

ON THE HEIGHT OF THE UNIVERSAL ABELIAN VARIETY

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ABSTRACT. In this paper we extend the arithmetic intersection theory of adelic divisors on quasi-projective varieties developed by X. Yuan and S. W. Zhang to cover certain adelic arithmetic divisors that are not nef nor integrable. The key concept used in this extension is the relative finite energy introduced by T. Darvas, E. Di Nezza, and C. H. Lu. As an application we compute the arithmetic self-intersection number of the line bundle of Siegel–Jacobi forms on the universal abelian variety endowed with its invariant hermitian metric. The techniques developed in this paper can be applied in many other situations like mixed Shimura varieties or the moduli space of stable marked curves.

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1. INTRODUCTION

1.1. **Background.** Let $\mathcal{A}_g(\mathbb{C})$ denote the moduli space of principally polarized abelian varieties of dimension g over the complex numbers and ω the Hodge bundle on $\mathcal{A}_g(\mathbb{C})$. The calculation of the geometric degree of ω has been of fundamental interest in the past. Since Chern–Weil theory holds on $\mathcal{A}_g(\mathbb{C})$, this degree computation amounts to calculating the volume of $\mathcal{A}_g(\mathbb{C})$ with respect to the natural invariant metric. The validity of Chern–Weil theory is due to the fact that the natural invariant metric has only mild logarithmic singularities when approaching the boundary of $\mathcal{A}_g(\mathbb{C})$. The volume computation has been carried out by C. L. Siegel in [Sie36] and leads to the formula

$$\deg(\omega) = (-1)^{d'} \zeta_{\mathbb{Q}}(1 - 2g) \cdot \zeta_{\mathbb{Q}}(3 - 2g) \cdot \dots \cdot \zeta_{\mathbb{Q}}(-1),$$

where $d' = g(g + 1)/2 = \dim_{\mathbb{C}}(\mathcal{A}_g(\mathbb{C}))$ and $\zeta_{\mathbb{Q}}(s)$ is the classical Riemann zeta function.

Shifting now our attention to arithmetic intersections, we need first to replace the complex varieties under consideration by arithmetic varieties. Thus, we consider the moduli stack \mathcal{A}_g of principally abelian schemes of dimension g over $\text{Spec}(\mathbb{Z})$. We point out that in order to avoid to work with stacks we will have to introduce level structures; however, for the sake of exposition, we do not use level structures here. Considering then the Hodge bundle $\overline{\omega} = (\omega, \|\cdot\|_{\text{inv}})$ on \mathcal{A}_g equipped with the natural invariant metric, it was striking when U. Kühn in [Küh01] and J.-B. Bost in [Bos99] independently computed in the case $g = 1$ the arithmetic degree $\widehat{\deg}(\overline{\omega})$ of

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$\overline{\omega}$, roughly speaking, as the logarithmic derivative of its geometric degree. More precisely, they obtained the formula

$$(1.1) \quad \widehat{\deg}(\overline{\omega}) = -\deg(\omega) \left(\frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + \frac{1}{2} \right).$$

This result was instrumental for various subsequent developments: Let us mention first the establishment of an arithmetic intersection theory generalizing the theory of H. Gillet and C. Soulé [GS90] in the series of papers [BGKK05] and [BGKK07] taking into account logarithmically singular metrics. Second, the result was important for the further development of the so-called “Kudla Program”, see for example [KRY06]. As a third ingredient, we refer to the conjectural generalization of formula (1.1) in the form (see also [MR02])

$$\widehat{\deg}(\overline{\omega}) = -\deg(\omega) \left(\frac{\zeta'_{\mathbb{Q}}(1-2g)}{\zeta_{\mathbb{Q}}(1-2g)} + \frac{\zeta'_{\mathbb{Q}}(3-2g)}{\zeta_{\mathbb{Q}}(3-2g)} + \dots + \frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + a \log(2) + b \right)$$

with $a, b \in \mathbb{Q}$; this conjectural formula has been verified for $g = 2$ (see [JvP22]) and for certain Hilbert modular surfaces (see [BBGK07]).

After having established an arithmetic intersection theory on the moduli stack \mathcal{A}_g , the question arises if such a theory could also be developed on the universal abelian scheme $\pi: \mathcal{B}_g \rightarrow \mathcal{A}_g$. An interesting example would be the calculation of the arithmetic degree of the line bundle \mathcal{L} corresponding to the principal polarization of \mathcal{B}_g , since it is related to ω by means of the “formule clé” stating that $\pi_* \mathcal{L} \cong \omega^{\otimes -1/2}$.

However, it turned out that even in the geometric setting the degree computations on the universal abelian variety are more complicated than on the moduli space of universal abelian varieties. This is the reason, why our next focus was to understand the situation on the universal abelian variety $\pi: \mathcal{B}_g(\mathbb{C}) \rightarrow \mathcal{A}_g(\mathbb{C})$ together with the line bundle $J = \pi^* \omega \otimes L$, where L is the rigidified symmetric line bundle corresponding to twice the principal polarization. As a first step, we investigated the case $g = 1$, where $\mathcal{B}_1(\mathbb{C})$ becomes an elliptic surface. There, it turns out that the natural invariant metric on L acquires singularities which are more severe than logarithmic ones, which leads to the fact that Chern–Weil theory no longer holds on $\mathcal{B}_1(\mathbb{C})$. In the respective investigations [BGKK16], we found that by replacing divisors on $\mathcal{B}_1(\mathbb{C})$ by b-divisors given by a limit of divisors on a sequence of blow-ups of $\mathcal{B}_1(\mathbb{C})$ along its boundary where the metric becomes singular, allows us to define sequences of intersection numbers which converge in such a way that Chern–Weil theory continues to hold in the limit. This work was generalized in [BBGHdJ21] to arbitrary genus $g \geq 1$ and the authors established the formula

$$(1.2) \quad \deg(\text{b-div}(J)) = 2^g \cdot \frac{d!}{d'!} \cdot \deg(\omega),$$

where $d = d' + g = \dim_{\mathbb{C}}(\mathcal{B}_g(\mathbb{C}))$ and $\text{b-div}(J)$ is the b-divisor associated to the line bundle J . This shows that the degree of the b-divisor associated to the line bundle J on $\mathcal{B}_g(\mathbb{C})$ is up to an elementary rational factor given by the degree of the Hodge bundle on $\mathcal{A}_g(\mathbb{C})$.

