On formal Fourier–Jacobi expansions revisited

Dedicated to Stephen Kudla at his 70th Birthday

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1 Introduction

1.1. Background. It is a well-known phenomenon that generating series arising from some arithmetic, geometric, or topological context, often turn out to correspond to Fourier expansions of elliptic modular forms or other types of automorphic forms. The most prominent example possibly is the Shimura–Taniyama conjecture proven by A. Wiles in [14]. Another prominent example has been provided by the generating series defined by intersection numbers of so-called Hirzebruch–Zagier cycles on Hilbert modular surfaces in the work [6] by F. Hirzebruch and D. Zagier. Subsequently, the latter example was vastly generalized in the work [10] on intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables by S. Kudla and J. Millson. More recently, in the framework of the so-called Kudla Program, generating series constructed by means of arithmetic intersection numbers have been investigated and conjecturally turn out to give rise to derivatives of automorphic forms. In summary, we speak in this context about modularity results or modularity conjectures.

In this note, we revisit the modularity conjecture for Shimura varieties associated to orthogonal groups of signature $(n, 2)$ stated by S. Kudla in [8] or, rather, its reformulation by W. Zhang in [15]. The reformulation by W. Zhang led to the question if a formal Fourier–Jacobi expansion satisfying some necessary symmetry condition gives in fact rise to a Siegel modular form. This question was positively answered in the work [3] by J. Bruinier and M. Raum. It was the author's curiosity to understand and reinterprete this modularity result in the framework of the arithmetic compactifications given by G. Faltings and C.-L. Chai in [4], which eventually led to this note. We claim no significant originality here, possibly the transfer of the modularity result from the complex category to the arithmetic setting might be of interest though the work [12] by M. Raum and O. Richter also sheds some light in this direction. Anyhow, we hope that our revisiting of this circle of problems finds the interest by the honoree.

1.2. Result of Bruinier–Raum. We let $\mathbb{H}_g$ denote the Siegel upper half-space of degree g, which allows an action of the Siegel modular group $\Gamma_g := \mathrm{Sp}_g(\mathbb{Z})$ by fractional linear transformations. It is known that the quotient space $\Gamma_g\backslash\mathbb{H}_g$ equals the moduli space of principally polarized abelian varieties of dimension g over $\mathbb{C}$. We denote by $M_k(\Gamma_g)$ the $\mathbb{C}$-vector space of Siegel modular forms of weight k for Γ_g, whose elements can be interpreted as global sections of the k-th power of the Hodge bundle on $\Gamma_g\backslash\mathbb{H}_g$.

A Siegel modular form $f \in M_k(\Gamma_g)$ is now known to have various expansions. For example, writing $\mathbb{H}_g \ni Z = \left(\begin{smallmatrix} \tau & z \\ z^t & \tau' \end{smallmatrix}\right)$ with $\tau \in \mathbb{H}_{g-1}, \tau' \in \mathbb{H}_1$, and $z \in \mathbb{C}^{g-1}$, then f has a so-called Fourier–Jacobi expansion (in codimension 1), which is of the form

$$f(Z) = \sum_{m\in\mathbb{N}} f_m(\tau, z)\, q'^m \quad (q' := \exp(2\pi i\tau')), \tag{1}$$

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with $f_m \in J_{k,m}(\Gamma_{g-1})$, where $J_{k,m}(\Gamma_{g-1})$ denotes the $\mathbb{C}$-vector space of Siegel–Jacobi forms of weight k and index m for the Jacobi group $\Gamma_{g-1} \ltimes \mathbb{Z}^{g-1}$, i.e., the f_m's are holomorphic functions on $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1}$ with a suitable transformation behavior with respect to $\Gamma_{g-1} \ltimes \mathbb{Z}^{g-1}$.

Substituting the Fourier expansions of the Siegel–Jacobi forms f_m, i.e.,

$$f_m(\tau,z) = \sum_{\substack{n\in\mathrm{Sym}_{g-1}(\mathbb{Q}),\,\text{half-integral}\\ r\in\mathbb{Z}^{g-1},\,4mn-rr^t\geq 0}} c_m(n,r)\exp(2\pi i(\mathrm{tr}(n\tau)+r^t z)), \tag{2}$$

into (1), yields another expansion of the Siegel modular form f under consideration, namely its Fourier expansion

$$f(Z) = \sum_{\substack{N\in\mathrm{Sym}_g(\mathbb{Q}),\,\text{half-integral}\\ N\geq 0}} c(N)\exp(2\pi i\mathrm{tr}(NZ)); \tag{3}$$

here we have set $c(N) := c_m(n,r)$ with $N := \begin{pmatrix} n & r/2 \\ r^t/2 & m\end{pmatrix}$. As a consequence of the modularity of f, we note that the Fourier coefficients in (3) satisfy the invariance condition

$$c(u^t N u) = c(N) \tag{4}$$

for all $u \in \mathrm{GL}_g(\mathbb{Z})$.

The main result of the paper [3] now states that given a *formal* Fourier–Jacobi expansion

$$\sum_{m\in\mathbb{N}} f_m(\tau,z)\,q'^m \tag{5}$$

with $f_m \in J_{k,m}(\Gamma_{g-1})$ and such that the Fourier coefficients $c_m(n,r)$ of their Fourier expansions (2) satisfy the invariance condition (4), then the formal expansion (5) is the Fourier–Jacobi expansion of a Siegel modular form f as described above.

The proof proceeds in the analytic category and starts with the observation that the formal expansion (5) transforms like a Siegel modular form of weight k for Γ_g. In order to prove the convergence of the formal expansion (5), the authors first establish its local convergence and subsequently its analytic continuation to $\mathbb{H}_g$ up to a divisor. The convergence is finally extended to the whole of $\mathbb{H}_g$ using that the Picard group of the minimal compactification of $\Gamma_g\backslash\mathbb{H}_g$ is generated by the Hodge bundle.

1.3. Main result. Our main result now states that the main theorem of J. Bruinier and M. Raum from [3] continues to hold in the arithmetic setting over the ring of integers $\mathbb{Z}$.

As mentioned in the beginning, our approach will be based on the arithmetic compactifications of the moduli space of principally polarized abelian varieties established by G. Faltings and C.-L. Chai in [4]. Therefore, in section 2, we will provide a review of the main ingredients for the construction of the arithmetic compactifications of such moduli spaces. In section 3, we will then recall the definition of arithmetic modular forms and arithmetic Jacobi forms, as well as the reinterpretation of the Fourier–Jacobi expansion of arithmetic modular forms given in [4]. With these recollections at hand, we will then be able to give the proof of our main result in section 4.

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2 Review of arithmetic compactifications

2.1. Toroidal compactification of $\mathscr{A}_g$. Our basic reference for this section is chapter IV of [4]. We let $\mathscr{A}_g/\mathbb{Z}$ denote the moduli stack of principally polarized abelian schemes of dimension g over $\mathbb{Z}$. We further let $\pi\colon \mathscr{B}_g \to \mathscr{A}_g$ be the universal abelian scheme over $\mathbb{Z}$, and $\varepsilon\colon \mathscr{A}_g \to \mathscr{B}_g$ the zero section. In order to describe a smooth toroidal compactification $\overline{\mathscr{A}}_g/\mathbb{Z}$ of $\mathscr{A}_g/\mathbb{Z}$, we need the following data.

