Effective sup-norm bounds on average for cusp forms of even weight

J.S. Friedman* J. Jorgenson† J. Kramer‡

January 14, 2019

Abstract

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind acting on the upper half-plane $\mathbb{H}$. Consider the d_{2k}-dimensional space of cusp forms $\mathcal{S}_{2k}^{\Gamma}$ of weight $2k$ for Γ, and let $\{f_1, \ldots, f_{d_{2k}}\}$ be an orthonormal basis of $\mathcal{S}_{2k}^{\Gamma}$ with respect to the Petersson inner product. In this paper we will give *effective* upper and lower bounds for the supremum of the quantity $S_{2k}^{\Gamma}(z) := \sum_{j=1}^{d_{2k}} |f_j(z)|^2 \, \mathrm{Im}(z)^{2k}$ as z ranges through $\mathbb{H}$.

1 Introduction

1.1 Statement of the main results

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind acting by fractional linear transformations on the upper half-plane $\mathbb{H}$, so that the quotient space $\Gamma\backslash\mathbb{H}$ has finite volume. For any integer $k \in \mathbb{N}_{\geq 1}$, we then consider the space $\mathcal{S}_{2k}^{\Gamma}$ of cusp forms of weight $2k$ for Γ, which is naturally equipped with the Petersson inner product. If d_{2k} denotes the dimension of the $\mathbb{C}$-vector space $\mathcal{S}_{2k}^{\Gamma}$, we let $\{f_1, \ldots, f_{d_{2k}}\}$ be an orthonormal basis of $\mathcal{S}_{2k}^{\Gamma}$. The purpose of this article is to determine *effective* upper and lower bounds for the supremum of the quantity

$$S_{2k}^{\Gamma}(z) := \sum_{j=1}^{d_{2k}} |f_j(z)|^2 \, \mathrm{Im}(z)^{2k} \tag{1.1}$$

as z ranges through $\mathbb{H}$. Optimal sup-norm bounds for the quantity (1.1) have been given in the case $k = 1$ in the articles [JK04] and [JK11], and for $k \geq 1$ in the paper [FJK16]. However, the sup-norm bounds obtained in these papers are not effective. The present article completes our previous investigations by now providing *effective* optimal sup-norm bounds for the quantity (1.1).

The main results of the paper are summarized in the following three theorems. When Γ is cocompact and torsionfree, we have the following result.

Theorem A. *Let Γ be cocompact and torsionfree, and let $k \in \mathbb{N}_{\geq 2}$. Then, the bounds*

$$\frac{2k-1}{4\pi} \leq \sup_{z \in \mathbb{H}} S_{2k}^{\Gamma}(z) \leq \frac{2k-1}{4\pi} + C_{\Gamma} \, e^{-\delta_{\Gamma} k}$$

*The views expressed in this article are the author's own and not those of the U.S. Merchant Marine Academy, the Maritime Administration, the Department of Transportation, or the United States government.

†The second named author acknowledges support from numerous PSC-CUNY grants.

‡The third named author acknowledges support from the DFG Graduate School *Berlin Mathematical School*.

hold, where the constants C_Γ and δ_Γ are effectively computable as

$$C_\Gamma = \frac{3\, e^{12\pi g_\Gamma/\ell_\Gamma}}{\pi(g_\Gamma - 1)} \frac{(\cosh(\ell_\Gamma)+1)^2}{\log((\cosh(\ell_\Gamma)+1)/2)} \quad \textit{and} \quad \delta_\Gamma = \frac{1}{2}\log\left(\frac{\cosh(\ell_\Gamma)+1}{2}\right);$$

here g_Γ and ℓ_Γ denote the genus and the length of the shortest closed geodesic on $\Gamma\backslash\mathbb{H}$, respectively.

In the general case, when Γ is cofinite, possibly with elliptic elements, we let $\mathcal{F}$ be a closed and connected fundamental domain for Γ. For $Y > 1$, we consider the neighborhoods $\mathcal{F}_j^Y$ of the j-th cusp of $\mathcal{F}$ ($j = 1, \ldots, h$), and we let $\mathcal{F}_Y$ denote the closure of the complement of the union of the cuspidal neighborhoods in $\mathcal{F}$, i.e., we have the decomposition

$$\mathcal{F} = \mathcal{F}_Y \cup \mathcal{F}_1^Y \cup \ldots \cup \mathcal{F}_h^Y .$$

We let $\mathcal{E} := \{e_1, \ldots, e_n\} \subset \mathcal{F}$ be the set of elliptic fixed points in $\mathcal{F}$ and denote the order of e_j by n_j ($j = 1, \ldots, n$). Then, we have the following result.

Theorem B. *Let Γ be cofinite, $k \in \mathbb{N}_{\geq 2}$, and $Y_0 \geq 8/\sqrt{15}$. Then, with the above notations, we have the following statements:*

(1) *For $Y > 1$, there exist effectively computable constants B_Y and σ_Y (depending on Y) such that the upper bound*

$$\sup_{z \in \mathcal{F}_Y} S_{2k}^\Gamma(z) \leq \frac{2k-1}{4\pi}\left(1 + 6\sum_{e_j \in \mathcal{E}}(n_j - 1)\right) + 12(2k-1)B_Y\,\sigma_Y^{-(k-2)}$$

holds.

(2) *For $2 \leq k \leq 4\pi Y_0$ and $Y \geq 2Y_0$, there exist effectively computable constants B_Y and σ_Y (depending on Y) such that the upper bounds*

$$\sup_{z \in \mathcal{F}_j^Y} S_{2k}^\Gamma(z) \leq \frac{2k-1}{4\pi}\left(1 + 6\sum_{e_j \in \mathcal{E}}(n_j - 1)\right) + 12(2k-1)B_Y\,\sigma_Y^{-(k-2)}$$

hold for $j = 1, \ldots, h$.

(3) *For $k > 4\pi Y_0$ and $Y = 2Y_0$, there exists an effectively computable constant B_{k,Y_0} (depending on k and Y_0) such that the upper bounds*

$$\sup_{z \in \mathcal{F}_j^Y} S_{2k}^\Gamma(z) \leq \frac{2k-1}{4\pi} + \frac{3(2k-1)}{2\pi}\left(B_{k,Y_0} + \frac{\sqrt{k}\,e^{5/4}}{\sqrt{\pi}}\right)$$

hold for $j = 1, \ldots, h$.

The constant σ_Y is given in Definition 3.3 and effectively bounded from below in Lemma 3.4. The constants B_Y and B_{k,Y_0} are given in Definition 3.5 and in Definition 4.3, respectively.

As an example, we provide in Subsection 5.4 explicit upper bounds for the supremum of the quantity $S_{2k}^\Gamma(z)$ in the case when Γ is the modular group. This example shows how the present investigations give rise to an algorithm to determine effective upper bounds for the supremum of the quantity $S_{2k}^\Gamma(z)$ for more general Fuchsian subgroups Γ.

Theorem C. *Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, $k \in \mathbb{N}$, and $Y = 16/\sqrt{15} = 4.131...$ Then, the upper bounds*

$$S_{2k}^\Gamma(z) \leq \begin{cases} \dfrac{31(2k-1)}{4\pi} + 1090\,(2k-1)1.014^{-(k-2)} & \textit{if } k \geq 2,\ z \in \mathcal{F}_Y, \\ \dfrac{31(2k-1)}{4\pi} + 1090\,(2k-1)1.014^{-(k-2)} & \textit{if } 2 \leq k \leq 25,\ z \in \mathcal{F}_1^Y, \\ \dfrac{2k-1}{4\pi} + \dfrac{9(2k-1)\sqrt{k}}{2\pi} & \textit{if } k \geq 26,\ z \in \mathcal{F}_1^Y, \end{cases}$$

hold.

In addition to the main results listed above, we also provide lower bounds for the supremum of the quantity $S_{2k}^\Gamma(z)$ when $k \in \mathbb{N}_{\geq 2}$. The corresponding bounds in the case $k = 1$ are discussed separately.

1.2 Results related to this paper

As mentioned above, the present article is the completion of our previous investigations [JK04], [JK11], and [FJK16] to determine sup-norm bounds for cusp forms on average. Our primary motivation for these studies originated from the article [Sil86], where the author determined the arithmetic degree of a modular parametrization of an elliptic curve defined over $\mathbb{Q}$ in terms of various quantities, including the Petersson norm of the cusp form of weight 2 associated to this parametrization. Following Silverman's article, the authors of [AU95] proved for the congruence subgroups $\Gamma = \Gamma_0(N)$ (N squarefree; $2, 3 \nmid N$) and $k = 1$ that for any $\varepsilon > 0$, one has the bound

$$\sup_{z \in \mathbb{H}} S_2^{\Gamma_0(N)}(z) = O(N^{2+\varepsilon}),$$

which was improved in [MU98] to $O(N^{1+\varepsilon})$. In [JK04], this bound was further improved by establishing a bound of the form $O(1)$, which holds uniformly for all subgroups Γ of finite index of a fixed Fuchsian subgroup Γ_0 of the first kind. The methodology of [JK04] was to study and employ the long-time asymptotic behavior of the heat kernel associated to the hyperbolic Laplacian acting on smooth functions on $\Gamma\backslash\mathbb{H}$; in [JK11] the main result of [JK04] was re-proved by relating it to special values of non-holomorphic elliptic, hyperbolic, as well as parabolic Eisenstein series.

Again a heat kernel approach was developed in [FJK16] in order to obtain bounds for the supremum of the quantity $S_{2k}^{\Gamma}(z)$ for Fuchsian subgroups Γ of the first kind and for $k \in \mathbb{N}_{\geq 1}$, ultimately leading to uniform sup-norm bounds with *ineffective* constants. In the present paper, we exploit knowledge of the resolvent kernel in order to obtain uniform sup-norm bounds with *effective* constants as stated in Theorem A and Theorem B. We mention here also results related to this paper obtained in [AMM16].

In a different direction, numerous authors have studied sup-norm bounds for individual holomorphic modular forms and non-holomorphic Maass forms. One of the main motivations for these investigations is the fact that a certain sup-norm bound for Maass forms implies the Lindelöf hypothesis for certain L-functions (see [Iwa02, p. 178]). We refer the reader to the articles [BH10], [HT13], [Tem15], and the references therein for some of the most recent results. As discussed in [FJK16], the results for sup-norm bounds on average and the results for bounds on individual sup-norms should be viewed as complementary since neither result implies the other.

Finally, we mention that effective sup-norm bounds of the type considered in this paper continue to prove to be useful in arithmetic geometry as, for example, the articles [BF14], [Jav14], [Jav16], or [JK14] show.

1.3 Outline of the paper

In the next section we setup the basic notation and recall the main results needed in the sequel of the paper. After providing a couple of technical lemmas, the main goal of the third section is to give upper bounds for certain Poincaré series, when z is ranging through the compact domain $\mathcal{F}_Y$. In the fourth section we give upper bounds for the Poincaré series under consideration, when z ranges through the cuspidal neighborhoods $\mathcal{F}_j^Y$. Based on the bounds established in the third and fourth section, the main results of the paper, in particular Theorems A, B, and C, are proven in the fifth section. The last section, presented as an appendix, collects various materials which support the understanding of the paper.

2 Preliminaries

In this section we setup the basic notation and recall the main results needed in the sequel of the paper.

Quotient spaces.

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind acting by fractional linear transformations on the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \,|\, z = x + iy\,,\, y > 0\}$. Let M be the quotient space $\Gamma\backslash\mathbb{H}$ and g_Γ the genus of M. In the sequel, we identify M with a fundamental domain $\mathcal{F} \subset \mathbb{H}$ for the group Γ, which we assume to be closed and connected. We denote the set of geodesic line segments which form the boundary $\partial\mathcal{F}$ of $\mathcal{F}$ by $\mathcal{S}$.

Denote by

$$\mathcal{P} = \{p_1, \ldots, p_h\}$$

the set of cusps of $\mathcal{F}$. Let $\sigma_j \in \mathrm{PSL}_2(\mathbb{R})$ be a scaling matrix of the cusp p_j, that is, $p_j = \sigma_j i\infty$ with stabilizer subgroup Γ_{p_j} described as

$$\sigma_j^{-1}\Gamma_{p_j}\sigma_j = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \qquad (j = 1, \ldots, h).$$

For $Y > 1$, we let $\mathcal{F}_j^Y \subset \mathcal{F}$ denote the neighborhood of the cusp p_j characterized by

$$\sigma_j^{-1}\mathcal{F}_j^Y = \{z = x + iy \in \mathbb{H} \,|\, -1/2 \leq x \leq 1/2,\, y \geq Y\} \qquad (j = 1, \ldots, h).$$

With these notations, we define $\mathcal{F}_Y$ to be the closure of the complement of the union $\mathcal{F}_1^Y \cup \ldots \cup \mathcal{F}_h^Y$ in $\mathcal{F}$, i.e.,

$$\mathcal{F}_Y := \mathrm{cl}\big(\mathcal{F} \setminus \big(\mathcal{F}_1^Y \cup \ldots \cup \mathcal{F}_h^Y\big)\big),$$

which is compact; we note that $\mathcal{F}_Y = \mathcal{F}$, if Γ is cocompact.

We note that by our choice of scaling matrices, Theorem 2.3.4 in [EGM98] shows that with the assumption $Y > 1$, the cuspidal neighborhoods $\mathcal{F}_j^Y$ are pairwise disjoint and that the boundary of each cuspidal neighborhood is contained in the boundary of the compact part $\mathcal{F}_Y$. We will thus assume that $Y > 1$ throughout the paper.

We choose $0 < m_Y < M_Y$ such that for all $z \in \mathcal{F}_Y$ the inequalities

$$m_Y \leq \mathrm{Im}(\sigma_j^{-1}z) \leq M_Y$$

hold for all $j = 1, \ldots, h$; we note that m_Y and M_Y depend on the choice of Y.

Denote by

$$\mathcal{E} = \{e_1, \ldots, e_n\}$$

the set of of elliptic fixed points of $\mathcal{F}$, let n_j denote the order of e_j, and let $\theta_j := 2\pi/n_j$ be the rotation angle of the corresponding primitive elliptic element ($j = 1, \ldots, n$). We put

$$\theta_\Gamma := \min_{j=1,\ldots,n} \theta_j;$$

note that $\theta_\Gamma > 0$.

Hyperbolic metric.