Of course, it is now immediate to ask if there is a framework such that formula (1.2) carries over to the arithmetic setting. Thus, we next consider the universal abelian scheme \mathcal{B}_g with the line bundle $\overline{\mathcal{J}} = \pi^* \overline{\omega} \otimes \overline{\mathcal{L}}$, where, as before, $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|_{\text{inv}})$ is the rigidified symmetric line bundle on \mathcal{B}_g corresponding to twice the principal polarization equipped with the natural invariant metric. As mentioned above, Chern–Weil theory fails to hold on $\mathcal{B}_g(\mathbb{C})$, which is one of the main reasons that the generalized arithmetic intersection theory developed in [BGKK05] no longer applies to define arithmetic intersections on \mathcal{B}_g . Thus, the question arises, if there is a further extension of arithmetic intersection theory that allows one to define and compute the arithmetic degree of the line bundle $\overline{\mathcal{J}}$. More specifically, one might ask, if formula (1.2) has an arithmetic analog. We are able to answer both questions affirmatively in this paper.

1.2. Main results. In order to find a promising attempt to establish a generalized arithmetic intersection theory for hermitian line bundles equipped with singular metrics on \mathcal{B}_g , it is worth noticing that the use of b-divisors in the geometric setting as described above can be extended to

the arithmetic setting by means of the theory of adelic arithmetic line bundles over quasi-projective arithmetic varieties developed by X. Yuan and S. W. Zhang in [YZ21]. In their theory, X. Yuan and S. W. Zhang defined arithmetic intersection numbers for so-called *adelic* arithmetic divisors as limits of “classical” arithmetic intersection numbers as long as the adelic arithmetic divisors are sufficiently positive, i. e., nef as arithmetic divisors. The arithmetic intersection numbers can be extended by linearity to *integrable* adelic arithmetic divisors, that are differences of nef adelic arithmetic divisors.

It turns out that the hermitian line bundle $\overline{\mathcal{L}}$ discussed above, defines an adelic arithmetic divisor on \mathcal{B}_g , which is nef so that its arithmetic self-intersection number is well-defined. On the other hand, even though the hermitian line bundle $\overline{\omega}$ also gives rise to an adelic arithmetic divisor on \mathcal{A}_g , we are not able to show its arithmetical nefness (in fact, we suspect that it is not even integrable), which prevents us from defining its arithmetic self-intersection number using the theory of X. Yuan and S. W. Zhang, although the existence of such an intersection number is known from $\overline{\mathcal{J}}$ [BGKK07]. Therefore, in order to have a well-defined arithmetic self-intersection product of $\overline{\mathcal{J}}$, we need to “merge” the results of [BGKK07] with those of [YZ21]. To this end, in our first main result, we extend the theory of X. Yuan and S. W. Zhang in Theorem 4.4 to be able to define arithmetic self-intersection numbers for certain adelic arithmetic divisors, which are not necessarily nef like $\overline{\omega}$. This extension uses two recent analytical tools: The theory of non-pluripolar products of positive currents developed in [BEGZ10] and the theory of relative energy introduced in [DDNL18b] (that uses non-pluripolar products in an essential way). We refer to the recent survey [DDNL23] and the references therein.

The relation between the theory developed in [YZ21] and the relative energy is as follows. Let $\overline{\mathcal{D}} = (\mathcal{D}, g_{\mathcal{D}})$ and $\overline{\mathcal{D}}' = (\mathcal{D}, g'_{\mathcal{D}})$ be two nef adelic arithmetic divisors on an arithmetic variety \mathcal{X} that share the same geometric part and only differ in the Green function’s part. Then, by choosing a “sufficiently large” arithmetic reference divisor $\overline{\mathcal{E}} = (\mathcal{E}, g_{\mathcal{E}})$ and by putting

$$\omega_E = \text{dd}^c g_E + \delta_E, \quad \varphi = g_{\mathcal{D}} - g_E, \quad \varphi' = g'_{\mathcal{D}} - g_E,$$

the functions φ and φ' turn out to be ω_E -plurisubharmonic on $\mathcal{X}(\mathbb{C})$ with φ having finite relative energy $I_{\varphi'}(\varphi)$ with respect to φ' . We are now able to establish the relationship

$$\overline{\mathcal{D}}^{d+1} = \overline{\mathcal{D}}'^{d+1} + I_{\varphi'}(\varphi)$$

between the corresponding arithmetic self-intersection numbers; here d is the relative dimension of \mathcal{X} . The relative energy $I_{\varphi'}(\varphi)$ is computed by means of non-pluripolar products. This shows that we can define an arithmetic self-intersection number for an adelic arithmetic divisor $(\mathcal{D}, g''_{\mathcal{D}})$, even if it is not nef nor integrable, as long as the function $\varphi'' = g''_{\mathcal{D}} - g_E$ is ω_E -plurisubharmonic and φ'' has finite relative energy with respect to φ' . One can argue that this is the largest possible extension of the theory given in [YZ21] to semipositive adelic divisors since one can say that a divisor with infinite relative energy should also have infinite arithmetic self-intersection. Many singular metrics that appear in practice are examples of metrics with finite relative energy, so we expect that this theory can be applied in many different situations.

Our second main result in Theorem 5.26 then states that the arithmetic self-intersection number of the adelic arithmetic line bundle $\overline{\mathcal{J}} = \pi^* \overline{\omega} \otimes \overline{\mathcal{L}}$, i. e., its arithmetic degree, is a well-defined real number. We compute this number as the third main result in Theorem 5.27 and obtain the formula

$$\widehat{\text{deg}}(\overline{\mathcal{J}}) = 2^g \cdot \frac{(d+1)!}{(d'+1)!} \cdot \widehat{\text{deg}}(\overline{\omega}),$$

which confirms that the arithmetic analog of formula (1.2) holds.

1.3. Outline. In Section 2, we gather the analytic prerequisites needed for the remainder of the paper. Namely, we review the theory of plurisubharmonic functions and its relation with semipositive hermitian metrics and Green functions. We also discuss the non-pluripolar product of positive currents as well as the notion of relative energy. We end the section by recalling the notions of algebraic and almost algebraic singularities.

In Section 3, we give a summary of the theory of adelic arithmetic divisors. We divide the discussion into three cases: The geometric case, the local arithmetic case, and the global arithmetic case. We also discuss the relationship between adelic arithmetic line bundles and adelic arithmetic divisors, and finally investigate the functorial properties that are needed for the computation of arithmetic self-intersection numbers (arithmetic degrees).

Section 4 is devoted to the extension of the theory to the finite relative energy case. To this end, we first show that the difference of two arithmetic self-intersection numbers can be expressed in terms of the relative energy. We then give a definition of arithmetic self-intersection numbers in terms of relative energy. We end the section by showing that some of the functorial properties are also satisfied by this extended arithmetic intersection product.

Finally, in Section 5, we recall the theory of Siegel–Jacobi forms and compute the arithmetic self-intersection number of the arithmetic line bundle of Siegel–Jacobi forms.