We let $X := \mathbb{Z}^g$ and $X_\mathbb{R} := X \otimes_\mathbb{Z} \mathbb{R}$. We denote by $B(X_\mathbb{R})$ the space of all symmetric bilinear forms on $X_\mathbb{R}$ with integral structure determined by the lattice $B(X)$ of all integral-valued symmetric bilinear forms on X. We denote by $C(X) \subset B(X_\mathbb{R})$ the convex cone of all positive semi-definite symmetric bilinear forms on $X_\mathbb{R}$ whose radicals are defined over $\mathbb{Q}$. Furthermore, we let $C(X)^\circ$ denote the interior of $C(X)$, i.e., the convex open cone of all positive definite symmetric bilinear forms on $X_\mathbb{R}$. We note that the natural action of $\mathrm{GL}(X) = \mathrm{GL}_g(\mathbb{Z})$ on X induces actions on $B(X)$ and $B(X_\mathbb{R})$, which stabilize $C(X)$, $C(X)^\circ$, and preserve the integral structure.

A $\mathrm{GL}(X)$-*admissible polyhedral cone decomposition* $\mathfrak{C}$ *of* $C(X)$ consists of a collection $\{\sigma_\alpha\}_{\alpha\in J}$ for some index set J such that the following properties are satisfied:

- Each $\sigma_\alpha \in \mathfrak{C}$ is a non-degenerate rational polyhedral cone such that there exist $v_1, \ldots, v_k \in B(X) \otimes_\mathbb{Z} \mathbb{Q}$ with $\sigma_\alpha = \mathbb{R}_{>0} \cdot v_1 + \ldots + \mathbb{R}_{>0} \cdot v_k$ and σ_α does not contain any line.
- The convex cone $C(X)$ is the disjoint union of all the σ_α's with α running through J. The closure of each σ_α is a disjoint union of finitely many σ_β's.
- The collection $\{\sigma_\alpha\}_{\alpha\in J}$ is invariant under the action of $\mathrm{GL}(X)$ on $B(X_\mathbb{R})$ and there are only finitely many cones modulo the action of $\mathrm{GL}(X)$.

Moreover, we ask that our $\mathrm{GL}(X)$-admissible polyhedral cone decomposition $\mathfrak{C} = \{\sigma_\alpha\}_{\alpha\in J}$ of $C(X)$ is *smooth*, which means that each σ_α is generated by part of a $\mathbb{Z}$-basis of $B(X)$, i.e., $\sigma_\alpha = \mathbb{R}_{>0} \cdot v_1 + \ldots + \mathbb{R}_{>0} \cdot v_k$, where $v_1, \ldots, v_k \in B(X)$ and $v_1, \ldots, v_k$ can be extended to a basis of $B(X)$.

A smooth toroidal compactification $\overline{\mathscr{A}}_g/\mathbb{Z}$ of $\mathscr{A}_g/\mathbb{Z}$ then depends on the choice of a $\mathrm{GL}(X)$-admissible polyhedral cone decomposition $\mathfrak{C} = \{\sigma_\alpha\}_{\alpha\in J}$ of $C(X)$, which is also smooth. We therefore write $\overline{\mathscr{A}}_g = \overline{\mathscr{A}}_g(\{\sigma_\alpha\})$.

2.2. More detailed description of the boundary components of $\overline{\mathscr{A}}_g$. In order to describe the boundary components of $\overline{\mathscr{A}}_g$ in more detail, we need the following additional notations explained with more details in section IV.3 of [4]: For a cone $\sigma \in \mathfrak{C}$, we let $X \to X_\sigma$ denote the smallest quotient of X such that all elements of σ factor through $X_\mathbb{R} \to X_{\sigma,\mathbb{R}} := X_\sigma \otimes_\mathbb{Z} \mathbb{R}$; by abuse of notation, we view $B(X_{\sigma,\mathbb{R}})$ as a subset of $B(X_\mathbb{R})$. We let $C(X_\sigma) := C(X) \cap B(X_{\sigma,\mathbb{R}})$ be the cone of all positive semi-definite symmetric bilinear forms on $X_{\sigma,\mathbb{R}}$ whose radicals are defined over $\mathbb{Q}$ with interior $C(X_\sigma)^\circ$. The cone $C(X_\sigma)$ becomes a rational boundary component of $C(X)$ and, by reduction theory, the set $\mathfrak{C}_\sigma := \{\sigma_\beta \in \mathfrak{C} \,|\, \sigma_\beta \subset B(X_{\sigma,\mathbb{R}})\}$ gives a $\mathrm{GL}(X_\sigma)$-admissible polyhedral cone decomposition of $C(X_\sigma)$. In order to simultaneously treat all the cones $\sigma \in \mathfrak{C}$ with smallest quotient X_σ of X of a fixed $\mathbb{Z}$-rank r ($0 \le r \le g$), we write $X_\xi := \mathbb{Z}^r$ to represent the isomorphism class determined by the $\mathbb{Z}$-modules X_σ, and we further introduce the notation

$$C(X_\xi) := \bigcup_{\substack{\sigma\in\mathfrak{C} \\ \mathrm{rk}_\mathbb{Z}(X_\sigma)=r}} C(X_\sigma) \qquad \text{and} \qquad \mathfrak{C}_\xi := \bigcup_{\substack{\sigma\in\mathfrak{C} \\ \mathrm{rk}_\mathbb{Z}(X_\sigma)=r}} \mathfrak{C}_\sigma,$$

and call $C(X_\sigma)$ a rational boundary component of rank r of $C(X)$. In other words, $C(X_\xi)$ is the union of all rational boundary components of rank r of $C(X)$, which is equipped with the $\mathrm{GL}(X_\xi)$-admissible polyhedral cone decomposition $\mathfrak{C}_\xi$.