We denote by $\mathrm{d}s^2_{\mathrm{hyp}}(z)$ the line element and by $\mu_{\mathrm{hyp}}(z)$ the volume form corresponding to the hyperbolic metric on $\mathbb{H}$, which is compatible with the complex structure of $\mathbb{H}$ and has constant curvature equal to -1. Locally on $\mathbb{H} \setminus \Gamma\mathcal{E}$, we have

$$\mathrm{d}s^2_{\mathrm{hyp}}(z) = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} \quad \text{and} \quad \mu_{\mathrm{hyp}}(z) = \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}\,.$$

For $z, w \in \mathbb{H}$, we let $\mathrm{dist}_{\mathrm{hyp}}(z,w)$ denote the hyperbolic distance between these two points. For later purposes, it is useful to introduce the displacement function

$$\sigma(z,w) := \cosh^2\left(\frac{\mathrm{dist}_{\mathrm{hyp}}(z,w)}{2}\right) = \frac{|z-\bar{w}|^2}{4\,\mathrm{Im}(z)\mathrm{Im}(w)}\,. \tag{2.1}$$

We denote the hyperbolic length of the shortest closed geodesic on M by ℓ_Γ. Finally, for a domain $D \subset \mathbb{H}$, we denote its hyperbolic diameter by $\mathrm{diam}_{\mathrm{hyp}}(D)$ and its hyperbolic volume by $\mathrm{vol}_{\mathrm{hyp}}(D)$.

Cusp forms of higher weights.

For $k \in \mathbb{N}_{\geq 1}$, we let $\mathcal{S}_{2k}^{\Gamma}$ denote the space of cusp forms of weight $2k$ for Γ, i.e., the space of holomorphic functions $f\colon \mathbb{H} \longrightarrow \mathbb{C}$, which have the transformation behavior

$$f(\gamma z) = (cz+d)^{2k} f(z)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, and which vanish at all the cusps of M. The space $\mathcal{S}_{2k}^{\Gamma}$ is equipped with the Petersson inner product

$$\langle f_1, f_2 \rangle := \int\limits_M f_1(z)\overline{f_2(z)}\, y^{2k} \mu_{\mathrm{hyp}}(z) \qquad (f_1, f_2 \in \mathcal{S}_{2k}^{\Gamma}).$$

By letting $d_{2k} := \dim_{\mathbb{C}}(\mathcal{S}_{2k}^{\Gamma})$ and choosing an orthonormal basis $\{f_1, \ldots, f_{d_{2k}}\}$ of $\mathcal{S}_{2k}^{\Gamma}$, we define the quantity

$$S_{2k}^{\Gamma}(z) := \sum_{j=1}^{d_{2k}} |f_j(z)|^2\, y^{2k}.$$

We note that the quantity $S_{2k}^{\Gamma}(z)$ is invariant under the action of the Fuchsian subgroup Γ.

Maass forms of higher weights.

Following [Roe66], [Fay77], or [Fis87], we introduce for any $k \in \mathbb{N}_{\geq 1}$, the space $\mathcal{V}_k^{\Gamma}$ of functions $\varphi\colon \mathbb{H} \longrightarrow \mathbb{C}$, which have the transformation behavior

$$\varphi(\gamma z) = \left(\frac{cz+d}{c\bar{z}+d}\right)^k \varphi(z) = e^{2ik\arg(cz+d)}\varphi(z)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. For $\varphi \in \mathcal{V}_k^{\Gamma}$, we set

$$\|\varphi\|^2 := \int\limits_M |\varphi(z)|^2 \mu_{\mathrm{hyp}}(z),$$

whenever it is defined. We then introduce the Hilbert space

$$\mathcal{H}_k^{\Gamma} := \left\{\varphi \in \mathcal{V}_k^{\Gamma} \,\middle|\, \|\varphi\| < \infty\right\}$$

equipped with the inner product

$$\langle \varphi_1, \varphi_2 \rangle := \int\limits_M \varphi_1(z)\overline{\varphi_2(z)}\mu_{\mathrm{hyp}}(z) \qquad (\varphi_1, \varphi_2 \in \mathcal{H}_k^{\Gamma}).$$

The generalized Laplacian

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 2iky\frac{\partial}{\partial x}$$

acts on the smooth functions of $\mathcal{H}_k^\Gamma$ and extends to an essentially self-adjoint linear operator acting on a dense subspace of $\mathcal{H}_k^\Gamma$.

From [Fay77] or [Fis87], we quote that the eigenvalues for the equation

$$\Delta_k \varphi(z) = \lambda\varphi(z) \qquad (\varphi \in \mathcal{H}_k^\Gamma)$$

satisfy the inequality $\lambda \geq k(1-k)$.

Furthermore, if $\lambda = k(1-k)$, then the corresponding eigenfunction φ is of the form $\varphi(z) = f(z)y^k$, where f is a cusp form of weight $2k$ for Γ, i.e., we have an isomorphism of $\mathbb{C}$-vector spaces

$$\ker\big(\Delta_k - k(1-k)\big) \cong \mathcal{S}_{2k}^\Gamma.$$

Resolvent kernel.

From [Fis87], we recall that for $k \in \mathbb{N}_{\geq 1}$, the resolvent kernel on $\mathbb{H}$ associated to Δ_k is the integral kernel $G_k(s; z, w)$, which inverts the operator $\Delta_k - s(1-s)\mathrm{id}$, where $s \in W_k := \mathbb{C} \setminus \{k-n,\, -k-n \,|\, n \in \mathbb{N}\}$ and $z, w \in \mathbb{H}$. We note that the function $G_k(s; z, w)$ depends only on $\sigma(z, w)$ and is denoted by $\mathfrak{K}_s(\sigma(z, w))$ in [Fis87].

When $z = w$, the resolvent kernel has a singularity, which we cancel out by considering the difference

$$G_k(s; z, w) - G_k(t; z, w)$$

for $s, t \in W_k$. In particular, by taking $t = s+1$, we define for $s \in W_k$ and $z, w \in \mathbb{H}$ the function

$$g_k(s; z, w) := G_k(s; z, w) - G_k(s+1; z, w). \tag{2.2}$$

For an explicit formula for the resolvent kernel and further properties of the functions $G_k(s; z, w)$ and $g_k(s; z, w)$, we refer to Subsection 6.1 of the Appendix.

Spectral expansion.

Let $\{\lambda_j\}_{j=0}^\infty$ be the set of eigenvalues of Δ_k acting on the Hilbert space $\mathcal{H}_k^\Gamma$, let $\{\varphi_j\}_{j\geq 0}$ denote the corresponding orthonormal basis of eigenfunctions, and let $E_j(\cdot, s')$ be the Eisenstein series associated to the cusp p_j ($j = 1, \ldots, h$); for the precise definition, see [Fis87, § 1.5].

Lemma 2.1. *Let $s \in W_k \cap \mathbb{R}$, so that $s > 1$, and let $t := s+1$. Then, letting $\lambda := s(1-s)$ and $\mu := t(1-t)$, we have*

$$\sum_{j=0}^{\infty}\left(\frac{1}{\lambda_j - \lambda} - \frac{1}{\lambda_j - \mu}\right)|\varphi_j(z)|^2 + \frac{1}{4\pi}\sum_{j=1}^{h}\int_{-\infty}^{\infty}\left(\frac{1}{\frac14 + r^2 - \lambda} - \frac{1}{\frac14 + r^2 - \mu}\right)\left|E_j\left(z, \frac12 + ir\right)\right|^2 \mathrm{d}r$$
$$= -\frac{1}{4\pi}\big(\psi(s+k) + \psi(s-k) - \psi(t+k) - \psi(t-k)\big) + \sum_{\substack{\gamma \in \Gamma\setminus\{\mathrm{id}\} \\ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}}\left(\frac{cz+d}{c\bar{z}+d}\right)^k\left(\frac{\gamma z - \bar{z}}{z - \gamma\bar{z}}\right)^k g_k(s; z, \gamma z); \tag{2.3}$$

here $\psi(\cdot)$ is the digamma function. Furthermore, all sums and integrals in the above formula converge uniformly for $s, t \in W_k$ as chosen above and $z \in \mathbb{H}$.

Proof. For the proof, we refer to [Fis87, p. 46, eq. (2.1.4)]. □

Note that in [Fis87] subgroups of $\mathrm{SL}_2(\mathbb{R})$ are used instead of $\mathrm{PSL}_2(\mathbb{R})$. Hence, the difference by a factor of $1/2$. Also, one needs to apply Dini's theorem to [Fis87, p. 46, eq. (2.1.4)] to obtain the uniform convergence.

3 Effective estimates in the compact domain $\mathcal{F}_Y$

The main goal of this section is to give an upper bound for Poincaré series of the type

$$P_{k,\varepsilon}^{\Gamma}(z) := \sum_{\gamma\in\Gamma\setminus\{\mathrm{id}\}} \sigma(z,\gamma z)^{-(k+\varepsilon)}$$

for $k \in \mathbb{N}_{\geq 2}$ and $\varepsilon > 0$, when z is ranging through the compact domain $\mathcal{F}_Y$. To obtain this bound, we first need a couple of technical lemmas.

3.1 The displacement lemma

In this subsection, we will give a lower bound for the displacement function $\sigma(z, \gamma z)$, when $\gamma \in \Gamma$ has no elliptic fixed points in the fundamental domain $\mathcal{F}$ and z is ranging through the compact domain $\mathcal{F}_Y$. We start with the following definition.

Definition 3.1. Recalling that $\mathcal{E}$ is the set of elliptic fixed points in the fundamental domain $\mathcal{F}$ and that the boundary $\partial\mathcal{F}$ consists of the geodesic line segments in the set $\mathcal{S}$, we define the quantity

$$\mu_\Gamma := \inf_{\substack{S\in\mathcal{S}\\ e\in\mathcal{E}\setminus S}} \mathrm{dist}_{\mathrm{hyp}}(S, e), \tag{3.1}$$

which will be bounded in the next lemma.

Lemma 3.2. *With the notations of Definition 3.1, the inequality*

$$\mu_\Gamma \leq \mathrm{dist}_{\mathrm{hyp}}(\mathcal{F}, \Gamma\mathcal{E} \setminus \mathcal{F})$$

holds.

Proof. We may assume that we have

$$\mathrm{dist}_{\mathrm{hyp}}(\mathcal{F}, \Gamma\mathcal{E} \setminus \mathcal{F}) = \mathrm{dist}_{\mathrm{hyp}}(z, e),$$

where $z \in \partial\mathcal{F}$ and $e \in \Gamma\mathcal{E}\setminus\mathcal{F}$. We show that the elliptic fixed point e lies in a translate of $\mathcal{F}$, which borders $\mathcal{F}$. To show this, we assume the contrary. So, let $\mathcal{F}_1 = \gamma_1\mathcal{F}$ be a translate of $\mathcal{F}$, which borders $\mathcal{F}$, and let $\mathcal{F}_2 = \gamma_2\mathcal{F}$ be a translate of $\mathcal{F}$, which borders $\mathcal{F}_1$, but not $\mathcal{F}$, and containing e; here $\gamma_1, \gamma_2 \in \Gamma$. The geodesic line joining z with e of hyperbolic length $\mathrm{dist}_{\mathrm{hyp}}(z, e)$, then leaves $\mathcal{F}_1$ and enters $\mathcal{F}_2$ in a point z_1. We thus obtain the bound

$$\mathrm{dist}_{\mathrm{hyp}}(\gamma_1^{-1}z_1, \gamma_1^{-1}e) = \mathrm{dist}_{\mathrm{hyp}}(z_1, e) < \mathrm{dist}_{\mathrm{hyp}}(z, e).$$

However, since $\gamma_1^{-1}z_1 \in \mathcal{F}$ and $\gamma_1^{-1}e \in \Gamma\mathcal{E} \setminus \mathcal{F}$, this leads to a contradiction, and hence we can assume that $e \in \mathcal{F}_1$.

To complete the proof, we realize that $z \in S_1$ for some $S_1 \in \mathcal{S}$, which necessarily has the property $S_1 \subset \mathcal{F} \cap \mathcal{F}_1$. This shows that $\gamma_1^{-1}z \in S$ for some suitable other $S \in \mathcal{S}$ (namely, $S = \gamma_1^{-1}S_1$). Furthermore, since $\gamma_1^{-1}e \in \mathcal{E}$, but $\gamma_1^{-1}e \notin S$ (otherwise, we would have $e \in S_1 \subset \mathcal{F}$, which is not the case), we obtain

$$\inf_{\substack{S\in\mathcal{S}\\ e\in\mathcal{E}\setminus S}} \mathrm{dist}_{\mathrm{hyp}}(S, e) \leq \mathrm{dist}_{\mathrm{hyp}}(\gamma_1^{-1}z, \gamma_1^{-1}e) = \mathrm{dist}_{\mathrm{hyp}}(z, e) = \mathrm{dist}_{\mathrm{hyp}}(\mathcal{F}, \Gamma\mathcal{E} \setminus \mathcal{F}),$$

which proves the claimed inequality. □

Definition 3.3. Let $\Gamma_\mathcal{E} := \Gamma_{e_1} \cup \ldots \cup \Gamma_{e_n}$ and $Y > 1$. Then, we define the quantity

$$\sigma_Y := \inf_{\substack{z\in\mathcal{F}_Y\\ \gamma\in\Gamma\setminus\Gamma_\mathcal{E}}} \sigma(z, \gamma z), \tag{3.2}$$

which will be bounded in the next lemma.

Lemma 3.4. *With the notations of Section 2, μ_Γ given in Definition 3.1, and σ_Y given in Definition 3.3, the inequalities*

$$\sigma_Y \geq \min\left\{\frac{\cosh(\ell_\Gamma)+1}{2},\, \sinh^2(\mu_\Gamma)\,\sin^2\left(\frac{\theta_\Gamma}{2}\right)+1,\, \frac{m_Y^2}{4}+1,\, \frac{1}{4M_Y^2}+1\right\} \geq 1$$

hold.