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2. ANALYTICAL PREREQUISITES

In this section we summarize the analytical prerequisites that are fundamental for this paper. Here, X will denote a projective complex manifold of dimension d .

We refer to [Dem92a], [Dem92b], and [NO90], but also to [Bou04], [BFJ16], and [Kim10] for definitions and proofs of the analytic properties discussed in this section.

2.1. Notation. We define the twisted differential d^c by the formula

$$d^c := \frac{i}{4\pi}(\bar{\partial} - \partial).$$

In this way, we obtain

$$dd^c = \frac{i}{2\pi} \partial\bar{\partial}.$$

This is the usual convention used in Arakelov theory as, for example, in [Sou92]. However, we point out that it differs by a factor of $1/2$ from the convention used in [Dem12].

2.1. Plurisubharmonic functions.

2.2. Definition. Let V be an open coordinate subset of X , which we identify with an open subset of \mathbb{C}^n . A function $\varphi: V \rightarrow \mathbb{R} \cup \{-\infty\}$ is called *plurisubharmonic*, if it satisfies the following two conditions:

- (i) The function φ is upper semi-continuous and not identically equal to $-\infty$ on every connected component of V .
- (ii) For every $z \in V$, every $\varepsilon > 0$ such that the closed ball $B(z, \varepsilon)$ is contained in V , and every $w \in \mathbb{C}^n$ with $|w| \leq \varepsilon$, the inequality

$$\varphi(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(z + we^{it}) dt$$

holds.

A function $\varphi: V \rightarrow \mathbb{R} \cup \{-\infty\}$ on an arbitrary open subset V of X is called *plurisubharmonic*, if V can be covered by open coordinate subsets $\{V_j\}_{j \in J}$ such that $\varphi|_{V_j}$ is plurisubharmonic on V_j for all $j \in J$.

2.3. Remark. Let $V \subseteq X$ denote an open subset of X . As usual, we then denote by $L_{\text{loc}}^1(V)$ the set of functions defined on V that are locally integrable on V . If $\varphi \in L_{\text{loc}}^1(V)$, then φ defines a distribution on V and thus a $(0, 0)$ -current on V , which we denote again by φ . Viewing the locally integrable function φ as a $(0, 0)$ -current, allows the formation $dd^c\varphi$, which then becomes a closed $(1, 1)$ -current.

Plurisubharmonic functions can be characterized in terms of positive currents. For the notion of positive currents and their properties, we refer to [Dem12, Chapter III]. The next result is [Dem12, Chapter I, Theorem 5.8].

2.4. Proposition. *Let $V \subseteq X$ denote an open subset and $\varphi: V \rightarrow \mathbb{R} \cup \{-\infty\}$ be a function.*

- (i) *If φ is plurisubharmonic on V , then φ belongs to $L^1_{\text{loc}}(V)$ and $\text{dd}^c\varphi$ is a closed positive $(1,1)$ -current.*
- (ii) *Conversely, if φ belongs to $L^1_{\text{loc}}(V)$ and $\text{dd}^c\varphi$ is a closed positive $(1,1)$ -current, then there exists a unique plurisubharmonic function φ_0 that agrees almost everywhere on V with φ .*

2.5. Remark. We note that part (ii) of Proposition 2.4 is the best we can achieve so that altering a function in a set of measure zero does not change the associated distribution, but may result that the function is no longer plurisubharmonic. Thus, in the sequel, when talking about plurisubharmonic functions, we will tacitly use the L^1_{loc} -topology, i. e., we will identify two functions that define the same distribution.

Since X is compact, any plurisubharmonic function on the whole X is known to be constant. To have a richer global theory, we need to allow some more flexibility.

2.6. Definition. Let ω_0 be a smooth closed positive $(1,1)$ -form on X . A function $\varphi: V \rightarrow \mathbb{R} \cup \{-\infty\}$ defined on an open subset $V \subseteq X$ is called ω_0 -*plurisubharmonic*, if it satisfies the following two conditions:

- (i) The function φ is upper semi-continuous and locally integrable on V .
- (ii) The current $\text{dd}^c\varphi + \omega_0$ is a closed positive $(1,1)$ -current on V .

The space of ω_0 -plurisubharmonic functions on X is denoted by $\text{PSH}(X, \omega_0)$.

When working in the L^1_{loc} -topology, there is no need to insist on the upper-semicontinuity. In fact, a consequence of Proposition 2.4 (ii) is the following global result.

2.7. Corollary. *Let $V \subseteq X$ be an open subset, ω_0 be a smooth closed positive $(1,1)$ -form, and $\varphi: V \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be a locally integrable function such that $\text{dd}^c\varphi + \omega_0$ is a (closed) positive $(1,1)$ -form. Then, there exists a unique ω_0 -plurisubharmonic function that agrees with φ almost everywhere.*

The plurisubharmonic functions can be classified by their type of singularity.

2.8. Definition. Let φ_1 and φ_2 be two ω_0 -plurisubharmonic functions on X . We say that φ_1 is *more singular than* φ_2 , if there is a constant C such that $\varphi_1 \leq \varphi_2 + C$. We say that φ_1 and φ_2 *have equivalent singularities*, if at the same time φ_1 is more singular than φ_2 and φ_2 is more singular than φ_1 . An equivalence class of singularities is called a *singularity type*. If φ is an ω_0 -plurisubharmonic function, then $[\varphi]$ denotes its singularity type. If φ_1 is more singular than φ_2 , we write $[\varphi_1] \leq [\varphi_2]$.

Note that $[\varphi_1] = [\varphi_2]$ if and only if the difference $\varphi_1 - \varphi_2$ is bounded.

2.9. Remark. Definition 2.8 is tailored along the assumption that X is compact. When X is not compact, we have to replace the positive constant C above by a locally bounded function.

2.2. Semipositive hermitian metrics.

2.10. Remark. If L is a line bundle on X , we recall that a smooth hermitian metric $\|\cdot\|_0$ on L is given by hermitian metrics on the fibers of L that vary smoothly across X . Moreover, if s denotes a non-trivial rational section of L , we have the equation

$$\text{dd}^c(-\log \|s\|_0^2) + \delta_{\text{div}(s)} = \omega_0,$$

where ω_0 is a closed smooth real $(1,1)$ -form on X , the so-called curvature form of the hermitian line bundle $\bar{L} = (L, \|\cdot\|_0)$, which is known to be independent of the choice of s .

2.11. Definition. Let L be a line bundle on X equipped with a smooth hermitian metric $\|\cdot\|_0$. We call a hermitian metric $\|\cdot\|$ a *singular hermitian metric* on L , if it is of the form

$$\|\cdot\| = \|\cdot\|_0 \cdot e^{-\varphi/2},$$

where φ is a locally integrable function on X . Two such metrics will be identified if they agree almost everywhere on X .

