathbb{C}$, and write $\tau = \xi + i\eta$ and $z = x + iy$. Let $p\colon \mathbb{H}\times\mathbb{C}\to E^0(N)$ be the quotient map. By the explicit description of the translation invariant metric of Definition 2.10, a direct computation yields

$$p^* c_1(L_{4,4,N}, \|\cdot\|)^{\wedge 2} = \frac{16}{\pi}\frac{\mathrm{d}\,x\wedge\mathrm{d}\,y}{\eta}\wedge\frac{\mathrm{d}\,\xi\wedge\mathrm{d}\,\eta}{\eta^2}.$$

Thus,

$$\int_{E^0(N)} c_1(L_{4,4,N}, \|\cdot\|)^{\wedge 2} = \frac{[\mathrm{SL}_2(\mathbb{Z}):\Gamma(N)]}{2}\int_K \frac{16}{\pi}\frac{\mathrm{d}\,x\wedge\mathrm{d}\,y}{\eta}\wedge\frac{\mathrm{d}\,\xi\wedge\mathrm{d}\,\eta}{\eta^2},$$

where $K\subset\mathbb{H}\times\mathbb{C}$ is the subset containing the pairs (τ, z) with $|\Re\tau|\le 1/2$, $|\tau|>1$ and z in the parallelogram with vertices $0, 1, \tau, 1+\tau$. Observe that the pre-factor $[\mathrm{SL}_2(\mathbb{Z}):\Gamma(N)]/2$ is the degree of the map $X(N)\to X(1)$.

The above integral is easily done, in fact it is a classical integral, and gives

$$\int_{E^0(N)} c_1(L_{4,4,N}, \|\cdot\|)^{\wedge 2} = \frac{[\mathrm{SL}_2(\mathbb{Z}):\Gamma(N)]}{2}\frac{16}{\pi}\frac{\pi}{3} = \frac{16Np_N}{3}.$$

□

Remark 5.5. Recall the function

$$f_{1,1}(x,y) = \frac{\log(x\overline{x})\log(y\overline{y})}{\log(x\overline{x})+\log(y\overline{y})}.$$

The heart of the first proof of Theorem 5.2 is the relation

$$-\operatorname{Res}_{(0,0)}\left(\frac{1}{(2\pi i)^2}\partial f_{1,1}\wedge\partial\bar{\partial} f_{1,1}\right) = \frac{1}{3} = \sum_{\substack{n>0,\, m>0\\ (n,m)=1}} \frac{1}{n^2 m^2 (n+m)^2}$$

between the residue at $(0,0)$ of the differential form $\frac{1}{(2\pi i)^2}\partial f_{1,1} \wedge \partial\bar{\partial} f_{1,1}$ and the harmonic double value $\zeta(2,2;2)/\zeta(6)$. This gives us a geometric interpretation of this harmonic double value.

Note also that the second proof relates the value of $\text{b-div}(\theta_{1,1}^8)^2$ with the volume of the modular curve and hence with the zeta value

$$\zeta(-1) = -\frac{1}{4\pi}\int_{\mathcal{F}} \frac{\mathrm{d}\,\xi \wedge \mathrm{d}\,\eta}{\eta^2},$$

where $\mathcal{F}$ is the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. In fact, the formula in Theorem 4.11 can be rewritten as

$$\text{b-div}(\theta_{1,1}^8)^2 = 4 \cdot 4 \cdot \frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)]}{2}(-4\zeta(-1)),$$

yielding the obvious formula

$$\zeta(2,2;2) = -4 \cdot \zeta(6) \cdot \zeta(-1).$$

Intersections with curves. Similarly, we also note that the intersection of $\text{b-div}(\theta_{1,1}^8)$ with a complete curve can also be computed using the differential form $c_1(L_{4,4,N}, \|\cdot\|)$.

To a proper curve C contained in $E(N)$, we associate the b-divisor that, on each $X \in \mathcal{BIR}'(E(N))$ consist on the strict transform of C on X. We will denote this divisor by $\text{b-div}(C)$. Note that this b-divisor is not integrable because, by taking successive blow-ups in points of C, the self-intersection of the strict transform of C can be made arbitrarily negative. Assume that C is irreducible and is not contained in $D = E(N) \setminus E^0(N)$. Then, the product $\text{b-div}(\theta_{1,1}^8) \cdot \text{b-div}(C)$ is well defined because after a finite number of blow-ups on the double points of D and of its total transforms, the strict transform of C will not meet any double point of the total transform of D.