Proof. Letting $z \in \mathcal{F}_Y$ and $\gamma \in \Gamma \setminus \Gamma_{\mathcal{E}}$, we need to distinguish and investigate the following four cases:

Case 1. Let γ be a hyperbolic element. Then we obviously have that $\mathrm{dist}_{\mathrm{hyp}}(z,\gamma z) \geq \ell_\Gamma$, from which we conclude

$$\sigma(z,\gamma z) \geq \cosh^2\left(\frac{\ell_\Gamma}{2}\right) = \frac{\cosh(\ell_\Gamma)+1}{2}\,.$$

Case 2. Let γ be an elliptic element associated to an elliptic fixed point $e \notin \mathcal{F}$. Denoting by θ the rotation angle of the corresponding primitive elliptic element, we obtain from [Bea95, p. 174, Theorem 7.35.1]

$$\begin{aligned}\sinh\left(\frac{\mathrm{dist}_{\mathrm{hyp}}(z,\gamma z)}{2}\right) &= \sinh\big(\mathrm{dist}_{\mathrm{hyp}}(z,e)\big)\sin\left(\frac{\theta}{2}\right)\\ &\geq \sinh\big(\mathrm{dist}_{\mathrm{hyp}}(\mathcal{F},\Gamma\mathcal{E}\setminus\mathcal{F})\big)\sin\left(\frac{\theta_\Gamma}{2}\right) \geq \sinh(\mu_\Gamma)\sin\left(\frac{\theta_\Gamma}{2}\right),\end{aligned}$$

where the last inequality is justified by Lemma 3.2. From this we immediately get

$$\sigma(z,\gamma z) = \sinh^2\left(\frac{\mathrm{dist}_{\mathrm{hyp}}(z,\gamma z)}{2}\right)+1 \geq \sinh^2(\mu_\Gamma)\sin^2\left(\frac{\theta_\Gamma}{2}\right)+1.$$

Case 3. Let γ be a parabolic element associated to a cusp $p \notin \mathcal{P}$. Then, we have $\gamma \in \Gamma_p$ and there exists a $\gamma' \in \Gamma$ such that $p = \gamma' p_j$ for some $j \in \{1,\ldots,h\}$. For the stabilizer subgroup Γ_p, we then find

$$\gamma'^{-1}\Gamma_p\gamma' = \Gamma_{p_j}, \quad \text{hence} \quad \sigma_j^{-1}\gamma'^{-1}\Gamma_p\gamma'\sigma_j = \sigma_j^{-1}\Gamma_{p_j}\sigma_j = \left\langle \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right\rangle.$$

Therefore, by setting

$$\delta := \sigma_j^{-1}\gamma'^{-1}\sigma_j = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \sigma_j^{-1}\Gamma\sigma_j,$$

we find

$$\delta\sigma_j^{-1}\gamma\sigma_j\delta^{-1} = \begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$$

with some $n \in \mathbb{Z}$. Letting $z' := \sigma_j^{-1}z$, we now compute

$$\begin{aligned}\sigma(z,\gamma z) &= \cosh^2\left(\frac{\mathrm{dist}_{\mathrm{hyp}}(z,\gamma z)}{2}\right) = \cosh^2\left(\frac{\mathrm{dist}_{\mathrm{hyp}}(\delta\sigma_j^{-1}z,\delta\sigma_j^{-1}\gamma z)}{2}\right)\\ &= \cosh^2\left(\frac{\mathrm{dist}_{\mathrm{hyp}}(\delta z',\delta\sigma_j^{-1}\gamma\sigma_j\delta^{-1}\delta z')}{2}\right) = \cosh^2\left(\frac{\mathrm{dist}_{\mathrm{hyp}}(\delta z',\delta z'+n)}{2}\right)\\ &= \frac{|\delta z' - \delta\bar{z}' - n|^2}{4\,\mathrm{Im}(\delta z')^2} = \frac{\frac{4\,\mathrm{Im}(z')^2}{|cz'+d|^4}+n^2}{4\,\mathrm{Im}(\delta z')^2} = \frac{n^2\,|cz'+d|^4}{4\,\mathrm{Im}(z')^2}+1.\end{aligned}$$

Taking into account that $c \neq 0$ (since otherwise we would have $p \in \mathcal{P}$), Shimizu's lemma gives the bound $|c| \geq 1$. Thus, the latter quantity can be bounded as

$$\sigma(z, \gamma z) \geq \frac{\big((c\operatorname{Re}(z') + d)^2 + c^2 \operatorname{Im}(z')^2\big)^2}{4\operatorname{Im}(z')^2} + 1 \geq \frac{c^4 \operatorname{Im}(z')^2}{4} + 1 \geq \frac{m_Y^2}{4} + 1.$$

Case 4. Let γ be a parabolic element associated to a cusp $p_j \in \mathcal{P}$. By proceding as in the previous case with $\gamma' = \mathrm{id}$ and hence $\delta = \mathrm{id}$, we have

$$\sigma_j^{-1} \gamma \sigma_j = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

with some $n \in \mathbb{Z}$. Letting $z' := \sigma_j^{-1} z$, we compute as in the previous case

$$\sigma(z, \gamma z) = \cosh^2\left(\frac{\mathrm{dist}_{\mathrm{hyp}}(z, \gamma z)}{2}\right) = \frac{|z' - \bar{z}' - n|^2}{4\operatorname{Im}(z')^2} = \frac{n^2}{4\operatorname{Im}(z')^2} + 1 \geq \frac{1}{4M_Y^2} + 1.$$

This completes the proof of the lemma, observing that the second claimed inequality is clear. ☐

3.2 Upper bounds for Poincaré series in the compact domain $\mathcal{F}_Y$

In this subsection, we will give an upper bound for the Poincaré series $P^{\Gamma}_{k,\varepsilon}(z)$ for $k \in \mathbb{N}_{\geq 2}$ and $\varepsilon > 0$, when z is ranging through the compact domain $\mathcal{F}_Y$. We start with the following definition.

Definition 3.5. Recalling that $\mathrm{diam}_{\mathrm{hyp}}(\mathcal{F}_Y)$ denotes the hyperbolic diameter of $\mathcal{F}_Y$, we define the quantity

$$B_Y := \frac{e^{3\,\mathrm{diam}_{\mathrm{hyp}}(\mathcal{F}_Y)/2}}{\mathrm{vol}_{\mathrm{hyp}}(\mathcal{F}_Y)}, \tag{3.3}$$

which will be useful in the next lemma.

Lemma 3.6. *For $z \in \mathcal{F}_Y$ and $r \geq 1$, let $\pi_{\mathcal{F}_Y}(z, r)$ denote the counting function*

$$\pi_{\mathcal{F}_Y}(z, r) := \#\big\{\gamma \in \Gamma \,|\, \sigma(z, \gamma z) \leq r\big\}.$$

Then, with B_Y given in Definition 3.5, the upper bound

$$\pi_{\mathcal{F}_Y}(z, r) \leq 4\pi\, B_Y r$$

holds.

Proof. By choosing $\rho \geq 0$ such that $r = \cosh^2(\rho/2)$, we have

$$\pi_{\mathcal{F}_Y}(z, r) = \#\big\{\gamma \in \Gamma \,|\, \mathrm{dist}_{\mathrm{hyp}}(z, \gamma z) \leq \rho\big\}.$$

Fix now $z_0 \in \mathcal{F}_Y$ such that the disk $B_{z_0}(r_0)$ of hyperbolic radius $r_0 := \mathrm{diam}_{\mathrm{hyp}}(\mathcal{F}_Y)/2$ centered at z_0 covers $\mathcal{F}_Y$. Then, we claim that as γ runs through the set $\{\gamma \in \Gamma \,|\, \mathrm{dist}_{\mathrm{hyp}}(z, \gamma z) \leq \rho\}$ for fixed $z \in \mathcal{F}_Y$, the translates $\gamma\mathcal{F}_Y$ cover parts of the disk $B_{z_0}(3r_0 + \rho)$ of hyperbolic radius $3r_0 + \rho$ centered at z_0. To see this, we observe that for $\gamma \in \Gamma$ such that $\mathrm{dist}_{\mathrm{hyp}}(z, \gamma z) \leq \rho$ and $w \in \mathcal{F}_Y$, we find by our assumptions

$$\begin{aligned}
\mathrm{dist}_{\mathrm{hyp}}(\gamma w, z_0) &\leq \mathrm{dist}_{\mathrm{hyp}}(\gamma w, \gamma z) + \mathrm{dist}_{\mathrm{hyp}}(\gamma z, z) + \mathrm{dist}_{\mathrm{hyp}}(z, z_0) \\
&= \mathrm{dist}_{\mathrm{hyp}}(w, z) + \mathrm{dist}_{\mathrm{hyp}}(z, \gamma z) + \mathrm{dist}_{\mathrm{hyp}}(z, z_0) \\
&\leq 2r_0 + \rho + r_0 = 3r_0 + \rho,
\end{aligned}$$

which proves the claim.

Noting that the intersections of two different translates $\gamma\mathcal{F}_Y$ and $\gamma'\mathcal{F}_Y$ of the type considered above are of measure 0, we are led to the upper bound

$$\pi_{\mathcal{F}_Y}(z,r)\cdot\mathrm{vol}_{\mathrm{hyp}}(\mathcal{F}_Y)\leq\mathrm{vol}_{\mathrm{hyp}}\big(B_{z_0}(3r_0+\rho)\big)=4\pi\sinh^2\left(\frac{3r_0+\rho}{2}\right)$$
$$\leq 4\pi\cosh^2\left(\frac{3r_0+\rho}{2}\right)\leq 4\pi\, e^{3r_0}\cosh^2\left(\frac{\rho}{2}\right).$$

This immediately implies the claimed upper bound recalling that $r_0=\mathrm{diam}_{\mathrm{hyp}}(\mathcal{F}_Y)/2$ and $r=\cosh^2(\rho/2)$. ☐

Lemma 3.7. *Let $Y>1$ and $\delta>1$. Then, with B_Y given in Definition 3.5, the upper bound*

$$\sum_{\gamma\in\Gamma}\sigma(z,\gamma z)^{-\delta}\leq 4\pi\, B_Y\frac{\delta}{\delta-1}$$

holds for $z\in\mathcal{F}_Y$.

Proof. Letting $R>1$ and rewriting the Poincaré series under consideration as a Stieltjes integral using the counting function $\pi_{\mathcal{F}_Y}(z,r)$ from Lemma 3.6, we get after integrating by parts

$$\sum_{\substack{\gamma\in\Gamma\\ \sigma(z,\gamma z)\leq R}}\sigma(z,\gamma z)^{-\delta}=\int_1^R r^{-\delta}\,\mathrm{d}\pi_{\mathcal{F}_Y}(z,r)=r^{-\delta}\,\pi_{\mathcal{F}_Y}(z,r)\Big|_1^R+\delta\int_1^R r^{-\delta-1}\,\pi_{\mathcal{F}_Y}(z,r)\,\mathrm{d}r.$$

Using Lemma 3.6, we find upon setting $\widetilde{B}_Y:=4\pi B_Y$ the bound

$$\sum_{\substack{\gamma\in\Gamma\\ \sigma(z,\gamma z)\leq R}}\sigma(z,\gamma z)^{-\delta}\leq R^{-\delta}\widetilde{B}_Y\,R+\delta\int_1^R r^{-\delta-1}\widetilde{B}_Y\,r\,\mathrm{d}r=\widetilde{B}_Y R^{-\delta+1}+\widetilde{B}_Y\,\delta\left(\frac{R^{-\delta+1}}{-\delta+1}-\frac{1}{-\delta+1}\right).$$

Letting $R\to\infty$, we thus obtain the upper bound

$$\sum_{\gamma\in\Gamma}\sigma(z,\gamma z)^{-\delta}\leq 4\pi\, B_Y\frac{\delta}{\delta-1}\,,$$

as claimed. ☐

Proposition 3.8. *Let $k\in\mathbb{N}_{\geq 2}$, $\varepsilon>0$, and $Y>1$. Then, with σ_Y given in Definition 3.3 and B_Y given in Definition 3.5, the upper bound*

$$\sum_{\gamma\in\Gamma\setminus\{\mathrm{id}\}}\sigma(z,\gamma z)^{-(k+\varepsilon)}\leq 4\pi\,\frac{2+\varepsilon}{1+\varepsilon}\,B_Y\,\sigma_Y^{-(k-2)}+\sum_{e_j\in\mathcal{E}}(n_j-1)$$

holds for $z\in\mathcal{F}_Y$.

Proof. Recalling that $\Gamma_{\mathcal{E}}=\Gamma_{e_1}\cup\ldots\cup\Gamma_{e_n}$, we have the decomposition

$$\sum_{\gamma\in\Gamma\setminus\{\mathrm{id}\}}\sigma(z,\gamma z)^{-(k+\varepsilon)}=\sum_{\gamma\in\Gamma\setminus\Gamma_{\mathcal{E}}}\sigma(z,\gamma z)^{-(k+\varepsilon)}+\sum_{\gamma\in\Gamma_{\mathcal{E}}\setminus\{\mathrm{id}\}}\sigma(z,\gamma z)^{-(k+\varepsilon)}.$$

Since $\sigma(z,\gamma z)\geq\sigma_Y\geq 1$ for $z\in\mathcal{F}_Y$ and $\gamma\in\Gamma\setminus\Gamma_{\mathcal{E}}$, and since $k\in\mathbb{N}_{\geq 2}$, Lemma 3.7 allows to bound the first summand as

$$\sum_{\gamma\in\Gamma\setminus\Gamma_{\mathcal{E}}}\sigma(z,\gamma z)^{-(k+\varepsilon)}=\sum_{\gamma\in\Gamma\setminus\Gamma_{\mathcal{E}}}\sigma(z,\gamma z)^{-(k-2)}\,\sigma(z,\gamma z)^{-(2+\varepsilon)}$$
$$\leq\sigma_Y^{-(k-2)}\sum_{\gamma\in\Gamma\setminus\Gamma_{\mathcal{E}}}\sigma(z,\gamma z)^{-(2+\varepsilon)}\leq\sigma_Y^{-(k-2)}\,4\pi\,B_Y\,\frac{2+\varepsilon}{1+\varepsilon}\,.$$

Since $\sigma(z, \gamma z) \geq 1$ for $z \in \mathcal{F}_Y$ and $\gamma \in \Gamma_{\mathcal{E}}$, we easily estimate the second summand as

$$\sum_{\gamma \in \Gamma_{\mathcal{E}} \setminus \{\mathrm{id}\}} \sigma(z, \gamma z)^{-(k+\varepsilon)} \leq \sum_{e_j \in \mathcal{E}} (n_j - 1).$$

This completes the proof of the proposition. □

4 Effective estimates in the cuspidal neighborhoods $\mathcal{F}_j^Y$

The main goal of this section is to give an upper bound for the Poincaré series $P^{\Gamma}_{k,\varepsilon}(z)$ for $k \in \mathbb{N}_{\geq 2}$ and $\varepsilon > 0$, when z ranges through the cuspidal neighborhoods $\mathcal{F}_j^Y$. It will turn out that we can restrict ourselves to the case when $Y < k/(2\pi)$.

4.1 A lemma of Faddeev

In this subsection, we first show that bounding $S^{\Gamma}_{2k}(z)$ in the cuspidal neighborhoods $\mathcal{F}_j^Y$ can be reduced to estimating this quantity in suitable compact sets depending on $Y \geq k/(2\pi)$ or $Y < k/(2\pi)$. Then, we will prove a lemma due to L.D. Faddeev [Fad69], which will be crucial for the next subsection.

Lemma 4.1. *Let $k \in \mathbb{N}$ and $Y_0 \geq 8/\sqrt{15}$. Then, for $j = 1, \ldots, h$, we have the following two statements:*

(1) *For $1 \leq k \leq 4\pi Y_0$ and $Y \geq 2Y_0$, the equality*

$$\sup_{z \in \mathcal{F}_j^Y} S^{\Gamma}_{2k}(z) \leq \sup_{z \in \mathcal{F}_Y} S^{\Gamma}_{2k}(z)$$

holds.

(2) *For $k > 4\pi Y_0$ and $2Y_0 \leq Y < k/(2\pi)$, the equality*

$$\sup_{z \in \mathcal{F}_j^Y} S^{\Gamma}_{2k}(z) = \sup_{z \in \mathrm{cl}(\mathcal{F}_j^Y \setminus \mathcal{F}_j^{k/(2\pi)})} S^{\Gamma}_{2k}(z)$$

holds; here $\mathrm{cl}(\,\cdot\,)$ refers to the topological closure.