Using the the $\mathrm{GL}(X_\xi)$-admissible polyhedral cone decomposition $\mathfrak{C}_\xi$ constructed above and considering the $r(r+1)/2$-dimensional torus $E_\xi := B(X_\xi) \otimes_\mathbb{Z} \mathbb{G}_\mathrm{m}$, we obtain the torus embedding $\overline{E}_\xi := \bigcup_{\sigma_\beta\in\mathfrak{C}_\xi} E(\sigma_\beta)$ with $E(\sigma_\beta)$ being the E_ξ-invariant torus embedding defined by σ_β, and the locally closed subscheme $Z_\xi := \bigcup_{\sigma_\beta\in\mathfrak{C}_\xi} Z(\sigma_\beta)$ given by the E_ξ-orbits $Z(\sigma_\beta)$ determined by σ_β. We then use these data to construct a suitable E_ξ-bundle $\mathscr{E}_\xi$ over the r-fold product of the universal abelian scheme $\mathscr{B}_{g-r}$ with itself, and further define (using contraction products)

$$\overline{\mathscr{E}}_\xi := \mathscr{E}_\xi \times^{E_\xi} \overline{E}_\xi = \bigcup_{\sigma_\beta\in\mathfrak{C}_\xi} \mathscr{E}(\sigma_\beta), \qquad \text{where} \qquad \mathscr{E}(\sigma_\beta) := \mathscr{E}_\xi \times^{E_\xi} E(\sigma_\beta),$$

and

$$\mathscr{Z}_\xi := \mathscr{E}_\xi \times^{E_\xi} Z_\xi = \bigcup_{\sigma_\beta \in \mathfrak{C}_\xi} \mathscr{Z}(\sigma_\beta), \qquad \text{where} \qquad \mathscr{Z}(\sigma_\beta) := \mathscr{E}_\xi \times^{E_\xi} Z(\sigma_\beta).$$

We note that on $\overline{\mathscr{E}}_\xi$ there is the locally closed subscheme given by the union of all $\mathscr{Z}(\sigma_\beta)$, where σ_β is running through $\mathfrak{C}_\xi$ such that $\sigma_\beta \subset C(X_\xi)^\circ$.

After these preliminary constructions, it is then shown in sections IV.4 and IV.5 of [4] that the arithmetic toroidal compactification $\overline{\mathscr{A}}_g$ is a proper algebraic stack over $\mathbb{Z}$ containing $\mathscr{A}_g$ as an open dense algebraic substack having a stratification $\overline{\mathscr{A}}_g = \coprod_\sigma \mathscr{Z}(\sigma)$, where $\sigma \in \mathfrak{C}$ is running through a complete set of representatives of the finitely many $\mathrm{GL}(X)$-orbits of $\mathfrak{C}$. Moreover, the formal completion of $\overline{\mathscr{A}}_g$ along the σ-stratum $\mathscr{Z}(\sigma)$ is isomorphic to the formal algebraic stack $\widehat{\overline{\mathscr{E}}}_\sigma/\Gamma_\sigma$, where $\widehat{\overline{\mathscr{E}}}_\sigma$ equals the formal completion of $\overline{\mathscr{E}}_\xi$ along the σ-stratum $\mathscr{Z}(\sigma)$ and Γ_σ is the subgroup of $\mathrm{GL}(X_\sigma)$ stabilizing σ.

Let us end this subsection by looking at the special cases of the above construction when $r = 0$ and $r = 1$. In case $r = 0$, we have $\sigma = \{0\}$ and thus $\mathscr{Z}(\{0\}) = \mathscr{A}_g$. In case $r = 1$, the cone σ is a ray of $\mathfrak{C}$ and thus the σ-stratum $\mathscr{Z}(\sigma)$ can be identified with the universal abelian scheme $\mathscr{B}_{g-1}$.

3 Modular forms and Jacobi forms

3.1. Arithmetic modular forms. Our basic reference for this section is chapter V of [4]. Using the notation of section 2, the Hodge bundle ω on $\mathscr{A}_g$ is defined as the line bundle $\varepsilon^* \det \Omega^1_{\mathscr{B}_g/\mathscr{A}_g}$. The $\mathbb{Z}$-module $M_k(\Gamma_g, \mathbb{Z})$ of *arithmetic modular forms of weight k for Γ_g* is then defined as the space of global sections $\Gamma(\mathscr{A}_g, \omega^{\otimes k})$ of the k-th power of the Hodge bundle ω over $\mathscr{A}_g$. After base change to the complex numbers, we find that

$$M_k(\Gamma_g) = M_k(\Gamma_g, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C},$$

where $M_k(\Gamma_g)$ is the space of Siegel modular forms of weight k for Γ_g.

The construction of the toroidal compactification $\overline{\mathscr{A}}_g$ of $\mathscr{A}_g$ shows that the universal abelian scheme $\pi\colon \mathscr{B}_g \to \mathscr{A}_g$ extends to a semi-abelian scheme $\overline{\pi}\colon \mathscr{G}_g \to \overline{\mathscr{A}}_g$ with zero section $\overline{\varepsilon}$. This extension gives rise to an extension $\overline{\omega}$ of the Hodge bundle ω given by

$$\overline{\omega} = \overline{\varepsilon}^* \det \Omega^1_{\mathscr{G}_g/\overline{\mathscr{A}}_g}.$$

It can be shown that

$$\Gamma\big(\overline{\mathscr{A}}_g, \overline{\omega}^{\otimes k}\big) = \Gamma\big(\mathscr{A}_g, \omega^{\otimes k}\big),$$

which can be seen as the Koecher principle. Viewing an arithmetic modular form $f \in M_k(\Gamma_g, \mathbb{Z})$ as a functorial assignment, which associates to any principally polarized abelian scheme A over some base ring R of relative dimension g an element $f(A) \in \Gamma(A, \omega^{\otimes k}_{A/R})$, it is shown on pp. 138/139 of [4] that by evaluating an arithmetic modular form $f \in M_k(\Gamma_g, \mathbb{Z})$ at Mumford families using a top-dimensional cone $\sigma \in \mathfrak{C}$ (i.e., by evaluating f at principally polarized abelian schemes degenerating to tori obtained by Mumford's construction over the power series ring $\mathbb{Z}[\![S^2(X) \cap \sigma^\vee]\!]$, where $S^2(X)$ is the second symmetric power of X and $\sigma^\vee$ is the dual cone of σ), one obtains its Fourier expansion corresponding to the Fourier expansion (3). It is furthermore shown there that the construction is independent on the choice of the top-dimensional cone $\sigma \in \mathfrak{C}$ (in fact, independent on the choice of any top-dimensional rational cone of $C(X)$) and that the resulting Fourier coefficients are invariant under the action induced by $\mathrm{GL}(X)$. By construction, the Fourier coefficients now turn out to be integral.

3.2. Arithmetic Jacobi forms. We continue to use the notation of section 2. Taking into account the principal polarization, the Poincaré bundle $\mathscr{P}_{\mathscr{B}_g}$ is the tautological invertible sheaf on the product $\mathscr{B}_g \times_{\mathscr{A}_g} \mathscr{B}_g$. By means of the diagonal morphism

$$\Delta\colon \mathscr{B}_g \longrightarrow \mathscr{B}_g \times_{\mathscr{A}_g} \mathscr{B}_g,$$

we obtain via pull-back the distinguished line bundle $\mathscr{L} := \Delta^{*}\mathscr{P}_{\mathscr{B}_g}$, which is π-ample and symmetric. The line bundle of Jacobi forms of weight k and index m on $\mathscr{B}_g$ is now defined as

$$\mathscr{J}_{k,m} := \mathscr{L}^{\otimes m} \otimes \pi^{*}\omega^{\otimes k}.$$