2.12. Remark. Let L be a line bundle on X equipped with a singular hermitian metric $\|\cdot\| = \|\cdot\|_0 \cdot e^{-\varphi/2}$ with $\varphi \in L^1_{\text{loc}}(X)$ as in the above definition. Then, for any open subset $V \subseteq X$ and any non-vanishing section s of L over V , we observe that

$$-\log \|s\|^2 = -\log \|s\|_0^2 + \varphi,$$

which shows that $-\log \|s\|^2 \in L^1_{\text{loc}}(V)$.

2.13. Definition. Let L be a line bundle on X equipped with a singular hermitian metric $\|\cdot\|$. Then, the singular metric $\|\cdot\|$ is called *semipositive*, if for any open subset $V \subseteq X$ and any non-vanishing section s of L over V , the locally integrable function $-\log \|s\|^2$ is plurisubharmonic on V .

2.14. Remark. Let L be a line bundle on X equipped with a singular hermitian metric $\|\cdot\|$. Let then s and s' be non-vanishing sections of L over V and V' , respectively. Since s and s' differ only by a unit on the intersection $V \cap V'$, we arrive at the equality of currents

$$\text{dd}^c(-\log \|s\|^2)|_{V \cap V'} = \text{dd}^c(-\log \|s'\|^2)|_{V \cap V'}$$

on $V \cap V'$. This shows that the $(1,1)$ -current $\text{dd}^c(-\log \|s\|^2)$ gives rise to a $(1,1)$ -current on the whole of X .

2.15. Lemma. Let L be a line bundle on X equipped with a smooth hermitian metric $\|\cdot\|_0$ with curvature form ω_0 . Then, there is a bijective correspondence

$$\{\text{semipositive metrics } \|\cdot\| \text{ on } L\} \longleftrightarrow \{\omega_0\text{-plurisubharmonic functions } \varphi \text{ on } X\},$$

given by the assignment $\|\cdot\| \mapsto \varphi$, where φ is locally, on a open subset $V \subseteq X$, determined by

$$\varphi = -\log \left(\frac{\|s\|^2}{\|s\|_0^2} \right),$$

with s being a non-vanishing section of L over V .

Proof. We start by noting that the local definition of φ globalizes, thus φ is defined on the whole of X . Next, given a semipositive metric $\|\cdot\|$ on L , we have for any open subset $V \subseteq X$ and any non-vanishing section s of L over V that

$$\varphi = -\log \|s\|^2 + \log \|s\|_0^2$$

on V . Since the locally integrable function $-\log \|s\|^2$ is plurisubharmonic on V by Definition 2.13, part (i) of Definition 2.6 is fulfilled. Furthermore, since $-\text{dd}^c \log \|s\|_0^2 = \omega_0$, we find that

$$\text{dd}^c \varphi + \omega_0 = \text{dd}^c(-\log \|s\|^2) + \text{dd}^c(\log \|s\|_0^2) + \omega_0 = \text{dd}^c(-\log \|s\|^2),$$

which, again using the plurisubharmonicity of $-\log \|s\|^2$, is a positive $(1,1)$ -current on V , and thus verifies part (ii) of Definition 2.6. In total, we have proven that φ is an ω_0 -plurisubharmonic function on X .

On the other hand, if φ is an ω_0 -plurisubharmonic function on X , we define $\|\cdot\| = \|\cdot\|_0 \cdot e^{-\varphi/2}$ and easily check that $\|\cdot\|$ becomes a semipositive metric on L . \square

2.3. Green functions. For our purposes, it will be convenient to translate the above results into the language of divisors and Green functions. Note that for us a Green function will be more appropriately called a Green current (of degree zero).

2.16. Definition. Let D be a divisor on X . A *Green function for D* is the current associated to a locally integrable function $g: X \rightarrow \mathbb{R} \cup \{\pm\infty\}$. A Green function g for D is called *of smooth, of continuous, of locally bounded, or of plurisubharmonic type*, if g can be chosen in such a way that for any open subset $V \subseteq X$ and any local equation f_D for D on V , the function

$$g + \log |f_D|^2$$

is smooth, continuous, locally bounded, or plurisubharmonic on V , respectively.

Note that two locally integrable functions on X define the same Green function for D if and only if they agree almost everywhere.

2.17. Definition. Let g be a Green function for the divisor D . Then, we define the $(1, 1)$ -current

$$\omega_D(g) := dd^c g + \delta_D.$$

If g is a Green function for D of smooth type, then $\omega_D(g)$ is a smooth $(1, 1)$ -form. Moreover, if g is a Green function for D of plurisubharmonic type, then $\omega_D(g)$ is a closed positive $(1, 1)$ -current.

2.18. Lemma. Let D be a divisor on X , $L = \mathcal{O}(D)$ the corresponding line bundle, and s a section of L with $\text{div}(s) = D$. Then, there is a bijective correspondence

$$\{\text{singular hermitian metrics } \|\cdot\| \text{ on } L\} \longleftrightarrow \{\text{Green functions } g \text{ for } D\},$$

given by the assignment $\|\cdot\| \mapsto -\log \|s\|^2$. Moreover, this correspondence sends smooth, continuous, or semipositive hermitian metrics on L to Green functions for D of smooth, continuous, or plurisubharmonic type, respectively.

Proof. The proof of the lemma follows immediately from Definition 2.16 and the preceding discussion. \square

2.19. Remark. Note that by Definition 2.16 any locally integrable function on X is a Green function for any divisor. For example, we can start with a Green function g for a divisor D of smooth, continuous, or plurisubharmonic type. We can then think of g as being a Green function for another divisor D' . But now the function g , viewed as a Green function for D' , may become singular or may no longer be of plurisubharmonic type.

The next lemma shows that positivity is always preserved when increasing the divisor.

2.20. Lemma. Let D and D' be divisors on X such that the divisor $C := D' - D$ is effective. If g is a Green function for D of plurisubharmonic type, then g also is a Green function for D' of plurisubharmonic type.

Proof. Let $V \subseteq X$ be an open subset and let f_D , $f_{D'}$, and f_C be local equations on V for D , D' , and C , respectively, satisfying $f_{D'} = f_D \cdot f_C$. Since C is effective, the function f_C is holomorphic on V and hence the function $\log |f_C|^2$ is plurisubharmonic on V . Now, since the function $g + \log |f_D|^2$ is plurisubharmonic on V by hypothesis, and since the sum of plurisubharmonic functions is again plurisubharmonic, we deduce that

$$g + \log |f_{D'}|^2 = g + \log |f_D|^2 + \log |f_C|^2$$

is also plurisubharmonic on V . This proves the claim. \square

Let us next rephrase the result of Lemma 2.20 in terms of metrized line bundles. To do this, we need the following remark.