Proof. (1) Without loss of generality, we may assume that $p_j = i\infty$ with scaling matrix $\sigma_j = \mathrm{id}$, so that we have

$$\mathcal{F}_j^Y = \{z = x + iy \in \mathbb{H} \mid -1/2 \leq x \leq 1/2,\ y \geq Y\}.$$

By then focussing on a single cusp form $f \in \mathcal{S}^{\Gamma}_{2k}$ with Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z},$$

we have to estimate the expression

$$|f(z)|^2\, y^{2k} = \left| \frac{f(z)}{e^{2\pi i z}} \right|^2 e^{-4\pi y} y^{2k}$$

in the strip $\mathcal{F}_j^Y$. Since the function $|f(z)/e^{2\pi i z}|^2$ is bounded and subharmonic in $\mathcal{F}_j^Y$, the strong maximum principle for subharmonic functions implies that its maximum occurs when $y = Y$.

Next we consider the function $h_k(y) := e^{-4\pi y} y^{2k}$ for $y > 0$. Elementary calculus shows that

$$h_k'(y) = 2k e^{-4\pi y} y^{2k-1} - 4\pi e^{-4\pi y} y^{2k},$$

so then $h_k(y)$ achieves its maximum when $y = k/(2\pi) \leq 2Y_0 \leq Y$, by our assumptions. Therefore, by the monotonicity of the function $h_k(y)$, we find that

$$\max_{z\in\mathcal{F}_j^Y} |f(z)|^2\, y^{2k} = \max_{\substack{-1/2\leq x\leq 1/2\\ y=Y}} |f(z)|^2\, y^{2k} \leq \max_{z\in\mathcal{F}_Y} |f(z)|^2\, y^{2k}.$$

For the last inequality, we recall that by our choice of Y, the cuspidal neighborhoods $\mathcal{F}_j^Y$ are pairwise disjoint and that the boundary of each cuspidal neighborhood is contained in the boundary of the compact part $\mathcal{F}_Y$.

(2) Since $2Y_0 \leq Y < k/(2\pi)$ in this case, we have the proper decomposition

$$\mathcal{F}_j^Y = \mathcal{F}_j^{k/(2\pi)} \cup \big(\mathcal{F}_j^Y \setminus \mathcal{F}_j^{k/(2\pi)}\big).$$

Proceeding as in (1), we are then led to the equality

$$\max_{z\in\mathcal{F}_j^{k/(2\pi)}} |f(z)|^2\, y^{2k} = \max_{\substack{-1/2\leq x\leq 1/2\\ y=k/(2\pi)}} |f(z)|^2\, y^{2k}.$$

From this we immediately conclude that

$$\max_{z\in\mathcal{F}_j^Y} |f(z)|^2\, y^{2k} = \max_{z\in\mathrm{cl}(\mathcal{F}_j^Y\setminus\mathcal{F}_j^{k/(2\pi)})} |f(z)|^2\, y^{2k},$$

which proves the second part of the claim. □

The next lemma is due to L.D. Faddeev [Fad69]; for its proof, we follow [Lan85, p. 307].

Lemma 4.2. *Let p_j be a cusp of $\mathcal{F}$ with scaling matrix σ_j, $z_0 = x + iy_0 \in \mathbb{H}$, and $\delta_1 > 0$. Then, the inequality*

$$\sum_{\gamma\in\Gamma\setminus\Gamma_{p_j}} \sigma(\sigma_j z, \gamma\sigma_j z)^{-\delta_2} \leq \left(\frac{64}{15}\right)^{\delta_2-\delta_1-1} y_0^{-2\delta_1-2} y^{-2\delta_2+4\delta_1+4} \sum_{\gamma\in\Gamma\setminus\Gamma_{p_j}} \sigma(\sigma_j z_0, \gamma\sigma_j z_0)^{-\delta_1-1}$$

holds for $z = x + iy \in \mathbb{H}$ with $y \geq 2y_0$ and $\delta_2 \geq \delta_1 + 1$.

Proof. Since we have $\sigma(\sigma_j z, \gamma\sigma_j z) = \sigma(z, \sigma_j^{-1}\gamma\sigma_j z)$ and

$$\sigma_j^{-1}\Gamma_{p_j}\sigma_j = \left\langle \begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix} \right\rangle,$$

we may assume without loss of generality that $p_j = i\infty$ and $\sigma_j = \mathrm{id}$. For any

$$\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma \setminus \Gamma_{i\infty},$$

we then have $|c| \geq 1$ by Shimizu's lemma. Using

$$u(z,w) := \sigma(z,w) - 1 = \sinh^2\left(\frac{\mathrm{dist_{hyp}}(z,w)}{2}\right) = \frac{|z-w|^2}{4\,\mathrm{Im}(z)\mathrm{Im}(w)}$$

with $w = \gamma z$, a direct calculation shows that

$$\begin{aligned} 4y^2 u(z,\gamma z) &= |cz^2 + dz - az - b|^2\\ &= (cx^2 + dx - ax - b)^2 + (cx+d)^2y^2 + (cx-a)^2y^2 + c^2y^4 - 2y^2; \end{aligned}$$

a similar equation holds for $z_0 = x + iy_0$. Hence, recalling that $y \geq 2y_0$, yields the inequality

$$4y^2 u(z,\gamma z) \geq 4y_0^2 u(z_0,\gamma z_0) + c^2y^4 - c^2y_0^4 - 2y^2 + 2y_0^2.$$

Next, adding $4y^2$ to both sides, gives (again, using $y \geq 2y_0$)

$$4y^2\sigma(z,\gamma z) \geq 4y_0^2\sigma(z_0,\gamma z_0) + c^2(y^4 - y_0^4) + 2y^2 - 2y_0^2 \geq 4y_0^2\sigma(z_0,\gamma z_0) + \frac{15}{16}c^2y^4.$$

After dividing both sides by y^4, we obtain

$$\frac{4}{y^2}\sigma(z,\gamma z) \geq \frac{4y_0^2}{y^4}\sigma(z_0,\gamma z_0) + \frac{15}{16}c^2.$$

Next, multiply both sides by $16/15$ and use $|c| \geq 1$ to get

$$\frac{64}{15y^2}\sigma(z,\gamma z) \geq \frac{64y_0^2}{15y^4}\sigma(z_0,\gamma z_0) + c^2 \geq 1.$$

Since both sides are at least one, we obtain after exponentiating with $\delta_2 \geq \delta_1 + 1 > 1$,

$$\left(\frac{64}{15y^2}\sigma(z,\gamma z)\right)^{\delta_2} \geq \left(\frac{64y_0^2}{15y^4}\sigma(z_0,\gamma z_0)\right)^{\delta_1+1}.$$

Rearranging terms leads to the inequality

$$\sigma(z,\gamma z)^{-\delta_2} \leq \left(\frac{64}{15}\right)^{\delta_2-\delta_1-1} y_0^{-2\delta_1-2}y^{-2\delta_2+4\delta_1+4}\sigma(z_0,\gamma z_0)^{-\delta_1-1},$$

which proves the claimed inequality after taking the sum over $\gamma \in \Gamma \setminus \Gamma_{i\infty}$. □

4.2 Upper bounds for Poincaré series in the cuspidal neighborhoods $\mathcal{F}_j^Y$

In this subsection, we will apply Faddeev's lemma to obtain an upper bound for the Poincaré series $P_{k,\varepsilon}^{\Gamma}(z)$ for $k \in \mathbb{N}_{\geq 2}$ and $\varepsilon > 0$, when z ranges through the cuspidal neighborhoods $\mathcal{F}_j^Y$ with $Y < k/(2\pi)$. We start with the following definition.

Definition 4.3. Let $k \in \mathbb{N}_{\geq 1}$, $\varepsilon > 0$, and $Y_0 > 1$. Then, we define the quantities

$$B_{k,Y_0}(\varepsilon) := \pi\, Y_0^{-4-2\varepsilon}\, B_{Y_0}\, 4^{-k+3}\, \frac{2+\varepsilon}{1+\varepsilon}\left(\frac{k}{2\pi}\right)^{4+2\varepsilon}, \tag{4.1}$$

and

$$B_{k,Y_0} := \lim_{\varepsilon\to 0} B_{k,Y_0}(\varepsilon) = 2\pi\, Y_0^{-4}\, B_{Y_0}\, 4^{-k+3}\left(\frac{k}{2\pi}\right)^{4}, \tag{4.2}$$

which will be useful for the next lemma.

Lemma 4.4. *Let $k \in \mathbb{N}_{\geq 2}$, $\varepsilon > 0$, $Y_0 > 1$, and $Y := \max\{2Y_0, 16/\sqrt{15}\}$; assume that $Y < k/(2\pi)$. Then, with $B_{k,Y_0}(\varepsilon)$ given in Definition 4.3, the upper bounds*

$$\sum_{\gamma\in\Gamma\setminus\Gamma_{p_j}} \sigma(z,\gamma z)^{-(k+\varepsilon)} \leq B_{k,Y_0}(\varepsilon) \qquad (j = 1,\ldots,h)$$

hold for $z \in \mathrm{cl}(\mathcal{F}_j^Y \setminus \mathcal{F}_j^{k/(2\pi)})$.

Proof. With the scaling matrix σ_j of the cusp p_j, we define $z' := \sigma_j^{-1}z$. We then employ Lemma 4.2 with $z' = x'+iy'$, $z_0 := x'+iY_0$ (note that $y' \geq 2Y_0$) and $\delta_1 := 1+\varepsilon$, $\delta_2 := k+\varepsilon$ (note that $\delta_2 \geq \delta_1+1$, since $k \in \mathbb{N}_{\geq 2}$), to get

$$\sum_{\gamma\in\Gamma\setminus\Gamma_{p_j}} \sigma(z,\gamma z)^{-(k+\varepsilon)} = \sum_{\gamma\in\Gamma\setminus\Gamma_{p_j}} \sigma(\sigma_j z',\gamma\sigma_j z')^{-(k+\varepsilon)}$$
$$\leq \left(\frac{64}{15}\right)^{k-2} Y_0^{-4-2\varepsilon}y'^{-2k+8+2\varepsilon} \sum_{\gamma\in\Gamma\setminus\Gamma_{p_j}} \sigma(\sigma_j z_0,\gamma\sigma_j z_0)^{-(2+\varepsilon)}.$$

Since we have $Y \geq 2 \cdot 8/\sqrt{15}$, we get $(64/15)^{k-2} \leq Y^{2k-4}/4^{k-2}$, which leads to the estimate

$$\sum_{\gamma\in\Gamma\setminus\Gamma_{p_j}} \sigma(z,\gamma z)^{-(k+\varepsilon)} \leq \frac{Y^{2k-4}}{4^{k-2}} Y_0^{-4-2\varepsilon} Y^{-2k+4} \left(\frac{k}{2\pi}\right)^{4+2\varepsilon} \sum_{\gamma\in\Gamma\setminus\Gamma_{p_j}} \sigma(\sigma_j z_0, \gamma\sigma_j z_0)^{-(2+\varepsilon)};$$

here we used that $1 < Y \leq y' \leq k/(2\pi)$. Observing now that we have by construction $\sigma_j z_0 \in \mathcal{F}_{Y_0}$, we can bound the latter sum from above by Lemma 3.7 as

$$\sum_{\gamma\in\Gamma\setminus\Gamma_{p_j}} \sigma(\sigma_j z_0, \gamma\sigma_j z_0)^{-(2+\varepsilon)} \leq 4\pi\, B_{Y_0}\, \frac{2+\varepsilon}{1+\varepsilon}.$$

All in all, this proves the claim. $\square$

Lemma 4.5. *Let $k \in \mathbb{N}_{\geq 1}$ and $\varepsilon > 0$; assume that $1 < Y < k/(2\pi)$. Then, the upper bounds*

$$\sum_{\gamma\in\Gamma_{p_j}\setminus\{\mathrm{id}\}} \sigma(z,\gamma z)^{-(k+\varepsilon)} \leq \frac{k\, e^{5/4}}{\sqrt{\pi}\,\sqrt{k+\varepsilon}} \qquad (j = 1, \ldots, h)$$

hold for $z \in \mathrm{cl}(\mathcal{F}_j^Y \setminus \mathcal{F}_j^{k/(2\pi)})$.

Proof. With the scaling matrix σ_j of the cusp p_j, we define $z' := \sigma_j^{-1} z$. Recalling that

$$\sigma_j^{-1}\Gamma_{p_j}\sigma_j = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle,$$

we compute using $z' = x' + iy'$ that

$$\sum_{\gamma\in\Gamma_{p_j}\setminus\{\mathrm{id}\}} \sigma(z,\gamma z)^{-(k+\varepsilon)} = \sum_{\gamma\in\Gamma_{p_j}\setminus\{\mathrm{id}\}} \sigma(\sigma_j z', \gamma\sigma_j z')^{-(k+\varepsilon)}$$
$$= \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \sigma(z', z'+n)^{-(k+\varepsilon)} = 2\sum_{n=1}^{\infty} \left(1 + \left(\frac{n}{2y'}\right)^2\right)^{-(k+\varepsilon)}.$$

By an integral test we obtain the upper bound (recalling formula 3.251.2 from [GR81])

$$\frac{1}{2y'}\sum_{n=1}^{\infty} \frac{1}{\left(1+\left(\frac{n}{2y'}\right)^2\right)^{k+\varepsilon}} \leq \int_0^{\infty} \frac{1}{(1+\nu^2)^{k+\varepsilon}}\,\mathrm{d}\nu = \frac{\sqrt{\pi}\,\Gamma(k-1/2+\varepsilon)}{2\,\Gamma(k+\varepsilon)}.$$

Using now that $Y \leq y' \leq k/(2\pi)$, we arrive at the upper bound

$$\sum_{\gamma\in\Gamma_{p_j}\setminus\{\mathrm{id}\}} \sigma(z,\gamma z)^{-(k+\varepsilon)} \leq \frac{k\,\Gamma(k-1/2+\varepsilon)}{\sqrt{\pi}\,\Gamma(k+\varepsilon)}.$$

An application of an effective version of Stirling's formula (see Lemma 6.3 of the Appendix) gives

$$\frac{k\,\Gamma(k-1/2+\varepsilon)}{\sqrt{\pi}\,\Gamma(k+\varepsilon)} \leq \frac{k\, e^{5/4}}{\sqrt{\pi}\,\sqrt{k+\varepsilon}},$$

which completes the proof of the lemma. $\square$

5 Main results

Based on the upper bounds for the Poincaré series $P_{k,\varepsilon}^{\Gamma}(z)$ established for z ranging through the compact domain $\mathcal{F}_Y$ in Subsection 3.2 and for z ranging through the cuspidal neighborhoods $\mathcal{F}_j^Y$ in Subsection 4.2, we are now in position to state and prove the main results of this paper providing upper bounds for the supremum of the quantity $S_{2k}^{\Gamma}(z)$ in the cocompact as well as in the cofinite setting. We also address the question of lower bounds for the quantity under consideration. We end this section with some explicit computations in the case of the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$.