The $\mathbb{Z}$-module $J_{k,m}(\Gamma_g, \mathbb{Z})$ of *arithmetic Jacobi forms of weight k and index m for Γ_g* is then given as the space of global sections $\Gamma(\mathscr{B}_g, \mathscr{J}_{k,m})$. As in the case of modular forms, by viewing an arithmetic Jacobi form $\varphi \in J_{k,m}(\Gamma_g, \mathbb{Z})$ as a functorial assignment, which associates to any pair of a principally polarized abelian scheme A over some base ring R of relative dimension g and an ample symmetric line bundle $\mathscr{L}_A = \Delta_A^{*}\mathscr{P}_A$ ($\Delta_A\colon A \to A \times_R A$ the diagonal morphism, $\mathscr{P}_A =$ Poincaré bundle on $A \times_R A$) an element $\varphi(A, \mathscr{L}_A) \in \Gamma(A, \mathscr{L}_A^{\otimes m} \otimes \omega_{A/R}^{\otimes k})$, it is shown on pp. 164/165 of [7] that by evaluating an arithmetic Jacobi form $\varphi \in J_{k,m}(\Gamma_g, \mathbb{Z})$ at Mumford families using a top-dimensional cone $\sigma \in \mathfrak{C}$, one obtains its Fourier expansion corresponding to the Fourier expansion (2). Furthermore, it is shown there that the construction is independent on the choice of the top-dimensional cone $\sigma \in \mathfrak{C}$ and that the resulting Fourier coefficients are invariant under the action induced by $\mathrm{GL}(X) \ltimes X$. Again, the Fourier coefficients turn out to be integral. The arithmetic theory of Jacobi forms has been investigated in detail in [7].

3.3. Fourier–Jacobi expansions. This subjection is devoted to summarize the derivation of the Fourier–Jacobi expansion for an arithmetic modular form as described on pp. 142–144 of [4]. To simplify the exposition, we restrict to the case where $r = 1$ and thus $X_\xi = \mathbb{Z}$. We then choose a cone $\sigma \subset C(X_\xi)^{\circ}$, i.e., $X_\sigma \cong X_\xi$. Given these data, we have the completion $\widehat{\overline{\mathscr{E}}}_\sigma$ of $\overline{\mathscr{E}}_\xi$ along the σ-stratum $\mathscr{Z}(\sigma)$ (see subsection 2.2); we let $\widehat{\overline{\omega}}_\sigma$ denote the pull-back of the line bundle $\overline{\omega}$ on $\overline{\mathscr{A}}_g$ to the completion $\widehat{\overline{\mathscr{E}}}_\sigma$. We then obtain the map

$$\mathrm{FJ}_\xi\colon \Gamma\big(\overline{\mathscr{A}}_g, \overline{\omega}^{\otimes k}\big) \longrightarrow \Gamma\big(\widehat{\overline{\mathscr{E}}}_\sigma, \widehat{\overline{\omega}}_\sigma^{\otimes k}\big),$$

assigning to an arithmetic modular form f of weight k for Γ_g the expansion $\sum_{m \in \mathbb{N}} f_m \chi^m$, which is obtained by expanding the pull-back of f to $\widehat{\overline{\mathscr{E}}}_\sigma$ according to the action of the characters χ^m of the torus $E_\xi = \mathbb{G}_{\mathrm{m}}$. The resulting expansion is called the *Fourier–Jacobi expansion of f*. It is mentioned on p. 143 of [4] that the formal functions f_m turn out to be algebraizable functions on $\mathscr{B}_{g-1}$ and that the construction is independent on the choice of the cone $\sigma \subset C(X_\xi)^{\circ}$. Furthermore, it also shown there that the map FJ_ξ is injective.

3.4. Lemma. *Let $f \in M_k(\Gamma_g, \mathbb{Z})$ with Fourier–Jacobi expansion $\sum_{m \in \mathbb{N}} f_m \chi^m$. Then, we have $f_m \in J_{k,m}(\Gamma_{g-1}, \mathbb{Z})$.*

Proof. We start by recalling from p. 143 of [4] that the algebraizable functions f_m are a global sections of the line bundle

$$\mathcal{L}_{g,1}^{k}(m; \mathbb{Z}) := \mathcal{L}(m) \otimes \pi^{*}\omega^{\otimes k},$$

where $\mathcal{L}(m)$ is a suitable line bundle on $\mathbf{Hom}_{\mathbb{Z}}(X_\xi, \mathscr{B}_{g-1}) = \mathscr{B}_{g-1}$ that we are going to describe next. We note in passing that subsequently we will identify X_ξ with its second symmetric power $S^2(X_\xi)$ via the assignment $m \mapsto [m \otimes 1]$. We claim that we have $\mathcal{L}(m) = \mathscr{L}^{\otimes m}$, which will indeed imply that $f_m \in J_{k,m}(\Gamma_{g-1}, \mathbb{Z})$ and that the map FJ_ξ corresponds to the classical Fourier–Jacobi expansion (1).

To prove this claim, we first recall from p. 105 of [4], the tautological $\mathbb{Z}$-homomorphism $c\colon X_\xi \to \mathscr{B}_{g-1}$ over $\mathbf{Hom}_{\mathbb{Z}}(X_\xi, \mathscr{B}_{g-1})$. Since we have that $\mathbf{Hom}_{\mathbb{Z}}(X_\xi, \mathscr{B}_{g-1}) = \mathscr{B}_{g-1}$, the $\mathbb{Z}$-homomorphisms from $X_\xi = \mathbb{Z}$ to $\mathscr{B}_{g-1}$ are determined by the assigment $\mathbb{Z} \ni 1 \mapsto b \in \mathscr{B}_{g-1}$. Thus, for integers m_1 and m_2, we obtain the morphisms

$$c(m_1) \times c(m_2)\colon \mathscr{B}_{g-1} \longrightarrow \mathscr{B}_{g-1} \times_{\mathscr{A}_{g-1}} \mathscr{B}_{g-1},$$

given by the assignment $b \mapsto (m_1 \cdot b, m_2 \cdot b)$ ($b \in \mathscr{B}_{g-1}$); here we observe in particular that the diagonal morphism Δ can be written as $\Delta = c(1) \times c(1)$.

As the second ingredient, we recall from pp. 142/143 of [4], that the evaluation map

$$\mathbf{ev}\colon S^2(X_\xi) \longrightarrow \mathrm{H}^1(\mathscr{B}_{g-1}, \mathbb{G}_\mathrm{m})$$

is induced by the assignment

$$[m_1 \otimes m_2] \mapsto (c(m_1) \times c(m_2))^* \mathscr{P}_{\mathscr{B}_{g-1}}.$$