Theorem 5.6. *The equality*

$$\text{b-div}(\theta_{1,1}^8) \cdot \text{b-div}(C) = \int_C c_1(L_{4,4,N}, \|\cdot\|)$$

holds.

Proof. Let $X \to E(N)$ be a birational morphism obtained by successive blow-ups on double points of D and of its total transforms and such that the strict transform of C in X, that we denote by C_X, does not meet any double point of the total transform of D to X. Then,

$$\text{b-div}(\theta_{1,1}^8) \cdot \text{b-div}(C) = \mathrm{div}_X(\theta_{1,1}^8) \cdot C_X.$$

Let $(e, \mathcal{L}, S, \alpha, \|\cdot\|)$ be a Mumford-Lear extension of $\overline{L} = (L_{4,4,N}, \|\cdot\|)$ to X. Denote by $s = \alpha(\theta_{1,1}^{8e})$ the rational section of $\mathcal{L}$ determined by $\theta_{1,1}^8$. Since the metric $\|\cdot\|$ is pre-log on $X \setminus S$, we deduce that

$$\mathrm{div}_X(\theta_{1,1}^8) \cdot C_X = \frac{1}{e}\,\mathrm{div}(s) \cdot C_X = \frac{1}{e}\int_C c_1(\mathcal{L}, \|\cdot\|) = \int_C c_1(L_{4,4,N}, \|\cdot\|).$$

$\square$

A TORIC ANALOGUE OF THE SINGULAR METRIC. We now give an interpretation of the harmonic double value $\zeta(2,2;2)/\zeta(6)$ in terms of toric varieties and the volume of a convex surface.

Consider the projective plane $\mathbb{P}^2$ with projective coordinates $(x_0 : x_1 : x_2)$ and the canonical line bundle $\mathcal{O}(1)$. On this line bundle we can put the canonical metric given by

$$\|x_0\|_{\text{can}} = \frac{|x_0|}{\max(|x_0|, |x_1|, |x_2|)}.$$

This metric is continuous. We have an open immersion $(\mathbb{C}^*)^2 \hookrightarrow \mathbb{P}^2$ that sends the point (z_1, z_2) to $(1 : z_1 : z_2)$. We define the valuation map $\text{val}\colon (\mathbb{C}^*)^2 \to \mathbb{R}^2$ by

$$\text{val}(z_1, z_2) = (-\log|z_1|, -\log|z_2|)$$

The function $\log(\|x_0\|_{\text{can}})$ is constant along the fibers of val. Thus, there exist a function $\Psi_{\text{can}}\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$\log\|x_0(p)\|_{\text{can}} = \Psi_{\text{can}}(\text{val}(p)).$$

This function is explicitly given by

$$\Psi_{\text{can}}(u, v) = \min(0, u, v).$$

The projective plane $\mathbb{P}^2$ is a toric variety with the action of $(\mathbb{C}^*)^2$ given by

$$(\lambda, \mu)(x_0 : x_1 : x_2) = (x_0 : \lambda x_1 : \mu x_2).$$

The theory of toric varieties tells us that the polytope associated to $\text{div}(x_0)$ is the stability set of Ψ_{can}:

$$\begin{aligned}\Delta = \{x \in (\mathbb{R}^2)^\vee \mid x(u,v) \geq \Psi_{\text{can}}(u,v),\ \forall (u,v) \in \mathbb{R}^2\}\\ = \text{conv}((0,0),(1,0),(0,1)).\end{aligned}$$

Moreover,

$$\text{div}(x_0)^2 = 2\,\text{Vol}(\Delta)) = 1,$$

where the volume is computed with respect to the Haar measure that gives $\mathbb{Z}^2$ covolume 1.

Now we want to modify the canonical metric to introduce a singularity of the same type as the singularity of the translation invariant metric on the Jacobi line bundle at the double points. We define the metric $\|\cdot\|_{\text{sing}}$ by

$$\log\|x_0\|_{\text{sing}} = \begin{cases} -\frac{\log(|x_1/x_0|)\log(|x_2/x_0|)}{\log(|x_1/x_0|)+\log(|x_2/x_0|)} & \text{if } |x_0| \geq \max(|x_1|,|x_2|), \\ -\max(\log(|x_1/x_0|), \log(|x_2/x_0|)) & \text{otherwise.}\end{cases}$$