5.1 Main results in the cocompact setting

In this subsection, we will give an effective upper bound for the supremum of the quantity $S_{2k}^{\Gamma}(z)$ for $k \in \mathbb{N}_{\geq 2}$, when z is ranging through the compact domain $\mathcal{F}_Y$. In particular, this will lead us to effective upper and lower bounds for the supremum of the quantity $S_{2k}^{\Gamma}(z)$, when Γ is cocompact and torsionfree. We start by establishing an upper bound for the quantity $S_{2k}^{\Gamma}(z)$ in terms of the Poincaré series $P_{k,\varepsilon}^{\Gamma}(z)$, which is valid for all $z \in \mathbb{H}$.

Proposition 5.1. *Let $k \in \mathbb{N}_{\geq 1}$ and $0 < \varepsilon < 1$. Then, the inequality*

$$S_{2k}^{\Gamma}(z) \leq \frac{(2k-1+\varepsilon)(1+\varepsilon)}{4\pi} + \frac{3(2k+\varepsilon)(2k-1+\varepsilon)(1+\varepsilon)}{4\pi(k+\varepsilon)} \sum_{\gamma \in \Gamma \setminus \{\mathrm{id}\}} \sigma(z, \gamma z)^{-(k+\varepsilon)}$$

holds for $z \in \mathbb{H}$.

Proof. Letting $\lambda = s(1-s)$ and $\mu = t(1-t)$ with $s, t \in W_k \cap \mathbb{R}$ such that $t > s > 1$, formula (2.3) of Lemma 2.1 states the equality

$$\sum_{j=0}^{\infty} \left(\frac{1}{\lambda_j - \lambda} - \frac{1}{\lambda_j - \mu} \right) |\varphi_j(z)|^2 + \frac{1}{4\pi} \sum_{j=1}^{h} \int_{-\infty}^{\infty} \left(\frac{1}{\frac{1}{4} + r^2 - \lambda} - \frac{1}{\frac{1}{4} + r^2 - \mu} \right) \left| E_j\left(z, \frac{1}{2} + ir\right) \right|^2 \mathrm{d}r$$
$$= -\frac{1}{4\pi} \big(\psi(s+k) + \psi(s-k) - \psi(t+k) - \psi(t-k) \big) + \sum_{\gamma \in \Gamma \setminus \{\mathrm{id}\}} \left(\frac{cz+d}{c\bar{z}+d} \right)^k \left(\frac{\gamma z - \bar{z}}{z - \gamma \bar{z}} \right)^k g_k(s; z, \gamma z).$$

Choosing now $s = k + \varepsilon$ and $t = s + 1 = k + 1 + \varepsilon$ and restricting the summation on the left-hand side of the above formula to the eigenvalue $\lambda_j = k(1-k)$, then yields the inequality after neglecting all the other summands and taking absolute values on the right-hand side

$$\sum_{j=1}^{d_{2k}} \left(\frac{1}{k(1-k) - s(1-s)} - \frac{1}{k(1-k) - t(1-t)} \right) |f_j(z)|^2 y^{2k}$$
$$\leq \frac{1}{4\pi} \big| \psi(s+k) + \psi(s-k) - \psi(t+k) - \psi(t-k) \big| + \sum_{\gamma \in \Gamma \setminus \{\mathrm{id}\}} |g_k(s; z, \gamma z)|.$$

For the left-hand side we then compute

$$\begin{aligned} r(k,\varepsilon) &:= \frac{1}{k(1-k) - s(1-s)} - \frac{1}{k(1-k) - t(1-t)} \\ &= \frac{1}{\varepsilon(2k-1+\varepsilon)} - \frac{1}{2k + \varepsilon(2k+1+\varepsilon)} \\ &= \frac{2(k+\varepsilon)}{\varepsilon(2k-1+\varepsilon)(2k+\varepsilon(2k+1+\varepsilon))} \\ &= \frac{2(k+\varepsilon)}{\varepsilon(2k+\varepsilon)(2k-1+\varepsilon)(1+\varepsilon)}\,. \end{aligned} \tag{5.1}$$

Furthermore, recalling that the digamma function $\psi(s)$ satisfies the functional equation $\psi(s+1) - \psi(s) = 1/s$, leads to the relation

$$\begin{aligned} &\psi(s+k) + \psi(s-k) - \psi(t+k) - \psi(t-k) \\ &\quad = \psi(2k+\varepsilon) + \psi(\varepsilon) - \psi(2k+1+\varepsilon) - \psi(1+\varepsilon) \\ &\quad = -\frac{1}{2k+\varepsilon} - \frac{1}{\varepsilon} = -\frac{2(k+\varepsilon)}{\varepsilon(2k+\varepsilon)}\,. \end{aligned} \tag{5.2}$$

Collecting the above calculations then gives the upper bound

$$\frac{2(k+\varepsilon)}{\varepsilon(2k+\varepsilon)(2k-1+\varepsilon)(1+\varepsilon)}\sum_{j=1}^{d_{2k}}|f_j(z)|^2y^{2k}\leq\frac{2(k+\varepsilon)}{4\pi\varepsilon(2k+\varepsilon)}+\sum_{\gamma\in\Gamma\setminus\{\mathrm{id}\}}|g_k(k+\varepsilon;z,\gamma z)|,$$

in other words, we have the upper bound

$$S^{\Gamma}_{2k}(z)\leq\frac{(2k-1+\varepsilon)(1+\varepsilon)}{4\pi}+\frac{\varepsilon(2k+\varepsilon)(2k-1+\varepsilon)(1+\varepsilon)}{2(k+\varepsilon)}\sum_{\gamma\in\Gamma\setminus\{\mathrm{id}\}}|g_k(k+\varepsilon;z,\gamma z)|,$$

which is valid for all $z\in\mathbb{H}$. Since $0<\varepsilon<1$, Lemma 6.2 of the Appendix applies and provides for all $z\in\mathbb{H}$ and $\gamma\in\Gamma$ the inequality

$$|g_k(k+\varepsilon;z,\gamma z)|\leq\frac{3}{2\pi\varepsilon}\sigma(z,\gamma z)^{-(k+\varepsilon)},$$

from which the claimed inequality follows. □

In the next theorem we prove the first part of Theorem B given in the introduction.

Theorem 5.2. *Let $k\in\mathbb{N}_{\geq2}$ and $Y>1$. Then, with σ_Y given in Definition 3.3 and B_Y given in Definition 3.5 , the upper bound*

$$\sup_{z\in\mathcal{F}_Y}S^{\Gamma}_{2k}(z)\leq\frac{2k-1}{4\pi}\bigg(1+6\sum_{e_j\in\mathcal{E}}(n_j-1)\bigg)+12(2k-1)B_Y\,\sigma_Y^{-(k-2)}$$

holds.

Proof. Given $0<\varepsilon<1$, Proposition 5.1 provides for all $z\in\mathbb{H}$ the upper bound

$$S^{\Gamma}_{2k}(z)\leq\frac{(2k-1+\varepsilon)(1+\varepsilon)}{4\pi}+\frac{3(2k+\varepsilon)(2k-1+\varepsilon)(1+\varepsilon)}{4\pi(k+\varepsilon)}\sum_{\gamma\in\Gamma\setminus\{\mathrm{id}\}}\sigma(z,\gamma z)^{-(k+\varepsilon)}\,. \tag{5.3}$$

By means of Proposition 3.8, we then obtain for $z\in\mathcal{F}_Y$ the upper bound

$$S^{\Gamma}_{2k}(z)\leq\frac{(2k-1+\varepsilon)(1+\varepsilon)}{4\pi}+\frac{3(2k+\varepsilon)(2k-1+\varepsilon)(1+\varepsilon)}{4\pi(k+\varepsilon)}\times$$
$$\times\bigg(4\pi\,\frac{2+\varepsilon}{1+\varepsilon}\,B_Y\,\sigma_Y^{-(k-2)}+\sum_{e_j\in\mathcal{E}}(n_j-1)\bigg).$$

Letting $\varepsilon\to0$, we thus arrive for $z\in\mathcal{F}_Y$ at the upper bound

$$S^{\Gamma}_{2k}(z)\leq\frac{2k-1}{4\pi}\bigg(1+6\sum_{e_j\in\mathcal{E}}(n_j-1)\bigg)+12(2k-1)B_Y\,\sigma_Y^{-(k-2)}\,,$$

which concludes the proof of the theorem. □

In the next theorem we prove Theorem A given in the introduction.

Theorem 5.3. *Let Γ be cocompact and torsionfree, and let $k\in\mathbb{N}_{\geq2}$. Then, the bounds*

$$\frac{2k-1}{4\pi}\leq\sup_{z\in\mathbb{H}}S^{\Gamma}_{2k}(z)\leq\frac{2k-1}{4\pi}+C_{\Gamma}\,e^{-\delta_{\Gamma}k}$$

hold, where the constants C_Γ and δ_Γ are effectively computable as

$$C_{\Gamma}=\frac{3\,e^{12\pi g_{\Gamma}/\ell_{\Gamma}}}{\pi(g_{\Gamma}-1)}\frac{(\cosh(\ell_{\Gamma})+1)^2}{\log((\cosh(\ell_{\Gamma})+1)/2)}\quad\textit{and}\quad\delta_{\Gamma}=\frac{1}{2}\log\bigg(\frac{\cosh(\ell_{\Gamma})+1}{2}\bigg).$$

Proof. The lower bound has been proven in [FJK16, Sec. 7.1]. As far as the proof of the upper bound is concerned, we recall that in the cocompact setting we have chosen $\mathcal{F}_Y = \mathcal{F}$, so that we obtain from Theorem 5.2

$$\sup_{z\in\mathbb{H}} S_{2k}^{\Gamma}(z) \leq \frac{2k-1}{4\pi} + 12(2k-1)B_Y\,\sigma_Y^{-(k-2)},$$

where we have from Lemma 3.4 and Definition 3.5 that

$$\sigma_Y \geq \frac{\cosh(\ell_\Gamma)+1}{2} \qquad \text{and} \qquad B_Y = \frac{e^{3\,\mathrm{diam}_{\mathrm{hyp}}(\mathcal{F})/2}}{\mathrm{vol}_{\mathrm{hyp}}(\mathcal{F})},$$

respectively; to simplify notations, we set $\sigma_\Gamma := (\cosh(\ell_\Gamma)+1)/2$. Using the inequality

$$\mathrm{diam}_{\mathrm{hyp}}(\mathcal{F}) \leq \frac{2\,\mathrm{vol}_{\mathrm{hyp}}(\mathcal{F})}{\ell_\Gamma} \leq \frac{8\pi g_\Gamma}{\ell_\Gamma}$$

proven in [Cha77], we can estimate B_Y as

$$B_Y \leq \frac{e^{12\pi g_\Gamma/\ell_\Gamma}}{4\pi(g_\Gamma-1)}\,.$$

Now, taking into account the inequalities

$$ax \leq e^{ax} \qquad \Longleftrightarrow \qquad xe^{-2ax} \leq \frac{e^{-ax}}{a}\,,$$

which are valid for $a>0$ and $x\geq 0$, we derive by choosing $a=\log(\sigma_\Gamma)$ and $x=k/2$ that

$$\frac{k}{2}\sigma_\Gamma^{-k} \leq \frac{e^{-\log(\sigma_\Gamma)k/2}}{\log(\sigma_\Gamma)}\,.$$

Since $k\in\mathbb{N}_{\geq 2}$, we conclude from the above that

$$\begin{aligned} 12(2k-1)B_Y\,\sigma_Y^{-(k-2)} &\leq 12(2k-1)B_Y\,\sigma_\Gamma^{-(k-2)} \\ &\leq 48B_Y\,\sigma_\Gamma^2\,\frac{k}{2}\,\sigma_\Gamma^{-k} \leq \frac{12\,e^{12\pi g_\Gamma/\ell_\Gamma}}{\pi(g_\Gamma-1)}\,\sigma_\Gamma^2\,\frac{e^{-\log(\sigma_\Gamma)k/2}}{\log(\sigma_\Gamma)}\,, \end{aligned}$$

which proves the claim with the constants C_Γ and δ_Γ as stated in the theorem. □

5.2 Main results in the cofinite setting

In this subsection, we will give an effective upper bound for the supremum of the quantity $S_{2k}^{\Gamma}(z)$ for $k\in\mathbb{N}_{\geq 2}$, when z is ranging through the cuspidal neighborhoods $\mathcal{F}_j^Y$. Letting $Y_0\geq 8/\sqrt{15}$ and $Y\geq 2Y_0$, we may assume that $k>4\pi Y_0$, since by Lemma 4.1 in the case $2\leq k\leq 4\pi Y_0$ the desired upper bound is also covered by Theorem 5.2. This will prove the second part of Theorem B given in the introduction.

Theorem 5.4. *Let $k\in\mathbb{N}$, $Y_0\geq 8/\sqrt{15}$, $k>4\pi Y_0$, and $Y=2Y_0$. Then, with B_{k,Y_0} given in Definition 4.3, the upper bounds*

$$\sup_{z\in\mathcal{F}_j^Y} S_{2k}^{\Gamma}(z) \leq \frac{2k-1}{4\pi} + \frac{3(2k-1)}{2\pi}\left(B_{k,Y_0} + \frac{\sqrt{k}\,e^{5/4}}{\sqrt{\pi}}\right) = O(k^{3/2})$$

hold for $j=1,\ldots,h$.

Proof. Since the inequality $\max\{2Y_0, 16/\sqrt{15}\} = 2Y_0 = Y < k/(2\pi)$ holds by assumption, the second part of Lemma 4.1 allows us to restrict the range for z from $\mathcal{F}_j^Y$ to $\mathrm{cl}(\mathcal{F}_j^Y \setminus \mathcal{F}_j^{k/(2\pi)})$ in the subsequent estimates.

Given $0 < \varepsilon < 1$, Proposition 5.1 provides for all $z \in \mathbb{H}$ the upper bound

$$S_{2k}^{\Gamma}(z) \leq \frac{(2k-1+\varepsilon)(1+\varepsilon)}{4\pi} + \frac{3(2k+\varepsilon)(2k-1+\varepsilon)(1+\varepsilon)}{4\pi(k+\varepsilon)} \sum_{\gamma \in \Gamma \setminus \{\mathrm{id}\}} \sigma(z, \gamma z)^{-(k+\varepsilon)} .$$

By means of the decomposition

$$\sum_{\gamma \in \Gamma \setminus \{\mathrm{id}\}} \sigma(z, \gamma z)^{-(k+\varepsilon)} \leq \sum_{\gamma \in \Gamma \setminus \Gamma_{p_j}} \sigma(z, \gamma z)^{-(k+\varepsilon)} + \sum_{\gamma \in \Gamma_{p_j} \setminus \{\mathrm{id}\}} \sigma(z, \gamma z)^{-(k+\varepsilon)} ,$$

we then obtain for $z \in \mathrm{cl}(\mathcal{F}_j^Y \setminus \mathcal{F}_j^{k/(2\pi)})$, using Lemma 4.4 and Lemma 4.5, that

$$\sum_{\gamma \in \Gamma \setminus \{\mathrm{id}\}} \sigma(z, \gamma z)^{-(k+\varepsilon)} \leq B_{k,Y_0}(\varepsilon) + \frac{k\, e^{5/4}}{\sqrt{\pi}\, \sqrt{k+\varepsilon}} ,$$

which yields the upper bound

$$S_{2k}^{\Gamma}(z) \leq \frac{(2k-1+\varepsilon)(1+\varepsilon)}{4\pi} + \frac{3(2k+\varepsilon)(2k-1+\varepsilon)(1+\varepsilon)}{4\pi(k+\varepsilon)} \left(B_{k,Y_0}(\varepsilon) + \frac{k\, e^{5/4}}{\sqrt{\pi}\, \sqrt{k+\varepsilon}} \right).$$

The proof of the theorem now follows by letting $\varepsilon \to 0$. □

The next proposition addresses the case $k = 1$.