Identifying $S^2(X_\xi)$ with X_ξ as described above, the line bundle $\mathcal{L}(m)$ then turns out to be the line bundle associated to the class $\mathbf{ev}([m \otimes 1]) \in \mathrm{H}^1(\mathscr{B}_{g-1}, \mathbb{G}_\mathrm{m})$.
In the third step, we finally show that $\mathcal{L}(m) = \mathscr{L}^{\otimes m}$. For this we first assume that m is a square, i.e., $m = m_1^2$ with $m_1 \in \mathbb{Z}$. By the above description, the line bundle $\mathcal{L}(m)$ is associated to the class $\mathbf{ev}([m \otimes 1]) = \mathbf{ev}([m_1 \otimes m_1]) \in \mathrm{H}^1(\mathscr{B}_{g-1}, \mathbb{G}_\mathrm{m})$. Thus, we have in case that $m = m_1^2$ the series of equalities

$$\begin{aligned}\mathcal{L}(m) &= (c(m_1) \times c(m_1))^* \mathscr{P}_{\mathscr{B}_{g-1}} = (m_1 \cdot c(1) \times m_1 \cdot c(1))^* \mathscr{P}_{\mathscr{B}_{g-1}} \\ &= ((c(1) \times c(1)) \circ [m_1])^* \mathscr{P}_{\mathscr{B}_{g-1}} = [m_1]^* \Delta^* \mathscr{P}_{\mathscr{B}_{g-1}} \\ &= [m_1]^* \mathscr{L} = \mathscr{L}^{\otimes m_1^2} = \mathscr{L}^{\otimes m}\,;\end{aligned}$$

here we have used the symmetry of $\mathscr{L}$ in conjunction with the theorem of the cube on $\mathscr{B}_{g-1}$ (as usual, $[m_1]$ here denotes the morphism given by multiplication by m_1 on $\mathscr{B}_{g-1}$). For general m, we simply write m as a sum of four squares, i.e., $m = m_1^2 + \ldots + m_4^2$ with $m_1, \ldots, m_4 \in \mathbb{Z}$, and then obtain

$$[m \otimes 1] = [m_1 \otimes m_1] + \ldots + [m_4 \otimes m_4],$$

whence

$$\mathbf{ev}([m \otimes 1]) = \mathbf{ev}([m_1 \otimes m_1]) + \ldots + \mathbf{ev}([m_4 \otimes m_4]),$$

from which we find using the above computation

$$\mathcal{L}(m) = \mathscr{L}^{\otimes m_1^2} \otimes \ldots \otimes \mathscr{L}^{\otimes m_4^2} = \mathscr{L}^{\otimes m}\,.$$

This proves the claim of the lemma. ☐

3.5. Remark. Using the notation of subsection 3.3, Lemma 3.4 leads to an injective map

$$\Phi\colon \Gamma\big(\widehat{\overline{\mathscr{E}}}_\sigma, \widehat{\overline{\omega}}_\sigma^{\otimes k}\big) \lhook\joinrel\longrightarrow \prod_{m \in \mathbb{N}} J_{k,m}(\Gamma_{g-1}, \mathbb{Z})$$

of $\mathbb{Z}$-modules given by the assignment

$$\sum_{m \in \mathbb{N}} f_m \chi^m \mapsto (f_m)_{m \in \mathbb{N}}\,.$$

We call a tuple $(f_m) \in \prod_{m \in \mathbb{N}} J_{k,m}(\Gamma_{g-1}, \mathbb{Z})$ invariant under $\mathrm{GL}(X)$, if the Fourier coefficients $c_m(n,r)$ of the f_m's satisfy the invariance condition (4); here, $n \in \mathrm{Sym}_{g-1}(\mathbb{Q})$ is half-integral and $r \in \mathbb{Z}^{g-1}$ such that the non-negativity condition $4mn - rr^t \geq 0$ holds.

3.6. Lemma. *The injective map* Φ *of Remark 3.5 induces an isomorphism*

$$\Gamma\big(\widehat{\overline{\mathscr{E}}}_\sigma, \widehat{\overline{\omega}}_\sigma^{\otimes k}\big) \cong \Big(\prod_{m \in \mathbb{N}} J_{k,m}(\Gamma_{g-1}, \mathbb{Z})\Big)^{\mathrm{GL}(X)} \tag{6}$$

of $\mathbb{Z}$*-modules, where the right-hand side of the isomorphism* (6) *refers to the tuples* $(f_m) \in \prod_{m \in \mathbb{N}} J_{k,m}(\Gamma_{g-1}, \mathbb{Z})$ *that are invariant under* $\mathrm{GL}(X)$.

Proof. Since the map Φ is injective, we have to describe its image in $\prod_{m\in\mathbb{N}} J_{k,m}(\Gamma_{g-1},\mathbb{Z})$ to prove the lemma. For this we pick an element

$$\sum_{m\in\mathbb{N}} f_m\chi^m \in \Gamma\big(\widehat{\overline{\mathscr{E}}}_\sigma, \widehat{\overline{\omega}}_\sigma^{\otimes k}\big). \tag{7}$$

By embedding σ into a top-dimensional cone $\sigma^{\mathrm{top}} \in C(X)^\circ$, we can determine the formal Fourier expansion of (7) by pulling it back to $\Gamma\big(\widehat{\overline{\mathscr{E}}}_{\sigma_{\mathrm{top}}}, \widehat{\overline{\omega}}_{\sigma_{\mathrm{top}}}^{\otimes k}\big)$. The natural isomorphisms (see end of subsection 3.1)

$$u^*\colon \Gamma\big(\widehat{\overline{\mathscr{E}}}_{u^*\sigma_{\mathrm{top}}}, \widehat{\overline{\omega}}_{u^*\sigma_{\mathrm{top}}}^{\otimes k}\big) \xrightarrow{\cong} \Gamma\big(\widehat{\overline{\mathscr{E}}}_{\sigma_{\mathrm{top}}}, \widehat{\overline{\omega}}_{\sigma_{\mathrm{top}}}^{\otimes k}\big)$$

then show that the resulting Fourier coefficients are invariant under the action induced by $u \in \mathrm{GL}(X)$. This shows that the Fourier coefficients $c_m(n,r)$ of the f_m's, where $n \in \mathrm{Sym}_{g-1}(\mathbb{Q})$ is half-integral and $r \in \mathbb{Z}^{g-1}$ such that the non-negativity condition $4mn - rr^t \geq 0$ holds, satisfy the invariance condition (4). This proves that

$$\Gamma\big(\widehat{\overline{\mathscr{E}}}_\sigma, \widehat{\overline{\omega}}_\sigma^{\otimes k}\big) \cong \Big(\prod_{m\in\mathbb{N}} J_{k,m}(\Gamma_{g-1},\mathbb{Z})\Big)^{\mathrm{GL}(X)},$$

and concludes the proof of the lemma. $\square$

3.7. Remark. Continuing to use the notation of subsection 3.3, we arrive by means of Lemma 3.6 at the injective map

$$\mathrm{FJ}_\xi\colon \Gamma\big(\overline{\mathscr{A}}_g, \overline{\omega}^{\otimes k}\big) \hookrightarrow \Gamma\big(\widehat{\overline{\mathscr{E}}}_\sigma, \widehat{\overline{\omega}}_\sigma^{\otimes k}\big) \cong \Big(\prod_{m\in\mathbb{N}} J_{k,m}(\Gamma_{g-1},\mathbb{Z})\Big)^{\mathrm{GL}(X)}.$$

From this we derive that the analog of the result of Bruinier–Raum in the present arithmetic setting will follow when we show that the injective map FJ_ξ is also surjective.