As before, the function $\log \|x_0\|_{\mathrm{sing}}$ is constant along the fibers of val and defines a function $\Psi_{\mathrm{sing}}\colon \mathbb{R}^2 \to \mathbb{R}$ that is given explicitly by

$$\Psi_{\mathrm{sing}}(u,v) = \begin{cases} \frac{uv}{u+v}, & \text{if } u,v \geq 0, \\ u, & \text{if } u \leq \min(0,v), \\ v, & \text{if } v \leq \min(0,u). \end{cases}$$

The function Ψ_{sing} is conic but is not piecewise linear. Assume that we can extend the theory of toric varieties to toric b-divisors. Then, to Ψ_{sing} we would associate the convex figure

$$\Delta_{\mathrm{sing}} = \{x \in (\mathbb{R}^2)^\vee \mid x(u,v) \geq \Psi_{\mathrm{sing}}(u,v),\ \forall (u,v) \in \mathbb{R}^2\},$$

and we should obtain

$$\text{b-div}(x_0, \|\cdot\|_{\mathrm{sing}})^2 = 2\operatorname{Vol}(\Delta_{\mathrm{sing}}). \tag{5.7}$$

We see that this is indeed the case.

Theorem 5.8. *The equation* (5.7) *holds.*

Proof. Arguing as in the proof of Theorem 4.11, we see that

$$\operatorname{div}(x_0)^2 - \text{b-div}(x_0, \|\cdot\|_{\mathrm{sing}})^2 = \sum_{\substack{n>0,\, m>0 \\ (n,m)=1}} \frac{1}{n^2 m^2 (n+m)^2} = \frac{1}{3}.$$

The stability set Δ_{sing} can be explicitly computed, and is given by

$$\Delta_{\mathrm{sing}} = \{(x,y) \in (\mathbb{R}^2)^\vee \mid x,y \geq 0,\ x+y \leq 1, \sqrt{x} + \sqrt{y} \geq 1.\}$$

Thus,

$$2\operatorname{Vol}(\Delta) - 2\operatorname{Vol}(\Delta_{\mathrm{sing}}) = 2\int_0^1 (1-\sqrt{x})^2 \,\mathrm{d}\,x = \frac{1}{3}.$$

□

Remark 5.9. In fact, since in the theory of toric varieties, the blow-ups have a explicit description in terms of fans, it is possible to interpret the equation $\zeta(2,2;2) = 1/3 \cdot \zeta(6)$ is terms of an infinite triangulation of $\Delta \setminus \Delta_{\mathrm{sing}}$.

Open questions. In this paper we have examined a particular example and observed, just by comparing numbers, that several classical results should be extendable to b-divisors and singular metrics with a shape similar to the one of the translation invariant metrics. We are in the process of investigating the following questions.

(1) Theorem 5.1 shows that the translation invariant metric encodes the asymptotic behavior of the space of Jacobi forms. It is possible to define global sections of a b-divisor. We can ask what is the exact relationship between the space of Jacobi forms and the global sections of the b-divisor $\operatorname{div}(\theta_{1,1}^8)$. Moreover,

once this is settled, we can ask whether there is a Riemann-Roch theorem or a Hilbert-Samuel theorem for b-divisors that imply directly Theorem 5.1.

(2) By Theorem 5.8, it is clear that much of the theory of toric varieties could be extended to toric b-divisors and singular metrics on toric varieties.

(3) Theorem 5.2 shows that Chern-Weil theory of singular metrics can be useful to study b-divisors. It would be interesting to generalize this theorem to higher dimensions. In this direction, with R. de Jong and D. Holmes, we have shown that the local integrability property extends, at least, to the case of toroidal compactifications of families of abelian varieties.

(4) The original motivation of this paper is to be able to define and study the height of cycles on the universal elliptic curve with respect to the Jacobi line bundle equipped with the translation invariant metric, extending the work in [18]. First, it is clear how to define the height of an algebraic point of $E^0(N)$ and one may wonder whether the new singularities are mild enough so that the Northcott property is still true. We can also define the height of an algebraic curve not contained in the divisor D. But it is not clear how to define the height of $E(N)$. The naive definition of that height would give the value $-\infty$, but it should be possible to extract a meaningful finite number. To this end, the study of toric varieties might be useful, because the techniques developed in [9] can be extended to the singular metrics of Theorem 5.8. In this case, we obtain that the stability set of the function associated to the metric is no longer a polytope but a convex set. In this case the regularized height should be defined from the integral along this convex set of the roof function, in analogy with [9, Theorem 5.2.5].

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