Proposition 5.5. *Let $0 < \varepsilon < 1$ and $Y \geq 16/\sqrt{15}$. Then, with B_Y given in Definition 3.5, the upper bound*

$$\sup_{z \in \mathbb{H}} S_2^{\Gamma}(z) \leq \frac{(1+\varepsilon)^2}{4\pi} + \frac{3(1+\varepsilon)^2(2+\varepsilon)}{\varepsilon} B_Y$$

holds.

Proof. From Proposition 5.1, we obtain the upper bound

$$S_2^{\Gamma}(z) \leq \frac{(1+\varepsilon)^2}{4\pi} + \frac{3(1+\varepsilon)(2+\varepsilon)}{4\pi} \sum_{\gamma \in \Gamma \setminus \{\mathrm{id}\}} \sigma(z, \gamma z)^{-(1+\varepsilon)}$$

for $z \in \mathbb{H}$. By means of Lemma 3.7, we then arrive at the upper bound

$$S_2^{\Gamma}(z) \leq \frac{(1+\varepsilon)^2}{4\pi} + \frac{3(1+\varepsilon)^2(2+\varepsilon)}{\varepsilon} B_Y$$

for $z \in \mathcal{F}_Y$. Furthermore, since we have by assumption that $Y \geq 16/\sqrt{15}$, Lemma 4.1, which is also valid for $k = 1$, shows that the same upper bound is valid for $z \in \mathcal{F}_j^Y$ for $j = 1, \ldots, h$. This proves the claim. □

5.3 Lower bounds for the sup-norm of $S_{2k}^{\Gamma}(z)$

In this subsection, we prove lower bounds for the supremum of the quantity $S_{2k}^{\Gamma}(z)$ for z ranging through the compact domain $\mathcal{F}_Y$, as well as for z ranging through the cuspidal neighborhoods $\mathcal{F}_j^Y$.

Proposition 5.6. *Let $g_\Gamma \geq 1$, $k \in \mathbb{N}_{\geq 7}$, and $Y \geq k/(2\pi)$. Then, the lower bound*

$$\sup_{z\in\mathcal{F}_Y} S_{2k}^{\Gamma}(z) \geq \frac{k-1}{2\pi}$$

holds.

Proof. We start from the obvious inequality

$$\sup_{z\in\mathcal{F}} S_{2k}^{\Gamma}(z)\cdot \mathrm{vol}_{\mathrm{hyp}}(M) \geq \int\limits_{\mathcal{F}} S_{2k}^{\Gamma}(z)\mu_{\mathrm{hyp}}(z) = d_{2k},$$

where we recall that

$$\mathrm{vol}_{\mathrm{hyp}}(M) = 2\pi\bigg((2g_\Gamma-2)+h+\sum_{e_j\in\mathcal{E}}\bigg(1-\frac{1}{n_j}\bigg)\bigg),$$
$$d_{2k} = (2k-1)(g_\Gamma-1)+(k-1)h+\sum_{e_j\in\mathcal{E}}\bigg\lfloor k\bigg(1-\frac{1}{n_j}\bigg)\bigg\rfloor.$$

Since $g_\Gamma \geq 1$, we arrive at the lower bound

$$d_{2k} \geq (2k-2)(g_\Gamma-1)+(k-1)h+\sum_{e_j\in\mathcal{E}}\bigg(k\bigg(1-\frac{1}{n_j}\bigg)-\bigg(1-\frac{1}{n_j}\bigg)\bigg)$$
$$=(k-1)\bigg((2g_\Gamma-2)+h+\sum_{e_j\in\mathcal{E}}\bigg(1-\frac{1}{n_j}\bigg)\bigg).$$

From this we derive

$$\sup_{z\in\mathcal{F}} S_{2k}^{\Gamma}(z) \geq \frac{d_{2k}}{\mathrm{vol}_{\mathrm{hyp}}(M)} \geq \frac{k-1}{2\pi}\,.$$

Since $Y \geq k/(2\pi) > 1$, Lemma 4.1 then shows that

$$\sup_{z\in\mathcal{F}_Y} S_{2k}^{\Gamma}(z) = \sup_{z\in\mathcal{F}} S_{2k}^{\Gamma}(z) \geq \frac{k-1}{2\pi}\,,$$

which concludes the proof of the proposition. □

Proposition 5.7. *Let $k \in \mathbb{N}_{\geq 2}$, $\delta > 0$, $Y_0 \geq 8/\sqrt{15}$, and $Y = 2Y_0$. Then, for $k \gg Y$, the lower bounds*

$$\sup_{z\in\mathcal{F}_j^Y} S_{2k}^{\Gamma}(z) = \Omega(k^{3/2-\delta})$$

hold for $j = 1, \ldots, h$.

Proof. Again, we work from formula (2.3) of Lemma 2.1 with $\lambda = s(1-s)$ and $\mu = t(1-t)$ with $s, t \in W_k \cap \mathbb{R}$ such that $t > s > 1$, which reads

$$\sum_{j=0}^{\infty}\bigg(\frac{1}{\lambda_j-\lambda}-\frac{1}{\lambda_j-\mu}\bigg)|\varphi_j(z)|^2+\frac{1}{4\pi}\sum_{j=1}^{h}\int\limits_{-\infty}^{\infty}\bigg(\frac{1}{\frac{1}{4}+r^2-\lambda}-\frac{1}{\frac{1}{4}+r^2-\mu}\bigg)\bigg|E_j\bigg(z,\frac{1}{2}+ir\bigg)\bigg|^2\mathrm{d}r$$
$$=-\frac{1}{4\pi}\big(\psi(s+k)+\psi(s-k)-\psi(t+k)-\psi(t-k)\big)+\sum_{\gamma\in\Gamma\setminus\{\mathrm{id}\}}\bigg(\frac{cz+d}{c\bar{z}+d}\bigg)^k\bigg(\frac{\gamma z-\bar{z}}{z-\gamma\bar{z}}\bigg)^k g_k(s;z,\gamma z).$$

Choosing $t = s+1$ and recalling that the smallest eigenvalue among the λ_j's equals $k(1-k)$, we find that the left-hand side of the above formula as a function of s has a simple pole of order 1 at $s = k$ arising from the summands corresponding to the eigenvalue $k(1-k)$. Therefore, letting

$s = k + \varepsilon$ with $\varepsilon > 0$ and $\lambda_j = k(1-k)$, we obtain after dividing both sides of the above formula by the quantity $r(k,\varepsilon)$ given by (5.1), for each cusp p_j ($j = 1, \ldots, h$) the equality

$$\begin{aligned} S_{2k}^{\Gamma}(z) = &- \lim_{\varepsilon \to 0} \frac{1}{4\pi r(k,\varepsilon)} \big(\psi(2k+\varepsilon) + \psi(\varepsilon) - \psi(2k+1+\varepsilon) - \psi(\varepsilon+1)\big) \\ &+ \lim_{\varepsilon \to 0} \frac{1}{r(k,\varepsilon)} \sum_{\gamma \in \Gamma \setminus \Gamma_{p_j}} \left(\frac{cz+d}{c\bar{z}+d}\right)^k \left(\frac{\gamma z - \bar{z}}{z - \gamma \bar{z}}\right)^k g_k(k+\varepsilon; z, \gamma z) \\ &+ \lim_{\varepsilon \to 0} \frac{1}{r(k,\varepsilon)} \sum_{\gamma \in \Gamma_{p_j} \setminus \{\mathrm{id}\}} \left(\frac{cz+d}{c\bar{z}+d}\right)^k \left(\frac{\gamma z - \bar{z}}{z - \gamma \bar{z}}\right)^k g_k(k+\varepsilon; z, \gamma z). \end{aligned}$$

Formulas (5.1) and (5.2) show that the first summand on the right-hand side of the above formula is of order $O(k)$. Furthermore, since we have assumed that $k \gg Y$, we can suppose that $Y < k/(2\pi)$, and Lemma 4.1 together with Lemma 4.4 shows that for $z \in \mathcal{F}_j^Y$, the second summand of the above formula is also of order $O(k)$. We are thus left to prove that the third summand is of order $\Omega(k^{3/2-\delta})$ for $k \gg Y$. To this end, we let $z = \sigma_j z'$ with the scaling matrix σ_j of the cups p_j, and compute

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{1}{r(k,\varepsilon)} \sum_{\gamma \in \Gamma_{p_j} \setminus \{\mathrm{id}\}} \left(\frac{cz+d}{c\bar{z}+d}\right)^k \left(\frac{\gamma z - \bar{z}}{z - \gamma \bar{z}}\right)^k g_k(k+\varepsilon; \sigma(z, \gamma z)) \\ &= \lim_{\varepsilon \to 0} \frac{1}{r(k,\varepsilon)} \sum_{\gamma' \in \sigma_j^{-1} \Gamma_{p_j} \sigma_j \setminus \{\mathrm{id}\}} \left(\frac{c'z'+d'}{c'\bar{z}'+d'}\right)^k \left(\frac{\gamma' z' - \bar{z}'}{z' - \gamma' \bar{z}'}\right)^k g_k(k+\varepsilon; \sigma(z', \gamma' z')) \\ &= \lim_{\varepsilon \to 0} \frac{1}{r(k,\varepsilon)} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left(\frac{z' - \bar{z}' + n}{z' - \bar{z}' - n}\right)^k g_k(k+\varepsilon; \sigma(z', z'+n)); \end{aligned}$$

recall that $g_k(k+\varepsilon; \sigma(z,\gamma z))$ depends only on $\sigma(z,\gamma z)$. We note now that the latter quantity is independent of the specific Fuchsian subgroup Γ. However, it has been shown in [FJK16, Sec. 7.2] for the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ that the latter quantity is of order $\Omega(k^{3/2-\delta})$ for $k \gg Y$. This completes the proof of the proposition. □

5.4 Explicit computations for the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$

In this subsection, we illustrate how Theorem 5.2 and Theorem 5.4 lead to effective upper bounds for the supremum of the quantity $S_{2k}^{\Gamma}(z)$ in the case of the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. The proof of this result gives rise to an algorithm to determine effective upper bounds for the supremum of the quantity $S_{2k}^{\Gamma}(z)$ for more general Fuchsian subgroups Γ; this algorithm is reproduced in Subsection 6.3 of the Appendix.

Theorem 5.8. *Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, $k \in \mathbb{N}$, and $Y = 16/\sqrt{15} = 4.131...$ Then, the upper bounds*

$$S_{2k}^{\Gamma}(z) \leq \begin{cases} \dfrac{31(2k-1)}{4\pi} + 1090\,(2k-1) 1.014^{-(k-2)} & \text{if } k \geq 2,\ z \in \mathcal{F}_Y, \\ \dfrac{31(2k-1)}{4\pi} + 1090\,(2k-1) 1.014^{-(k-2)} & \text{if } 2 \leq k \leq 25,\ z \in \mathcal{F}_1^Y, \\ \dfrac{2k-1}{4\pi} + \dfrac{9(2k-1)\sqrt{k}}{2\pi} & \text{if } k \geq 26,\ z \in \mathcal{F}_1^Y, \end{cases}$$

hold.

Proof. For the subsequent proof, the reader will have to recall various notations that have been introduced in Section 2.

(1) We start by choosing the standard fundamental domain for the quotient space $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$, which is given as

$$\mathcal{F} = \{z = x + iy \in \mathbb{C} \,|\, |z| \geq 1,\ -1/2 \leq x \leq +1/2\}.$$

Its boundary $\partial\mathcal{F}$ consists of the set of geodesic line segments $\mathcal{S} = \{S_1, S_2, S_3, S_4\}$, where

$$S_1 := \{z = -1/2 + iy \,|\, y \geq \sqrt{3}/2\}, \quad S_2 := \{z = x + iy \,|\, |z| = 1,\ -1/2 \leq x \leq 0\},$$
$$S_3 := \{z = +1/2 + iy \,|\, y \geq \sqrt{3}/2\}, \quad S_4 := \{z = x + iy \,|\, |z| = 1,\ 0 \leq x \leq +1/2\}.$$

Since the minimal positive trace of a hyperbolic element $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ must be at least 3 and since $\gamma = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ is an element having this trace, the length ℓ_Γ of the shortest closed geodesic on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ is easily computed to

$$\ell_\Gamma = 2\cosh^{-1}(3/2) = 1.924...$$

(2) The set of cusps of $\mathcal{F}$ is given by $\mathcal{P} = \{p_1\}$, where $p_1 := i\infty$; for the corresponding scaling matrix we have $\sigma_1 = \mathrm{id}$. The set of elliptic fixed points of $\mathcal{F}$ is given by $\mathcal{E} = \{e_1, e_2, e_3\}$, where

$$e_1 := \frac{-1 + i\sqrt{3}}{2}, \quad e_2 := i, \quad e_3 := \frac{1 + i\sqrt{3}}{2}\,;$$

from this we immediately get that $\theta_\Gamma = 2\pi/3$.