4 The main result

We begin by restating a result which originates from A. Borel's article [1] and its reformulation as Theorem 1′ in [13].

4.1. Lemma. *With the above notations, let $U \subseteq \overline{\mathscr{A}}_g$ be a non-empty, open subset and $f_U \in \Gamma(U, \overline{\omega}^{\otimes k})$. Then, the section f_U extends as a rational section of $\overline{\omega}^{\otimes k}$ to all of $\overline{\mathscr{A}}_g$, i.e., to a rational arithmetic modular form f of weight k for Γ_g, possibly with a character.*

Proof. We start with the following introductory recollections. Let $\mathscr{A}_g^*/\mathbb{Z}$ denote the minimal compactification of $\mathscr{A}_g/\mathbb{Z}$. Given any (smooth) toroidal compactification $\overline{\mathscr{A}}_g/\mathbb{Z}$, there is a canonical birational morphism $\varphi\colon \overline{\mathscr{A}}_g \to \mathscr{A}_g^*$ such that $\overline{\omega}^{\otimes k_0} = \varphi^*\mathcal{L}$ for some positive integer k_0 and some very ample line bundle $\mathcal{L}$ on $\mathscr{A}_g^*$. By the birationality of the morphism φ, we have $\varphi_*\mathscr{O}_{\overline{\mathscr{A}}_g} = \mathscr{O}_{\mathscr{A}_g^*}$ and an isomorphism of function fields $K(\overline{\mathscr{A}}_g) \cong K(\mathscr{A}_g^*)$ induced by the pull-back map φ^*. Furthermore, from section 5 in [11] and the references therein, we find that the Picard group $\mathrm{Pic}(\mathscr{A}_g^*)$ is generated by the (class of the) line bundle $\mathcal{L}$.

With these recollections at hand, the proof of the lemma proceeds as follows in the case $k_0 = 1$. Given the section $f_U \in \Gamma\big(U, \overline{\omega}^{\otimes k}\big)$, we get

$$f_U \in \Gamma\big(U, \overline{\omega}^{\otimes k}\big) = \Gamma\big(U, \varphi^*\mathcal{L}^{\otimes k}\big),$$

which shows that there is a non-empty, open subset $V \subseteq \mathscr{A}_g^*$ satisfying $V \supseteq \varphi(U)$ and a section $h_V \in \Gamma\big(V, \mathcal{L}^{\otimes k}\big)$ satisfying $\varphi^*(h_V)|_U = f_U$. We then let $D := \overline{\mathrm{div}(h_V)}$ denote the Zariski closure of $\mathrm{div}(h_V)$ in $\mathscr{A}_g^*$. Since $\mathrm{Pic}(\mathscr{A}_g^*)$ is generated by the class of $\mathcal{L}$, there exists $k_1 \in \mathbb{N}_{>0}$ such that the class of $\mathscr{O}_{\mathscr{A}_g^*}(D)$ equals the class of $\mathcal{L}^{\otimes k_1}$ in $\mathrm{Pic}(\mathscr{A}_g^*)$. We thus find $h' \in \Gamma\big(\mathscr{A}_g^*, \mathcal{L}^{\otimes k_1}\big)$ such that

$$\mathrm{div}(h') \sim D,$$

where $\sim$ means linear equivalence of divisors. Hence, we find a rational function $h_1' \in K(\mathscr{A}_g^*)^\times$ so that we have

$$D = \mathrm{div}(h' \cdot h_1').$$

Our construction shows that

$$\mathrm{div}(h' \cdot h_1')|_V = D|_V = \mathrm{div}(h_V).$$

We thus find

$$\frac{h_V}{(h' \cdot h_1')|_V} \in \Gamma\big(V, \mathscr{O}^\times_{\mathscr{A}_g^*}\big),$$

i.e., there is a further rational function $h_2' \in K(\mathscr{A}_g^*)^\times$ such that

$$\frac{h_V}{(h' \cdot h_1')|_V} = h_2'|_V.$$

With these data, we define $h := h' \cdot h_1' \cdot h_2'$, which satisfies the relation

$$h|_V = (h' \cdot h_1' \cdot h_2')|_V = h_V.$$

We then put

$$f := \varphi^*(h), \quad f' := \varphi^*(h'), \quad f_1' := \varphi^*(h_1'), \quad f_2' := \varphi^*(h_2'),$$

so that $f = f' \cdot f_1' \cdot f_2'$, where $f_1', f_2' \in K(\overline{\mathscr{A}}_g)^\times$ and

$$f' \in \Gamma\big(\overline{\mathscr{A}}_g, \varphi^*\mathcal{L}^{\otimes k_1}\big) = \Gamma\big(\overline{\mathscr{A}}_g, \overline{\omega}^{\otimes k_1}\big).$$

In other words, the section f is a rational arithmetic modular form of weight k_1 for Γ_g satisfying

$$f|_U = \varphi^*(h)|_U = \varphi^*(h|_V)|_U = \varphi^*(h_V)|_U = f_U.$$

Since $f_U \in \Gamma\big(U, \overline{\omega}^{\otimes k}\big)$, we conclude that $k_1 = k$, and the proof of the lemma follows in the case under consideration.

Let us now turn to the proof of the lemma in the general case. Given $f_U \in \Gamma(U, \overline{\omega}^{\otimes k})$ with arbitrary k, the above argument shows that its k_0-th power $f_U^{k_0}$ extends as a rational arithmetic modular form of weight $k \cdot k_0$ for Γ_g. Hence, the section f_U itself extends as a rational arithmetic modular form of weight k for Γ_g, possibly with a character. This completes the proof of the lemma. □

4.2. Lemma. *With the notation of subsection 3.3, there is an isomorphism*

$$\Gamma\big(\widehat{\overline{\mathscr{E}}}_\sigma, \widehat{\overline{\omega}}_\sigma^{\otimes k}\big) \cong \varprojlim_n \Gamma\big(\overline{\mathscr{A}}_g, \overline{\omega}^{\otimes k} / \mathscr{J}_\sigma^{\otimes n+1} \otimes \overline{\omega}^{\otimes k}\big),$$

where $\mathscr{J}_\sigma$ denotes the ideal sheaf of a σ-stratum $\mathscr{Z}(\sigma)$ of $\overline{\mathscr{A}}_g$ associated to the cone $\sigma \subset C(X_\xi)^\circ$ having $\mathbb{Z}$-rank $r = 1$.