2.21. Remark. Let D be a divisor on X and assume that the corresponding line bundle $\mathcal{O}(D)$ is equipped with a semipositive hermitian metric $\|\cdot\|$. Furthermore, let D' be another divisor on X such that the divisor $C := D' - D$ is effective. Therefore, we have an inclusion $\mathcal{O}(D) \subseteq \mathcal{O}(D')$ of line bundles. By definition, we can view the line bundles $\mathcal{O}(D)$ and $\mathcal{O}(D')$ as subsheaves of

the sheaf of rational functions \mathcal{K}_X of X . Thus, the unit function 1 of \mathcal{K}_X , which gives rise to a rational section of $\mathcal{O}(D)$, can also be viewed as a rational section of $\mathcal{O}(D')$. In this way, we can define by means of the semipositive hermitian metric $\|\cdot\|$ on $\mathcal{O}(D)$, a hermitian metric $\|\cdot\|'$ on $\mathcal{O}(D')$ through the formula

$$\|1\|' := \|1\|,$$

which will turn out to be again semipositive in the subsequent lemma.

2.22. Lemma. *Let D and D' be divisors on X such that the divisor $C := D' - D$ is effective. If $\|\cdot\|$ is a semipositive hermitian metric on $\mathcal{O}(D)$, then the hermitian metric $\|\cdot\|'$ on $\mathcal{O}(D')$ defined as in Remark 2.21 is also semipositive.*

Proof. By our assumptions, we have

$$\mathcal{O}(D') = \mathcal{O}(D) \otimes \mathcal{O}(C).$$

By now choosing V small enough as an open subset of X , we find non-vanishing sections s, s' , and s_C of the line bundles $\mathcal{O}(D), \mathcal{O}(D')$, and $\mathcal{O}(C)$ over V , respectively, such that we have

$$s' = s \cdot s_C.$$

Observing next that the inclusion $\mathcal{O}(C) \subseteq \mathcal{K}_X$ is provided by the assignment $s_C \mapsto f_C^{-1} \cdot h$, where $f_C \in \mathcal{K}_X(V)$ gives rise to a local equation for the divisor C on V and $h \in \mathcal{K}_X(V)^\times$ is a unit, we find

$$s' = s \cdot f_C^{-1} \cdot h.$$

From this equality we compute

$$\begin{aligned} -\log \|s'\|'^2 &= -\log \|s\|^2 - \log |f_C^{-1}|^2 - \log |h|^2 \\ &= -\log \|s\|^2 + \log |f_C|^2 - \log |h|^2. \end{aligned}$$

Arguing now as in the proof of Lemma 2.20, the plurisubharmonicity of $-\log \|s\|^2$ on V implies the plurisubharmonicity of $-\log \|s'\|'^2$ on V , which proves the semipositivity of the hermitian metric $\|\cdot\|'$ on $\mathcal{O}(D')$. \square

In the next lemma we give a criterion when the change of divisor preserves positivity even if the change of divisor is not increasing.

2.23. Lemma. *Let D and D' be divisors on X . If g_D is a Green function for D of plurisubharmonic type, and if for each point $x \in X$, there is a neighbourhood V of x and a local equation $f_{D'}$ of D' such that $g_D + \log |f_{D'}|^2$ is locally bounded above on V , then g_D also is a Green function for D' of plurisubharmonic type.*

Proof. We begin by choosing a covering of X by open subsets V such that in each open subset there are equations f_D and $f_{D'}$ for D and D' , respectively. Moreover, $g_D + \log |f_D|^2$ extends to a plurisubharmonic function on V and $g_D + \log |f_{D'}|^2$ is bounded above on V . On $V \setminus (|D| \cup |D'|)$, the function

$$g_D + \log |f_{D'}|^2 = g_D + \log |f_D|^2 + \log |f_{D'}/f_D|^2$$

is plurisubharmonic because the function $f_{D'}/f_D$ is holomorphic there. Since the function $g_D + \log |f_{D'}|^2$ is bounded above on V , by [Dem12, Theorem 5.24], it extends to a plurisubharmonic function on V , proving the lemma. \square

2.4. Divisors with real coefficients. Since later on we will be interested in different completions of spaces of divisors, it is natural to use real coefficients instead of rational or integral ones. We will write

$$\text{Div}_{\mathbb{R}}(X) := \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

The notion of Green function can be directly extended to divisors with real coefficients.

2.24. Definition. Let D be a divisor on X with real coefficients. A *Green function* g for D is the current associated to a locally integrable function $g: X \rightarrow \mathbb{R} \cup \{\pm\infty\}$. It is said to be *of smooth, of continuous, or of locally bounded type*, if there are decompositions

$$(2.1) \quad D = \sum_{j=1}^n a_j D_j \quad \text{and} \quad g = \sum_{j=1}^n a_j g_j,$$

with $D_j \in \text{Div}(X)$ and $a_j \in \mathbb{R}$ such that g_j are Green functions for D_j of smooth, of continuous, or of locally bounded type for $j = 1, \dots, n$, respectively. If D is a divisor on X with real coefficients and g is a Green function for D , we continue to write

$$\omega_D(g) = \text{dd}^c g + \delta_D.$$

The definition of a Green function of plurisubharmonic type does not rely on writing the divisor as a linear combination of ordinary divisors, but can be written directly.

2.25. Definition. Let D be a divisor on X with real coefficients. A Green function g for D is said to be *of plurisubharmonic type*, if the closed $(1, 1)$ -current $\omega_D(g)$ is positive.

The next lemma gives a characterization of Green functions of plurisubharmonic type.

2.26. Lemma. *Let D be a divisor on X with real coefficients and g_0 be a Green function for D of smooth type; put $\omega_0 = \omega_D(g_0)$. A Green function g for D is of plurisubharmonic type, if and only if the function $g - g_0$ is ω_0 -plurisubharmonic.*

Proof. We first compute

$$\omega_D(g) = \text{dd}^c g + \delta_D = \text{dd}^c g + \omega_0 - \omega_0 + \delta_D = \text{dd}^c g + \omega_0 - \text{dd}^c g_0 = \text{dd}^c(g - g_0) + \omega_0.$$

Therefore, if $g - g_0$ is ω_0 -plurisubharmonic, then $\omega_D(g)$ is positive, and hence g is of plurisubharmonic type. For the converse, assume that $\omega_D(g)$ is positive. Then, $\text{dd}^c(g - g_0) + \omega_0$ is positive. By Corollary 2.7, there is a unique ω_0 -plurisubharmonic function φ that agrees with $g - g_0$ almost everywhere. Then, the function $g' := g_0 + \varphi$ defines the same Green function as g and $g' - g_0$ is ω_0 -plurisubharmonic. \square