(3) We now choose $Y_0 = 8/\sqrt{15} = 2.065...$ and $Y = 2Y_0 = 16/\sqrt{15} = 4.131...$ From this and the above choices, we get $m_Y = \sqrt{3}/2$ and $M_Y = 4.131...$

(4) In this step we determine the quantity μ_Γ given by (3.1). With the notations of steps (1) and (2), we first need to calculate the hyperbolic distances $\mathrm{dist}_{\mathrm{hyp}}(S_j, e_h)$ for $j \in \{1,2,3,4\}$ and $h \in \{1,2,3\}$ subject to the condition that $e_h \notin S_j$. By symmetry, it suffices to consider the following three cases:

$$\mathrm{dist}_{\mathrm{hyp}}(S_1, e_2), \quad \mathrm{dist}_{\mathrm{hyp}}(S_3, e_1), \quad \mathrm{dist}_{\mathrm{hyp}}(S_4, e_1).$$

In the first case we compute

$$\begin{aligned}\cosh\big(\mathrm{dist}_{\mathrm{hyp}}(S_1, e_2)\big) &= \min_{y \geq \sqrt{3}/2}\left(1 + \frac{|-1/2 + iy - i|^2}{2y}\right)\\ &= \min_{y \geq \sqrt{3}/2}\left(\frac{y}{2} + \frac{5}{8y}\right) = \frac{\sqrt{5}}{2},\end{aligned}$$

which gives

$$\mathrm{dist}_{\mathrm{hyp}}(S_1, e_2) = 0.481...$$

In a similar way, we find in the remaining two cases

$$\begin{aligned}\mathrm{dist}_{\mathrm{hyp}}(S_3, e_1) &= \cosh^{-1}\left(\frac{\sqrt{7}}{\sqrt{3}}\right) = 0.986...\,,\\ \mathrm{dist}_{\mathrm{hyp}}(S_4, e_1) &= \mathrm{dist}_{\mathrm{hyp}}(e_2, e_1) = \cosh^{-1}\left(\frac{2\sqrt{3}}{3}\right) = 0.549...\end{aligned}$$

Using (3.1) we arrive at

$$\mu_\Gamma = \min\{0.481...\,, 0.986...\,, 0.549...\} = 0.481...$$

(5) In this step we estimate the quantity σ_Y given by (3.2) using Lemma 3.4. With the results of steps (1)–(4), we find

$$\begin{aligned}\sigma_Y &\geq \min\left\{\frac{\cosh(\ell_\Gamma) + 1}{2},\ \sinh^2(\mu_\Gamma)\,\sin^2\left(\frac{\theta_\Gamma}{2}\right) + 1,\ \frac{m_Y^2}{4} + 1,\ \frac{1}{4M_Y^2} + 1\right\}\\ &= \min\{2.248..., 1.187..., 1.187..., 1.014...\} \geq 1.014.\end{aligned}$$

(6) In this step we give crude upper bounds for the hyperbolic diameters of $\mathcal{F}_Y$ and $\mathcal{F}_{Y_0}$. In order to estimate $\mathrm{diam}_{\mathrm{hyp}}(\mathcal{F}_Y)$, we consider the rectangle $\mathcal{R} \subset \mathbb{H}$ with vertices

$$\{-1/2 + ia, +1/2 + ia, +1/2 + ib, -1/2 + ib\},$$

where $a = \sqrt{3}/2$ and $b = Y$. For $z, w \in \mathcal{R}$, we then have the bounds

$$|z-w|^2 \leq |(-1/2+ia) - (+1/2+ib)|^2 = 1 + (b-a)^2 \quad \text{and} \quad \mathrm{Im}(z)\mathrm{Im}(w) \geq a^2.$$

Using the formula

$$\cosh\big(\mathrm{dist}_{\mathrm{hyp}}(z,w)\big) = 1 + \frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)}\,,$$

we find the upper bound

$$\mathrm{dist}_{\mathrm{hyp}}(z,w) \leq \cosh^{-1}\left(1 + \frac{1+(b-a)^2}{2a^2}\right),$$

and thus can estimate the hyperbolic diameter of $\mathcal{F}_Y$ as

$$\mathrm{diam}_{\mathrm{hyp}}(\mathcal{F}_Y) \leq 2.861...$$

In a similar way, we find for the hyperbolic diameter of $\mathcal{F}_{Y_0}$ the upper bound

$$\mathrm{diam}_{\mathrm{hyp}}(\mathcal{F}_{Y_0}) \leq 1.620...$$

(7) In this step we compute the hyperbolic volumes of $\mathcal{F}_Y$ and $\mathcal{F}_{Y_0}$. For the hyperbolic volume of $\mathcal{F}_Y$, we obtain

$$\mathrm{vol}_{\mathrm{hyp}}(\mathcal{F}_Y) = \int\limits_{-1/2}^{1/2} \int\limits_{\sqrt{1-x^2}}^{16/\sqrt{15}} \frac{\mathrm{d}y \wedge \mathrm{d}x}{y^2} = \int\limits_{-1/2}^{1/2} \left(\frac{1}{\sqrt{1-x^2}} - \frac{\sqrt{15}}{16}\right)\mathrm{d}x = 0.805...$$

In a similar way, we find for the hyperbolic volume of $\mathcal{F}_{Y_0}$ the result

$$\mathrm{vol}_{\mathrm{hyp}}(\mathcal{F}_{Y_0}) = 0.563...$$

(8) In this step we estimate the quantities B_Y and B_{Y_0} given by (3.3). Applying the results obtained in steps (6) and (7), we get the upper bound

$$B_Y = \frac{e^{3\,\mathrm{diam}_{\mathrm{hyp}}(\mathcal{F}_Y)/2}}{\mathrm{vol}_{\mathrm{hyp}}(\mathcal{F}_Y)} \leq 90.777...$$

In a similar way, we find $B_{Y_0} \leq 20.178...$ Recalling (4.2), we derive from this

$$B_{k,Y_0} = 2\pi\, Y_0^{-4}\, B_{Y_0}\, 4^{-k+3}\left(\frac{k}{2\pi}\right)^4 = \frac{B_{Y_0}\, 4^{-k}\, k^4}{2\pi^3} \leq \frac{1}{2},$$

recalling that $k \geq 2$.

(9) For $k \geq 2$ and $z \in \mathcal{F}_Y$, or for $2 \leq k \leq 25$ and $z \in \mathcal{F}_1^Y$, we obtain from Theorem 5.2, taking into account the bounds obtained in steps (5) and (8), that the upper bound

$$S_{2k}^{\Gamma}(z) \leq \frac{31(2k-1)}{4\pi} + 1090\,(2k-1)1.014^{-(k-2)}$$

holds, which proves the first two parts of the theorem.

(10) Finally, for $k \geq 26$ and $z \in \mathcal{F}_1^Y$, we obtain from Theorem 5.4, taking into account the bounds obtained in step (8), that the upper bound

$$S_k^{\Gamma}(z) \leq \frac{2k-1}{4\pi} + \frac{3(2k-1)}{2\pi}\left(\frac{1}{2} + \frac{\sqrt{k}\,e^{5/4}}{\sqrt{\pi}}\right) \leq \frac{2k-1}{4\pi} + \frac{9(2k-1)\sqrt{k}}{2\pi}$$

holds, which proves the last part of the theorem. □

6 Appendix

For the sake of completeness we collect in this appendix some basic facts about the resolvent kernel and the heat kernel for the hyperbolic Laplacian Δ_k. Furthermore, we provide an effective version of Stirling's formula and end the appendix with an algorithm formalizing the proof of Theorem 5.8.

6.1 The resolvent kernel

In this subsection, we give the basic definitions of the resolvent kernel and the heat kernel for the hyperbolic Laplacian Δ_k, as well as the representation of the resolvent kernel as an integral transform of the heat kernel. Furthermore, we provide an upper bound for the resolvent kernel which is crucial for the main results of this paper.

Definition of the resolvent kernel.

Let $F(a,b;c;Z)$ be the hypergeometric series with variable Z and parameters $a,b,c \in \mathbb{C}$ such that $-c \in \mathbb{N}$ is allowed only if $-a \in \mathbb{N}$ and $a > c$, or if $-b \in \mathbb{N}$ and $b > c$. For $Z \in \mathbb{C}$ with $|Z| < 1$, the hypergeometric series then has the power series expansion (see [MOS66, p. 37])

$$F(a,b;c;Z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{Z^n}{n!}$$

with the Pochhammer symbols $(a)_n = \Gamma(a+n)/\Gamma(a)$ etc..

Following [Fis87, § 1.4] (see also [Els73]), the resolvent kernel $G_k(s;z,w)$ on $\mathbb{H}$ associated to Δ_k ($k \in \mathbb{N}_{\geq 1}$) is defined for $s \in W_k = \mathbb{C} \setminus \{k-n,\, -k-n \,|\, n \in \mathbb{N}\}$ and $z,w \in \mathbb{H}$ by the formula

$$G_k(s;z,w) := G_k(s;\sigma(z,w)),$$

where

$$\sigma(z,w) = \cosh^2\left(\frac{\mathrm{dist}_{\mathrm{hyp}}(z,w)}{2}\right)$$

is the displacement function (2.1) and

$$G_k(s;\sigma) := \frac{1}{\sigma^s} \frac{\Gamma(s+k)\Gamma(s-k)}{4\pi\Gamma(2s)} F\left(s+k, s-k; 2s; \frac{1}{\sigma}\right) \qquad (\sigma \geq 1).$$

We note that the function $G_k(s;z,w)$ is denoted by $\mathfrak{K}_s(\sigma(z,w))$ in [Fis87]; furthermore, we note that in [Fis87] the function

$$\frac{1}{2} \sum_{\substack{\gamma \in \Gamma \setminus \{\mathrm{id}\} \\ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}} \left(\frac{cz+d}{c\bar{z}+d}\right)^k \left(\frac{\gamma z - \bar{z}}{z - \gamma\bar{z}}\right)^k G_k(s;z,\gamma z)$$

is denoted by $G_{k\,\lambda}(z,w)$.

Definition of the heat kernel.

Following [Osh90], correcting a corresponding formula in [Fay77], the heat kernel $K_k(t;z,w)$ on $\mathbb{H}$ associated to Δ_k ($k \in \mathbb{N}_{\geq 1}$) is defined for $t \in \mathbb{R}_{\geq 0}$ and $z,w \in \mathbb{H}$ by the formula

$$K_k(t;z,w) := K_k(t;\mathrm{dist}_{\mathrm{hyp}}(z,w)),$$

where

$$K_k(t;\rho) := \frac{\sqrt{2}\, e^{-t/4}}{(4\pi t)^{3/2}} \int\limits_{\rho}^{\infty} \frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r)-\cosh(\rho)}}\, T_{2k}\left(\frac{\cosh(r/2)}{\cosh(\rho/2)}\right) \mathrm{d}r \qquad (\rho \geq 0),$$

with

$$T_{2k}(X) := \cosh(2k \operatorname{arccosh}(X))$$

denoting the $2k$-th Chebyshev polynomial. In [FJK16], we have shown that $K_k(t;\rho)$ is a monotone decreasing function of ρ and that the inequality

$$T_{2k}\big(\cosh(r/2)\big) \leq e^{kr} \tag{6.1}$$

holds. Using the upper bound (6.1), we then derive for later purposes for $\rho \geq 0$, the estimate

$$\begin{aligned} K_k(t;\rho) \leq K_k(t;0) &= \frac{\sqrt{2}\, e^{-t/4}}{(4\pi t)^{3/2}} \int\limits_0^{\infty} \frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r)-1}}\, T_{2k}\big(\cosh(r/2)\big)\, \mathrm{d}r \\ &= \frac{e^{-t/4}}{(4\pi t)^{3/2}} \int\limits_0^{\infty} \frac{re^{-r^2/(4t)}}{\sinh(r/2)}\, T_{2k}\big(\cosh(r/2)\big)\, \mathrm{d}r \leq \frac{e^{-t/4}}{(4\pi t)^{3/2}} \int\limits_0^{\infty} \frac{re^{-r^2/(4t)}}{\sinh(r/2)}\, e^{kr}\, \mathrm{d}r \\ &\leq \frac{C_\delta\, e^{-t/4}}{t^{3/2}} \int\limits_0^{\infty} e^{(\delta-1/2)r} e^{-r^2/(4t)} e^{kr}\, \mathrm{d}r \leq \frac{C'_\delta\, e^{-t/4}}{t}\, e^{t(k-1/2+\delta)^2}; \end{aligned} \tag{6.2}$$

here $\delta > 0$ is arbitrarily small and the positive constants C_δ, C'_δ depend solely on δ.

Resolvent kernel as an integral transform of the heat kernel.

The resolvent kernel $G_k(s;z,w)$ on $\mathbb{H}$ associated to Δ_k can be represented as an integral transform of the heat kernel $K_k(t;z,w)$ on $\mathbb{H}$ associated to Δ_k; the precise relationship is given as

$$G_k(s;\sigma) = \int\limits_0^{\infty} e^{-(s-1/2)^2 t} e^{t/4} K_k(t;\rho)\, \mathrm{d}t, \tag{6.3}$$

where $\sigma = \cosh^2(\rho/2)$. We note that by (6.2), formula (6.3) is valid for $\mathrm{Re}(s) \geq k + \varepsilon$ for any $\varepsilon > 0$. We emphasize that we will be able to obtain useful estimates for the resolvent kernel by viewing it as the integral transform (6.3) of the heat kernel and applying some of the estimates that have been established in [FJK16].

Next, we recall the function $g_k(s;z,w)$, which has been defined for $s \in W_k$ and $z, w \in \mathbb{H}$ by means of formula (2.2); in the present notation this leads to

$$g_k(s;\sigma) := G_k(s;\sigma) - G_k(s+1;\sigma).$$

Using (6.3), the function $g_k(s;\sigma)$ can be rewritten as

$$g_k(s;\sigma) = \int\limits_0^{\infty} \big(e^{-(s-1/2)^2 t} - e^{-(s+1/2)^2 t}\big) e^{t/4} K_k(t;\rho)\, \mathrm{d}t;$$

again, we have $\sigma = \cosh^2(\rho/2)$.

Estimates for the resolvent kernel.

Letting $a, b \in \mathbb{R}$ with $b \neq 0$ and using the formula

$$\int_0^\infty t^{-3/2} e^{-a^2 t - b^2/(4t)} \,\mathrm{d}t = \frac{2\sqrt{\pi} e^{-ab}}{b},$$

we compute for $s \geq k + \varepsilon$ with $\varepsilon > 0$

$$\begin{aligned} g_k(s;\sigma) &= \int_0^\infty \big(e^{-(s-1/2)^2 t} - e^{-(s+1/2)^2 t}\big) e^{t/4} K_k(t;\rho) \,\mathrm{d}t \\ &= \int_0^\infty \int_\rho^\infty \frac{\sqrt{2}}{(4\pi t)^{3/2}} \big(e^{-(s-1/2)^2 t} - e^{-(s+1/2)^2 t}\big) \frac{r e^{-r^2/(4t)}}{\sqrt{\cosh(r) - \cosh(\rho)}} T_{2k}\left(\frac{\cosh(r/2)}{\cosh(\rho/2)}\right) \mathrm{d}r \,\mathrm{d}t \\ &= \frac{1}{2\pi\sqrt{2}} \int_\rho^\infty \frac{e^{-(s-1/2)r} - e^{-(s+1/2)r}}{\sqrt{\cosh(r) - \cosh(\rho)}} T_{2k}\left(\frac{\cosh(r/2)}{\cosh(\rho/2)}\right) \mathrm{d}r. \end{aligned} \tag{6.4}$$

To establish the crucial upper bound for the function $g_k(s;\sigma)$, we need the following lemma.

Lemma 6.1. *Let $k \in \mathbb{N}_{\geq 1}$ and $0 < \varepsilon < 1$. Then, for $s = k + \varepsilon$, the upper bound*

$$\int_\rho^\infty \frac{\big(e^{-(s-1/2)r} - e^{-(s+1/2)r}\big) e^{kr}}{\sqrt{\cosh(r) - \cosh(\rho)}} \,\mathrm{d}r \leq \frac{3\sqrt{2}}{\varepsilon} e^{-\varepsilon\rho}$$

holds.