Proof. From the description of the arithmetic compactification $\overline{\mathscr{A}}_g$ of $\mathscr{A}_g$ in subsection 2.2, we recall that the completion $\widehat{\overline{\mathscr{A}}}_g$ of $\overline{\mathscr{A}}_g$ along the σ-stratum $\mathscr{Z}(\sigma)$ is isomorphic to $\widehat{\overline{\mathscr{E}}}_\sigma/\Gamma_\sigma$, where Γ_σ is the subgroup of $\mathrm{GL}(X_\sigma)$ stabilizing σ. Since $\mathrm{GL}(X_\sigma) = \mathrm{GL}_1(\mathbb{Z}) = \{\pm 1\}$ and both ± 1 act as the identity on σ, the subgroup Γ_σ becomes trivial. By denoting the pull-back of $\overline{\omega}$ to the completion $\widehat{\overline{\mathscr{A}}}_g$ by $\widehat{\overline{\omega}}$, we arrive at the isomorphism of global sections

$$\Gamma\big(\widehat{\overline{\mathscr{A}}}_g, \widehat{\overline{\omega}}^{\otimes k}\big) \cong \Gamma\big(\widehat{\overline{\mathscr{E}}}_\sigma, \widehat{\overline{\omega}}_\sigma^{\otimes k}\big). \tag{8}$$

Before continuing, we note that we have suppressed the dependence on σ in our notation for the completions of $\overline{\mathscr{A}}_g$ and $\overline{\omega}$ along the σ-stratum $\mathscr{Z}(\sigma)$.

We now apply A. Grothendieck's existence theorem on formal functions, i.e., Theorem 4.1.5 of [5], to the identity morphism of $\overline{\mathscr{A}}_g$ (for global sections) to obtain the isomorphism

$$\Gamma\big(\widehat{\overline{\mathscr{A}}}_g, \widehat{\overline{\omega}}^{\otimes k}\big) \cong \varprojlim_n \Gamma\big(\overline{\mathscr{A}}_g, \overline{\omega}^{\otimes k}/\mathscr{J}_\sigma^{\otimes n+1} \otimes \overline{\omega}^{\otimes k}\big). \tag{9}$$

Combining the isomorphisms (8) and (9), concludes the proof of the lemma. □

4.3. Lemma. *With the notations of subsection 3.3, the $\mathbb{Z}$-modules*

$$\varprojlim_n \Gamma\big(\overline{\mathscr{A}}_g, \overline{\omega}^{\otimes k}/\mathscr{J}_\sigma^{\otimes n+1} \otimes \overline{\omega}^{\otimes k}\big) \cong \Gamma\big(\widehat{\overline{\mathscr{E}}}_\sigma, \widehat{\overline{\omega}}_\sigma^{\otimes k}\big) \cong \Big(\prod_{m\in\mathbb{N}} J_{k,m}(\Gamma_{g-1},\mathbb{Z})\Big)^{\mathrm{GL}(X)}$$

are of finite rank over $\mathbb{Z}$.

Proof. The proof of this finiteness result is given in Lemma 3.6 of [3]. We only note that the finitely generatedness result of Theorem 3.8 (cited from B. Runge) needs to be replaced according to [2], but this can easily be fixed. □

4.4. Theorem. *With the notations of subsection 3.3, the analog of the result of Bruinier–Raum holds true in the arithmetic setting over the ring of integers $\mathbb{Z}$, in other words, the map*

$$\mathrm{FJ}_\xi \colon \Gamma\big(\overline{\mathscr{A}}_g, \overline{\omega}^{\otimes k}\big) \hookrightarrow \Gamma\big(\widehat{\overline{\mathscr{E}}}_\sigma, \widehat{\overline{\omega}}_\sigma^{\otimes k}\big)$$

is an isomorphism.

Proof. Since the map FJ_ξ is already known to be injective, we are left to prove its surjectivity. We thus choose an element

$$\sum_{m=0}^{\infty} f_m \chi^m \in \Gamma\big(\widehat{\overline{\mathscr{E}}}_\sigma, \widehat{\overline{\omega}}_\sigma^{\otimes k}\big), \tag{10}$$

for which we aim to construct a preimage in $\Gamma\big(\overline{\mathscr{A}}_g, \overline{\omega}^{\otimes k}\big)$. To carry out this construction, we start by recalling from Lemma 4.2 the isomorphism

$$\Gamma\big(\widehat{\overline{\mathscr{E}}}_\sigma, \widehat{\overline{\omega}}_\sigma^{\otimes k}\big) \cong \varprojlim_n \Gamma\big(\overline{\mathscr{A}}_g, \overline{\omega}^{\otimes k}/\mathscr{J}_\sigma^{\otimes n+1} \otimes \overline{\omega}^{\otimes k}\big).$$

We next choose a non-empty, affine, open subset $U \subset \overline{\mathscr{A}}_g$ intersecting the boundary divisor $\mathscr{Z}(\sigma)$. Since the higher cohomology groups vanish for affine schemes, we have for all $n \in \mathbb{N}$ that

$$\mathrm{H}^1\big(U, \mathscr{J}_\sigma^{\otimes n+1} \otimes \overline{\omega}^{\otimes k}\big) = 0,$$

from which we derive the short exact sequence

$$0 \longrightarrow \mathrm{H}^0\big(U, \mathscr{J}_\sigma^{\otimes n+1} \otimes \overline{\omega}^{\otimes k}\big) \longrightarrow \mathrm{H}^0\big(U, \overline{\omega}^{\otimes k}\big) \longrightarrow \mathrm{H}^0\big(U, \overline{\omega}^{\otimes k}/\mathscr{J}_\sigma^{\otimes n+1} \otimes \overline{\omega}^{\otimes k}\big) \longrightarrow 0,$$

which gives

$$\mathrm{H}^0\big(U, \overline{\omega}^{\otimes k}/\mathscr{J}_\sigma^{\otimes n+1} \otimes \overline{\omega}^{\otimes k}\big) \cong \mathrm{H}^0\big(U, \overline{\omega}^{\otimes k}\big)/\mathrm{H}^0\big(U, \mathscr{J}_\sigma^{\otimes n+1} \otimes \overline{\omega}^{\otimes k}\big).$$

Restricting the given element (10) to U thus gives rise to a compatible system

$$\big(\ldots,[h_U^{(n)}],\ldots,[h_U^{(1)}],[h_U^{(0)}]\big)\in\varprojlim_n\Gamma\big(U,\overline{\omega}^{\otimes k}\big)/\Gamma\big(U,\mathscr{J}_\sigma^{\otimes n+1}\otimes\overline{\omega}^{\otimes k}\big),$$

where $h_U^{(n)}\in\Gamma\big(U,\overline{\omega}^{\otimes k}\big)$ for all $n\in\mathbb{N}$ and the notation $[h_U^{(n)}]$ means the class of the section $h_U^{(n)}$ modulo $\Gamma\big(U,\mathscr{J}_\sigma^{\otimes n+1}\otimes\overline{\omega}^{\otimes k}\big)$.