2.27. Remark. If D is a divisor with integral coefficients, then both definitions of Green functions of plurisubharmonic type, namely Definition 2.16 and Definition 2.25, are equivalent, after identifying functions that agree almost everywhere. Indeed, if g is a Green function of plurisubharmonic type for D according to Definition 2.16, then for any open subset $V \subseteq X$ and any local equation f_D for D on V , we have that $g + \log |f_D|^2$ is plurisubharmonic. In particular, we have that

$$\omega_D(g)|_V = \text{dd}^c(g + \log |f_D|^2) \geq 0.$$

Hence, g is of plurisubharmonic type according to Definition 2.25. Conversely, assume that g satisfies Definition 2.25. Let g_0 be a Green function of smooth type for D and put $\omega_0 = \omega_D(g_0)$. Then, the proof of Lemma 2.26 shows that there exists an ω_0 -plurisubharmonic function φ that agrees with $g - g_0$ almost everywhere; in other words, the function $g' = g_0 + \varphi$ agrees with g almost everywhere. Locally, on V , we now have

$$g' + \log |f_D|^2 = g_0 + \log |f_D|^2 + \varphi.$$

Since $g_0 + \log |f_D|^2$ is smooth and φ is ω_0 -plurisubharmonic, we deduce that $g' + \log |f_D|^2$ is upper semicontinuous, does not take the value $+\infty$, and that $\text{dd}^c(g' + \log |f_D|^2)$ is positive on V . Thus, $g' + \log |f_D|^2$ is plurisubharmonic on V .

We can extend Lemma 2.20 to divisors with real coefficients.

2.28. Lemma. *Let D and D' be divisors on X with real coefficients such that the divisor $C := D' - D$ is effective. If g is a Green function for D of plurisubharmonic type, then g also is a Green function for D' of plurisubharmonic type.*

Proof. Since the divisor $C = D' - D$ is effective, the current δ_C is positive. Assume now that g is a Green function for D of plurisubharmonic type. Then, we have

$$\omega_{D'}(g) = dd^c g + \delta_{D'} = dd^c g + \delta_D + \delta_C = \omega_D(g) + \delta_C \geq 0.$$

Hence, g is a Green function for D' of plurisubharmonic type. \square

2.5. Non-pluripolar product of currents. In general, currents cannot be multiplied. Nevertheless, the Bedford–Taylor calculus allows us to define currents of the form

$$dd^c \varphi_1 \wedge \dots \wedge dd^c \varphi_p \quad \text{and} \quad \varphi_0 dd^c \varphi_1 \wedge \dots \wedge dd^c \varphi_p,$$

whenever the functions $\varphi_0, \varphi_1, \dots, \varphi_p$ are locally bounded plurisubharmonic functions on an open subset $V \subseteq X$. In order to extend these products to plurisubharmonic functions that are not locally bounded on V , one can follow two directions.

One direction consists in imposing conditions on the sets, where the functions $\varphi_0, \varphi_1, \dots, \varphi_p$ take the value $-\infty$. For this approach, we refer for instance to [Dem12, Chapter III, § 3].

The other direction is to “throw away” the part of the currents that is supported on the so-called pluripolar sets. This point of view is introduced in [BEGZ10]. We recall this construction next.

2.29. Definition. Let $Z \subseteq X$ be a subset of X . Then, Z is called a *complete pluripolar set*, if for each $x \in Z$, there exists a connected open subset $V_x \subseteq X$ containing x and a plurisubharmonic function $\varphi_x: V_x \rightarrow \mathbb{R} \cup \{-\infty\}$, not identically equal to $-\infty$, such that

$$Z \cap V_x = \{z \in V_x \mid \varphi_x(z) = -\infty\}.$$

We note that it can be shown that any proper analytic subvariety Z of X is a complete pluripolar set. Any subset of a complete pluripolar set is called a *pluripolar set*.

2.30. Definition. A function $\varphi: X \rightarrow \mathbb{R} \cup \{-\infty\}$ is said to *have small unbounded locus*, if there exists a complete pluripolar set Z such that φ is locally bounded outside Z .

2.31. Definition. For $j = 1, \dots, p$, let ω_j be closed smooth real $(1, 1)$ -forms on X and let φ_j be ω_j -plurisubharmonic functions on X having small unbounded locus. Then, the *non-pluripolar product of $(1, 1)$ -currents*

$$(2.2) \quad \langle (dd^c \varphi_1 + \omega_1) \wedge \dots \wedge (dd^c \varphi_p + \omega_p) \rangle$$

is the (p, p) -current defined as follows: For each $k \in \mathbb{N}_{>0}$, we introduce the set

$$U_k := \{x \in X \mid \varphi_j(x) \geq -k \text{ for } j = 1, \dots, p\};$$

we observe that the functions φ_j are locally bounded on U_k for $j = 1, \dots, p$. Then, letting $q := d - p$ and η be a smooth (q, q) -form on X , the non-pluripolar product (2.2) is defined as

$$(2.3) \quad \langle (dd^c \varphi_1 + \omega_1) \wedge \dots \wedge (dd^c \varphi_p + \omega_p) \rangle(\eta) := \lim_{k \rightarrow \infty} \int_{U_k} (dd^c \varphi_1 + \omega_1) \wedge \dots \wedge (dd^c \varphi_p + \omega_p) \wedge \eta,$$

where the right-hand side of equation (2.3) is defined using Bedford–Taylor theory.

The basic properties of the non-pluripolar product of $(1, 1)$ -currents just defined are the following. They follow from [BEGZ10, Proposition 1.6], [BEGZ10, Proposition 1.4], and [BEGZ10, Theorem 1.8], because we are assuming that X is projective, and hence is Kähler.

2.32. Theorem. *With the notations of Definition 2.31, we set $T_j := dd^c \varphi_j + \omega_j$ for $j = 1, \dots, p$. Then, we have the following statements:*

- (i) *The non-pluripolar product $\langle T_1 \wedge \dots \wedge T_p \rangle$ is well-defined and depends only on the currents T_j and not on the particular choice of functions φ_j for $j = 1, \dots, p$.*
- (ii) *The non-pluripolar product $\langle T_1 \wedge \dots \wedge T_p \rangle$ is symmetric. Moreover, it is multilinear in the following sense: If, for example, ω'_1 is another closed smooth real $(1, 1)$ -form on X , φ'_1 is an ω'_1 -plurisubharmonic function on X having small unbounded locus, $T'_1 := dd^c \varphi'_1 + \omega'_1$, and $\lambda_1, \lambda'_1 \in \mathbb{R}_{>0}$, then we have*

$$\langle (\lambda_1 T_1 + \lambda'_1 T'_1) \wedge \dots \wedge T_p \rangle = \lambda_1 \langle T_1 \wedge \dots \wedge T_p \rangle + \lambda'_1 \langle T'_1 \wedge \dots \wedge T_p \rangle.$$

(iii) The current $\langle T_1 \wedge \dots \wedge T_p \rangle$ is closed and positive.