Proof. Since $s = k + \varepsilon$, we have to estimate the function

$$F(\rho) := \int_\rho^\infty \frac{e^{-ar} - e^{-br}}{\sqrt{\cosh(r) - \cosh(\rho)}} \,\mathrm{d}r,$$

where $a := -1/2 + \varepsilon$ and $b := a + 1 = 1/2 + \varepsilon$. Using integration by parts, we then obtain

$$\begin{aligned} F(\rho) &= 2 \int_\rho^\infty \frac{e^{-ar} - e^{-br}}{\sinh(r)} \frac{\mathrm{d}}{\mathrm{d}r} \big(\cosh(r) - \cosh(\rho)\big)^{1/2} \,\mathrm{d}r \\ &= -4 \int_\rho^\infty \big(\cosh(r) - \cosh(p)\big)^{1/2} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{e^{-ar} - e^{-br}}{e^r - e^{-r}}\right) \mathrm{d}r. \end{aligned}$$

With the above choices of a and b, we compute

$$\frac{e^{-ar} - e^{-br}}{e^r - e^{-r}} = \frac{e^{-\varepsilon r}(e^{r/2} - e^{-r/2})}{e^r - e^{-r}} = \frac{e^{-\varepsilon r}}{e^{r/2} + e^{-r/2}}.$$

Hence, we get

$$F(\rho) = -2 \int_\rho^\infty \big(\cosh(r) - \cosh(\rho)\big)^{1/2} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{e^{-\varepsilon r}}{\cosh(r/2)}\right) \mathrm{d}r.$$

Observing that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{e^{-\varepsilon r}}{\cosh(r/2)}\right) \leq 0$$

for $\rho \leq r < \infty$, and using the estimate

$$\big(\cosh(r) - \cosh(\rho)\big)^{1/2} \leq \big(\cosh(r) - 1\big)^{1/2} = \sqrt{2}\sinh(r/2),$$

we arrive at the upper bound

$$\begin{aligned} F(\rho) &\leq -2\sqrt{2}\int_{\rho}^{\infty} \sinh(r/2)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{e^{-\varepsilon r}}{\cosh(r/2)}\right)\mathrm{d}r \\ &= 2\sqrt{2}\,e^{-\varepsilon\rho}\tanh(\rho/2) + \frac{\sqrt{2}\,e^{-\varepsilon\rho}}{\varepsilon}; \end{aligned}$$

for the last equality, we used integration by parts once again. Since $0 < \varepsilon < 1$, we complete the proof of the lemma by employing the crude upper bound $\tanh(\rho/2) < 1/\varepsilon$. □

Lemma 6.2. *Let $k \in \mathbb{N}_{\geq 1}$ and $0 < \varepsilon < 1$. Then, the upper bound*

$$|g_k(k+\varepsilon;\sigma)| \leq \frac{3}{2\pi\varepsilon}\,\sigma^{-(k+\varepsilon)}$$

holds.

Proof. In [FJK16], we have shown that

$$T_{2k}\left(\frac{\cosh(r/2)}{\cosh(\rho/2)}\right) = \cosh\left(2k\operatorname{arccosh}\left(\frac{\cosh(r/2)}{\cosh(\rho/2)}\right)\right) \leq \frac{e^{kr}}{\cosh^{2k}(\rho/2)}.$$

Hence, using formula (6.4), the above estimate, and Lemma 6.1, we obtain the upper bound

$$\begin{aligned} 0 \leq \; g_k(s,\sigma) &= \frac{1}{2\pi\sqrt{2}}\int_{\rho}^{\infty} \frac{e^{-(s-1/2)r} - e^{-(s+1/2)r}}{\sqrt{\cosh(r) - \cosh(\rho)}}\,T_{2k}\left(\frac{\cosh(r/2)}{\cosh(\rho/2)}\right)\mathrm{d}r \\ &\leq \frac{1}{2\pi\sqrt{2}}\int_{\rho}^{\infty} \frac{e^{-(s-1/2)r} - e^{-(s+1/2)r}}{\sqrt{\cosh(r) - \cosh(\rho)}}\,\frac{e^{kr}}{\cosh^{2k}(\rho/2)}\,\mathrm{d}r \\ &= \frac{1}{2\pi\sqrt{2}\cosh^{2k}(\rho/2)}\int_{\rho}^{\infty} \frac{\big(e^{-(s-1/2)r} - e^{-(s+1/2)r}\big)\,e^{kr}}{\sqrt{\cosh(r) - \cosh(\rho)}}\,\mathrm{d}r \\ &\leq \frac{3}{2\pi\varepsilon}\,\frac{e^{-\varepsilon\rho}}{\cosh^{2k}(\rho/2)}. \end{aligned}$$

Recalling that $\sigma = \cosh^2(\rho/2) \leq e^{\rho}$, we easily conclude the proof of the lemma. □

6.2 Effective version of Stirling's formula

In this subsection, we provide an effective version of Stirling's formula.

Lemma 6.3. *Let $Z \geq 1$. Then, the upper bound*

$$\frac{\Gamma(Z - 1/2)}{\Gamma(Z)} \leq \frac{e^{5/4}}{\sqrt{Z}}$$

holds.

Proof. We set

$$R(Z) := \log(\Gamma(Z)) - \left(Z - \frac{1}{2}\right)\log(Z) + Z - \frac{1}{2}\log(2\pi).$$

It follows that

$$\begin{aligned}\log\left(\frac{\Gamma(Z-1/2)}{\Gamma(Z)}\right)\\ &= R\big(Z-\tfrac{1}{2}\big) - R(Z) + Z\big(\log\left(Z-\tfrac{1}{2}\right) - \log(Z)\big) - \log\left(Z-\tfrac{1}{2}\right) + \tfrac{1}{2}\log(Z) + \tfrac{1}{2}\\ &= R\big(Z-\tfrac{1}{2}\big) - R(Z) + Z\big(\log\left(Z-\tfrac{1}{2}\right) - \log(Z)\big) + \log(Z) - \log\left(Z-\tfrac{1}{2}\right) - \tfrac{1}{2}\log(Z) + \tfrac{1}{2}.\end{aligned}$$

Now, we estimate the last expression as follows: From [AS72, p. 257, eq. (6.1.42)], we recall the inequality $0 < R(Z) \leq 1/(12Z)$, which gives for $Z \geq 1$

$$R\left(Z - \frac{1}{2}\right) - R(Z) \leq \left|R\left(Z - \frac{1}{2}\right) - R(Z)\right| \leq \frac{1}{12(Z-1/2)} + \frac{1}{12Z} \leq \frac{3}{12}.$$

Next, using the power series expansion of the logarithm, we get the estimate

$$\log\left(Z - \frac{1}{2}\right) - \log(Z) = \log\left(1 - \frac{1}{2Z}\right) \leq -\frac{1}{2Z}.$$

Finally, using that

$$\log(Z) - \log\left(Z - \frac{1}{2}\right) = -\log\left(1 - \frac{1}{2Z}\right) \leq \log(2) < 1,$$

we find the upper bound

$$\log\left(\frac{\Gamma(Z-1/2)}{\Gamma(Z)}\right) \leq \frac{1}{4} - \frac{Z}{2Z} + 1 - \frac{1}{2}\log(Z) + \frac{1}{2} = \frac{5}{4} - \frac{1}{2}\log(Z),$$

which concludes the proof of the lemma. □

6.3 The algorithm

In this subsection, following the proof of Theorem 5.8, we reproduce an algorithm that determines effective sup-norm bounds for $S_{2k}^{\Gamma}(z)$ for general Fuchsian subgroups Γ.

```
(1) Determine a closed and connected fundamental domain F of Γ.
(2) Determine the set S of geodesic line segments forming ∂F.
(3) Determine the length ℓ_Γ of the shortest closed geodesic on Γ\H.
(4) Determine the set P of cusps of F and their scaling matrices.
(5) Determine the set E of elliptic fixed points in F and their orders.
(6) Choose Y_0 ≥ 8/√15 and Y = 2Y_0, and determine m_Y and M_Y.
(7) Determine an upper bound for the quantity μ_Γ given by (3.1).
(8) Determine a lower bound for the quantity σ_Y given by (3.2).
(9) Determine the hyperbolic diameters of F_Y and F_{Y_0}.
(10) Determine the hyperbolic volumes of F_Y and F_{Y_0}.
```

(11) Determine upper bounds for the quantities B_Y and B_{Y_0} given by (3.3), as well as for the quantity B_{k,Y_0} given by (4.2).

(12) For $k \geq 2$ and $z \in \mathcal{F}_Y$, or for $2 \leq k \leq 4\pi Y_0$ and $z \in \mathcal{F}_j^Y$, determine an upper bound for $S_{2k}^{\Gamma}(z)$ using Theorem 5.2.

(13) For $k > 4\pi Y_0$ and $z \in \mathcal{F}_j^Y$, determine an upper bound for $S_{2k}^{\Gamma}(z)$ using Theorem 5.4.

References

[AU95] A. Abbes and E. Ullmo: *Comparaison des métriques d'Arakelov et de Poincaré sur $X_0(N)$*. Duke Math. J. **80** (1995), 295–307.

[AS72] M. Abramowitz and I.A. Stegun, et al.: *Handbook of mathematical functions*. http://people.math.sfu.ca/∼cbm/aands/abramowitz_and_stegun.pdf.

[AMM16] H. Auvray, X. Ma, and G. Marinescu: *Bergman kernels on punctured Riemann surfaces.* https://arxiv.org/abs/1604.06337.

[Bea95] A.F. Beardon: *The geometry of discrete groups*. Graduate Texts in Mathematics, vol. 91. Springer-Verlag, New York, 1995.

[BF14] R.J. Berman and G. Freixas i Montplet: *An arithmetic Hilbert–Samuel theorem for singular hermitian line bundles and cusp forms*. Compos. Math. **150** (2014), no. 10, 1703–1728.

[BH10] V. Blomer and R. Holowinsky: *Bounding sup-norms of cusp forms of large level*. Invent. Math. **179** (2010), 645–681.

[Cha77] F.R. Chang: *On the diameters of compact Riemann surfaces*. Proc. Amer. Math. Soc. **65** (1977), 274–276.

[Els73] J. Elstrodt: *Resolvent of eigenvalue problem of automorphic forms in hyperbolic planes. 1*. Math. Ann. **203** (1973), 295–330.

[EGM98] J. Elstrodt, F. Grunewald, and J. Mennicke: *Groups acting on hyperbolic space. Harmonic analysis and number theory.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998,

[Fad69] L.D. Faddeev: *The eigenfunction expansion of Laplace's operator on the fundamental domain of a discrete group on the Lobačevskiĭ plane*. Transactions Moscow Math. Soc. **17** (1969), 357–386.

[Fay77] J.D. Fay: *Fourier coefficients of the resolvent for a Fuchsian group*. J. reine angew. Math. **293** (1977), 143–203.

[Fis87] J. Fischer: *An approach to the Selberg trace formula via the Selberg zeta function*. Lecture Notes in Math., vol. 1253. Springer-Verlag, Berlin, 1987.

[FJK16] J.S. Friedman, J. Jorgenson, and J. Kramer: *Uniform sup-norm bounds on average for cusp forms of higher weights*. In: W. Ballmann et al. (eds.), "Arbeitstagung Bonn 2013. In memory of Friedrich Hirzebruch". Progress in Mathematics, vol. 319, 127–154. Birkhäuser, Basel, 2016.

[GR81] I. Gradshteyn and I. Ryzhik: *Tables of integrals, series, and products*. Academic Press, New York, 1981.

[HT13] G. Harcos and N. Templier: *On the sup-norm of Maass cusp forms of large level. III*. Math. Ann. **356** (2013), 209–216.

[Iwa02] H. Iwaniec: *Spectral methods of automorphic forms*. Graduate Studies in Mathematics, vol. 53. AMS, Providence, 2002.

[Jav14] A. Javanpeykar: *Polynomial bounds for Arakelov invariants of Belyi curves*. With an appendix by P. Bruin. Algebra Number Theory **8** (2014), no. 1, 89–140.

[Jav16] A. Javanpeykar: *An effective Arakelov-theoretic version of the hyperbolic isogeny theorem*. Math. Proc. Cambridge Philos. Soc. **160** (2016), no. 3, 463–476.

[JK04] J. Jorgenson and J. Kramer: *Bounding the sup-norm of automorphic forms*. Geom. Funct. Anal. **14** (2004), 1267–1277.

[JK11] J. Jorgenson and J. Kramer: *Sup-norm bounds for automorphic forms and Eisenstein series*. In: J. Cogdell et al. (eds.), "Arithmetic Geometry and Automorphic Forms". Advanced Lectures in Mathematics, vol. 19, 407–444. Higher Education Press and International Press, Beijing-Boston, 2011.

[JK14] J. Jorgenson and J. Kramer: *Effective bounds for Faltings's delta function*. Ann. Fac. Sci. Toulouse, Math., Sér. 6, Vol. XXIII (2014), 665–698.

[Lan85] S. Lang: $\mathrm{SL}_2(\mathbf{R})$. Graduate Texts in Mathematics, vol. 105. Springer-Verlag, New York, 1985.

[MOS66] W. Magnus, F. Oberhettinger, and R.P. Soni: *Formulas and theorems for the special functions of mathematical physics*. Grundlehren der mathematischen Wissenschaften, vol. 52, 3rd enlarged edition. Springer-Verlag, New York, 1966.

[MU98] P. Michel and E. Ullmo: *Points de petite hauteur sur les courbes modulaires* $X_0(N)$. Invent. Math. **131** (1998), 645–674.

[Osh90] K. Oshima: *Completeness relations for Maass Laplacians and heat kernels on the super Poincaré upper half-plane*. J. Math. Physics **31** (1990), no. 12, 3060–3063.

[Roe66] W. Roelcke: *Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene. I, II*. Math. Ann. **167** (1966), 292–337 and **168** (1966), 261–324.

[Sil86] J. Silverman: *Heights and elliptic curves*. In: G. Cornell and J. Silverman (eds.), "Arithmetic Geometry", 253–265. Springer-Verlag, New York,1986.

[Tem15] N. Templier: *Hybrid sup-norm bounds for Hecke-Maass cusp forms*. J. Eur. Math. Soc. (JEMS) **17** (2015), 2069–2082.

Joshua S. Friedman
Department of Mathematics and Science
UNITED STATES MERCHANT MARINE ACADEMY
300 Steamboat Road
Kings Point, NY 11024
U.S.A.
e-mail: FriedmanJ@usmma.edu

Jay Jorgenson
Department of Mathematics
The City College of New York
Convent Avenue at 138th Street
New York, NY 10031
U.S.A.
e-mail: jjorgenson@mindspring.com

Jürg Kramer
Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6
D-10099 Berlin
Germany
e-mail: kramer@math.hu-berlin.de