Now, we fix $n\in\mathbb{N}$ with $n\gg 1$. By Lemma 4.1, the element $h_U^{(n)}\in\Gamma\big(U,\overline{\omega}^{\otimes k}\big)$ extends to a rational arithmetic modular form $h^{(n)}$ of weight k for Γ_g, possibly with a character. By pulling $h^{(n)}$ back to $\widehat{\overline{\mathscr{E}}}_\sigma$ leads to the Fourier–Jacobi expansion

$$h^{(n)}=\sum_{m=-m_0}^{\infty}h_m^{(n)}\chi^m$$

with a suitable $m_0\in\mathbb{N}$. After restriction to U, we obtain

$$\sum_{m=-m_0}^{\infty}h_m^{(n)}\chi^m\big|_U\equiv\sum_{m=0}^{\infty}f_m\chi^m\big|_U\mod\Gamma\big(U,\mathscr{J}_\sigma^{\otimes n+1}\otimes\overline{\omega}^{\otimes k}\big),$$

which shows

$$\begin{aligned}h_m^{(n)}\big|_{\mathscr{Z}(\sigma)\cap U}&=0 &&\text{for } m=-m_0,\ldots,-1;\\ h_m^{(n)}\big|_{\mathscr{Z}(\sigma)\cap U}&=f_m\big|_{\mathscr{Z}(\sigma)\cap U} &&\text{for } m=0,\ldots,n.\end{aligned}$$

Since U is dense in $\overline{\mathscr{A}}_g$, the open set $\mathscr{Z}(\sigma)\cap U$ is dense in $\mathscr{Z}(\sigma)$. Thus, the above equalities extend to $\mathscr{Z}(\sigma)$ to give

$$\begin{aligned}h_m^{(n)}&=0 &&\text{for } m=-m_0,\ldots,-1;\\ h_m^{(n)}&=f_m &&\text{for } m=0,\ldots,n.\end{aligned}$$

This shows that $h^{(n)}\in\Gamma\big(\overline{\mathscr{A}}_g,\overline{\omega}^{\otimes k}\big)$ with Fourier–Jacobi expansion satisfying

$$h^{(n)}=\sum_{m=0}^{\infty}h_m^{(n)}\chi^m\equiv\sum_{m=0}^{\infty}f_m\chi^m\mod\Gamma\big(\overline{\mathscr{A}}_g,\mathscr{J}_\sigma^{n+1}\otimes\overline{\omega}^{\otimes k}\big).$$

The proof of the theorem can now be completed by recalling from Lemma 4.3 that the space of formal Fourier–Jacobi expansions satisfying the invariance condition (4) has finite $\mathbb{Z}$-rank, which implies that for $n\gg 1$, the above congruence becomes an equality, i.e., we indeed have

$$h^{(n)}=\sum_{m=0}^{\infty}f_m\chi^m.$$

This proves the claimed surjectivity of the map FJ_ξ. □

4.5. Remark. Lemma 4.2 allows us to rewrite the map FJ_ξ in the form

$$\mathrm{FJ}_\xi\colon\Gamma\big(\overline{\mathscr{A}}_g,\overline{\omega}^{\otimes k}\big)\hookrightarrow\varprojlim_n\Gamma\big(\overline{\mathscr{A}}_g,\overline{\omega}^{\otimes k}/\mathscr{J}_\sigma^{\otimes n+1}\otimes\overline{\omega}^{\otimes k}\big).$$

We note that this map, at least up to tensoring with $\mathbb{Q}$, can alternatively be obtained in the following way: Consider the short exact sequence of sheaves

$$0\longrightarrow\mathscr{J}_\sigma^{\otimes n+1}\otimes\overline{\omega}^{\otimes k}\longrightarrow\overline{\omega}^{\otimes k}\longrightarrow\overline{\omega}^{\otimes k}/\mathscr{J}_\sigma^{\otimes n+1}\otimes\overline{\omega}^{\otimes k}\longrightarrow 0$$

on $\overline{\mathscr{A}}_g$. It gives rise to the long exact sequence in cohomology

$$\begin{aligned}0\longrightarrow\mathrm{H}^0\big(\overline{\mathscr{A}}_g,\mathscr{J}_\sigma^{\otimes n+1}\otimes\overline{\omega}^{\otimes k}\big)\longrightarrow\mathrm{H}^0\big(\overline{\mathscr{A}}_g,\overline{\omega}^{\otimes k}\big)\longrightarrow\mathrm{H}^0\big(\overline{\mathscr{A}}_g,\overline{\omega}^{\otimes k}/\mathscr{J}_\sigma^{\otimes n+1}\otimes\overline{\omega}^{\otimes k}\big)\longrightarrow\\ \longrightarrow\mathrm{H}^1\big(\overline{\mathscr{A}}_g,\mathscr{J}_\sigma^{\otimes n+1}\otimes\overline{\omega}^{\otimes k}\big)\longrightarrow\mathrm{H}^1\big(\overline{\mathscr{A}}_g,\overline{\omega}^{\otimes k}\big)\longrightarrow\ldots\end{aligned}\tag{11}$$

We aim at applying the inverse limit functor to the long exact sequence (11). However, this functor is only exact if the Mittag–Leffler condition is satisfied. This condition, for example, is satisfied if we deal with finite dimensional vector space. Therefore, we tensor the long exact sequence (11) with $\mathbb{Q}$ and obtain after observing that

$$\varprojlim_n \mathrm{H}^0\big(\overline{\mathscr{A}}_g, \mathscr{J}_\sigma^{\otimes n+1} \otimes \overline{\omega}^{\otimes k}\big) = \bigcap_{n=0}^{\infty} \mathrm{H}^0\big(\overline{\mathscr{A}}_g, \mathscr{J}_\sigma^{\otimes n+1} \otimes \overline{\omega}^{\otimes k}\big) = 0,$$

the long exact sequence

$$0 \longrightarrow \mathrm{H}^0\big(\overline{\mathscr{A}}_g, \overline{\omega}^{\otimes k}\big)_{\mathbb{Q}} \longrightarrow \varprojlim_n \mathrm{H}^0\big(\overline{\mathscr{A}}_g, \overline{\omega}^{\otimes k} / \mathscr{J}_\sigma^{\otimes n+1} \otimes \overline{\omega}^{\otimes k}\big)_{\mathbb{Q}} \longrightarrow$$
$$\longrightarrow \varprojlim_n \mathrm{H}^1\big(\overline{\mathscr{A}}_g, \mathscr{J}_\sigma^{\otimes n+1} \otimes \overline{\omega}^{\otimes k}\big)_{\mathbb{Q}} \longrightarrow \mathrm{H}^1\big(\overline{\mathscr{A}}_g, \overline{\omega}^{\otimes k}\big)_{\mathbb{Q}} \longrightarrow \ldots$$

From the latter long exact sequence we obtain, at least after tensoring with $\mathbb{Q}$, the natural interpretation of the injective map FJ_ξ as the restriction of an arithmetic modular form $f \in \mathrm{H}^0(\overline{\mathscr{A}}_g, \overline{\omega}^{\otimes k})_{\mathbb{Q}}$ to all the $(n+1)$-thickenings of the boundary divisor $\mathscr{X}(\sigma)$. In connection with the surjectivity of the map FJ_ξ vanishing results for the inverse limit of cohomology groups

$$\varprojlim_n \mathrm{H}^1\big(\overline{\mathscr{A}}_g, \mathscr{J}_\sigma^{\otimes n+1} \otimes \overline{\omega}^{\otimes k}\big)_{\mathbb{Q}}$$

are of interest. This line of thoughts is presently under investigation and will be addressed elsewhere.

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