Since $\langle T_1 \wedge \dots \wedge T_p \rangle$ is closed, we will denote by $\text{cl}(\langle T_1 \wedge \dots \wedge T_p \rangle)$ the associated cohomology class in $H^{p,p}(X, \mathbb{C})$.

The following monotonicity criterion is contained in [BEGZ10, Theorem 1.16].

2.33. Theorem. For $j = 1, \dots, p$, let ω_j be closed smooth real $(1,1)$ -forms on X and let φ_j, φ'_j be ω_j -plurisubharmonic functions on X having small unbounded locus. If there are constants $c_j \in \mathbb{R}_{>0}$ such that $\varphi'_j \leq \varphi_j + c_j$, i. e., the functions φ'_j are more singular than the functions φ_j for $j = 1, \dots, p$, then the cohomology class

$$\text{cl}(\langle (\text{dd}^c \varphi_1 + \omega_1) \wedge \dots \wedge (\text{dd}^c \varphi_p + \omega_p) \rangle) - \text{cl}(\langle (\text{dd}^c \varphi'_1 + \omega_1) \wedge \dots \wedge (\text{dd}^c \varphi'_p + \omega_p) \rangle)$$

is strongly positive. This means that for any closed smooth and weakly positive form η , the inequality

$$\int_X \langle (\text{dd}^c \varphi_1 + \omega_1) \wedge \dots \wedge (\text{dd}^c \varphi_p + \omega_p) \rangle \wedge \eta - \int_X \langle (\text{dd}^c \varphi'_1 + \omega_1) \wedge \dots \wedge (\text{dd}^c \varphi'_p + \omega_p) \rangle \wedge \eta \geq 0$$

holds. In particular, if $p = d$, the inequality

$$\int_X \langle (\text{dd}^c \varphi'_1 + \omega_1) \wedge \dots \wedge (\text{dd}^c \varphi'_d + \omega_d) \rangle \leq \int_X \langle (\text{dd}^c \varphi_1 + \omega_1) \wedge \dots \wedge (\text{dd}^c \varphi_d + \omega_d) \rangle$$

holds.

The non-pluripolar product of $(1,1)$ -currents given in Definition 2.31 can be used to define a non-pluripolar product of $(1,1)$ -currents of the form $\omega_D(g)$ associated to Green functions g for a divisor D on X which are of plurisubharmonic type.

2.34. Definition. For $j = 1, \dots, p$, let D_j be divisors on X and let g_j be Green functions for D_j of plurisubharmonic type having small unbounded locus. Furthermore, let $g_{j,0}$ be Green functions for D_j of smooth type, and put $\omega_{j,0} := \omega_{D_j}(g_{j,0})$. Then, the difference $\varphi_j := g_j - g_{j,0}$ is an $\omega_{j,0}$ -plurisubharmonic function on X . We thus define

$$\langle \omega_{D_1}(g_1) \wedge \dots \wedge \omega_{D_p}(g_p) \rangle := \langle (\text{dd}^c \varphi_1 + \omega_{1,0}) \wedge \dots \wedge (\text{dd}^c \varphi_p + \omega_{p,0}) \rangle.$$

We note that by part (i) of Theorem 2.32, the non-pluripolar product introduced above is well-defined and does not depend on the choice of Green functions $g_{j,0}$ for D_j of smooth type ($j = 1, \dots, p$). What is surprising is that it only depends on the Green functions g_j , but not on the actual divisors D_j as long as g_j are Green functions for D_j of plurisubharmonic type ($j = 1, \dots, p$).

2.35. Proposition. For $j = 1, \dots, p$, let D_j, D'_j be divisors on X and let g_j be Green functions for D_j of plurisubharmonic type such that they are also Green functions for D'_j of plurisubharmonic type. Then, we have the equality

$$\langle \omega_{D'_1}(g_1) \wedge \dots \wedge \omega_{D'_p}(g_p) \rangle = \langle \omega_{D_1}(g_1) \wedge \dots \wedge \omega_{D_p}(g_p) \rangle.$$

Proof. We can find divisors D''_j such that $D''_j - D_j$ and $D''_j - D'_j$ are effective. It is then enough to prove the two equalities

$$\langle \omega_{D'_1}(g_1) \wedge \dots \wedge \omega_{D'_p}(g_p) \rangle = \langle \omega_{D''_1}(g_1) \wedge \dots \wedge \omega_{D''_p}(g_p) \rangle = \langle \omega_{D_1}(g_1) \wedge \dots \wedge \omega_{D_p}(g_p) \rangle.$$

Hence, without loss of generality, we can assume that we are in the case when $C_j := D'_j - D_j$ is effective for $j = 1, \dots, p$.

By construction, we have that $\omega_{D'_j}(g_j) = \omega_{D_j}(g_j) + \delta_{C_j}$ for $j = 1, \dots, p$. By part (ii) of Theorem 2.32, we are thus reduced to prove that

$$\langle \delta_{C_1} \wedge \omega_{D_2}(g_2) \wedge \dots \wedge \omega_{D_p}(g_p) \rangle = 0.$$

However, this follows directly from the definition of the non-pluripolar product of $(1,1)$ -currents: For this, let g be a Green function for C_1 of smooth type and let $\omega := \omega_{C_1}(g)$, so that ω is a smooth $(1,1)$ -form. Now, put $\varphi = -g$, so that φ becomes an ω -plurisubharmonic function on X satisfying $\text{dd}^c \varphi + \omega = \delta_{C_1}$. On any of the sets U_k introduced in Definition 2.31, the function φ

will be bounded below. Therefore, we have $U_k \cap C_1 = \emptyset$, which implies that after replacing φ_1 and ω_1 in equation (2.3) by φ and ω , respectively, all the integrals on the right-hand side of (2.3) will vanish. This concludes the proof of the proposition. \square

2.36. Example. Let D and D' be divisors on X such that the divisor $C := D' - D$ is effective. Let $\|\cdot\|$ be a semipositive hermitian metric on $\mathcal{O}(D)$ and define the hermitian metric $\|\cdot\|'$ on $\mathcal{O}(D')$ as in Remark 2.21; by Lemma 2.22, we know that $\|\cdot\|'$ is also semipositive. Choosing a non-trivial rational section s of $\mathcal{O}(D)$, which can also be viewed as a non-trivial rational section of $\mathcal{O}(D')$, we obtain the relation (see the proof of Lemma 2.22)

$$\mathrm{dd}^c(-\log \|s\|'^2) = \mathrm{dd}^c(-\log \|s\|^2) + \delta_C,$$

which shows that we have the following equality of non-pluripolar parts

$$\langle \mathrm{dd}^c(-\log \|s\|'^2) \rangle = \langle \mathrm{dd}^c(-\log \|s\|^2) \rangle.$$

Letting φ and φ' denote the ω_0 -plurisubharmonic functions on X corresponding to the semipositive hermitian metrics $\|\cdot\|$ and $\|\cdot\|'$, respectively, we can rewrite the above relation as

$$\mathrm{dd}^c \varphi' + \omega_0 = \mathrm{dd}^c \varphi + \omega_0 + \delta_C,$$

which shows that the ω_0 -plurisubharmonic function φ' is “more singular” than the ω_0 -plurisubharmonic function φ ; the equality of non-pluripolar parts translates as

$$\langle \mathrm{dd}^c \varphi' + \omega_0 \rangle = \langle \mathrm{dd}^c \varphi + \omega_0 \rangle.$$

2.6. Capacity and monotonicity of non-pluripolar products. The monotonicity result given in Theorem 2.33 has been strengthened in [DDNL18b] to obtain weak convergence of measures. We need some preliminary definitions (see [GZ17, Chapter IX]).

2.37. Definition. Let ω be a Kähler form on X . Write $V = \int_S \omega^d$. For every Borel subset $K \subseteq X$, the ω -capacity of K is defined as

$$\mathrm{Cap}_\omega(K) := \sup \left\{ \frac{1}{V} \int_K (\mathrm{dd}^c \varphi + \omega)^{\wedge d} \mid \varphi \in \mathrm{PSH}(X, \omega), 0 \leq \varphi \leq 1 \right\}.$$

If $K \subseteq X$ is any subset, the *outer* ω -capacity of K is defined as

$$\mathrm{Cap}_\omega^*(K) := \inf \{ \mathrm{Cap}_\omega(W) \mid W \text{ open}, K \subseteq W \}.$$

2.38. Definition. A sequence of functions $(\varphi_n)_{n \in \mathbb{N}}$ is said to *converge in capacity* to a function φ , if for any $\delta > 0$, we have

$$\lim_{n \rightarrow \infty} \mathrm{Cap}_\omega^* \{x \in X \mid |\varphi_n(x) - \varphi(x)| > \delta\} = 0.$$

2.39. Example. Let $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence of plurisubharmonic functions that converges monotonically almost everywhere to a plurisubharmonic function φ . Then, the sequence $(\varphi_n)_{n \in \mathbb{N}}$ converges in capacity to φ . See [GZ17, Proposition 4.25].

The following result is [DDNL18b, Theorem 1.2].

2.40. Theorem. For $j = 1, \dots, p$, let ω_j be closed smooth real $(1, 1)$ -forms on X and let φ_j be ω_j -plurisubharmonic functions on X . Furthermore, let $(\varphi_{j,n})_{n \in \mathbb{N}}$ be a sequence of ω_j -plurisubharmonic functions on X converging in capacity to φ_j and satisfying

$$\int_X \langle (\mathrm{dd}^c \varphi_1 + \omega_1) \wedge \dots \wedge (\mathrm{dd}^c \varphi_d + \omega_d) \rangle \geq \limsup_{n \rightarrow \infty} \int_X \langle (\mathrm{dd}^c \varphi_{1,n} + \omega_1) \wedge \dots \wedge (\mathrm{dd}^c \varphi_{d,n} + \omega_d) \rangle.$$

Then, the sequence of measures $(\langle (\mathrm{dd}^c \varphi_{1,n} + \omega_1) \wedge \dots \wedge (\mathrm{dd}^c \varphi_{d,n} + \omega_d) \rangle)_{n \in \mathbb{N}}$ converges weakly to the measure $\langle (\mathrm{dd}^c \varphi_1 + \omega_1) \wedge \dots \wedge (\mathrm{dd}^c \varphi_d + \omega_d) \rangle$.

2.7. Relative energy. In this subsection, we assume that ω is a Kähler form on X and that ω_0 is a smooth closed positive $(1,1)$ -form on X . We start by recalling the definitions of relative full mass and relative energy following [DDNL18b].

2.41. Definition. Let $\psi \in \text{PSH}(X, \omega_0)$ having small unbounded locus. The space $\mathcal{E}(X, \omega_0, \psi)$ of *functions with relative full mass (with respect to ψ)* is the subspace of $\text{PSH}(X, \omega_0)$ consisting of those functions φ , which are more singular than ψ and such that

$$\int_X \langle (\text{dd}^c \varphi + \omega_0)^{\wedge d} \rangle = \int_X \langle (\text{dd}^c \psi + \omega_0)^{\wedge d} \rangle.$$

2.42. Definition. Let $\psi \in \text{PSH}(X, \omega_0)$ and $\varphi \in \mathcal{E}(X, \omega_0, \psi)$ with $[\varphi] = [\psi]$, i. e., $\varphi - \psi$ is bounded, then the *relative energy of φ (with respect to ψ)* is defined as

$$I_\psi(\varphi) := \frac{1}{d+1} \sum_{k=0}^d \int_X (\varphi - \psi) \langle (\text{dd}^c \varphi + \omega_0)^{\wedge k} \wedge (\text{dd}^c \psi + \omega_0)^{\wedge (d-k)} \rangle.$$

Moreover, if $\varphi \in \text{PSH}(X, \omega_0)$ does not satisfy $[\varphi] = [\psi]$, its relative energy is defined as

$$I_\psi(\varphi) := \inf \{ I_\psi(u) \mid u \in \mathcal{E}(X, \omega_0, \psi), [u] = [\psi], \varphi \leq u \}.$$

With this notion at hand, the space $\mathcal{E}^1(X, \omega_0, \psi)$ of *functions of finite relative energy* is then defined as the subspace of $\mathcal{E}(X, \omega_0, \psi)$ consisting of those functions whose relative energy is finite.

We have the following characterization of functions with finite relative energy, whose proof is given in [DDNL23, Lemma 5.6].

2.43. Lemma. *Let $\varphi \in \text{PSH}(X, \omega_0)$ with $[\varphi] \leq [\psi]$. Then, $\varphi \in \mathcal{E}^1(X, \omega_0, \psi)$ if and only if $\varphi \in \mathcal{E}(X, \omega_0, \psi)$ and*

$$\int_X (\varphi - \psi) \langle (\text{dd}^c \varphi + \omega_0)^{\wedge d} \rangle > -\infty.$$

2.44. Remark. The spaces $\mathcal{E}(X, \omega_0, \psi)$ and $\mathcal{E}^1(X, \omega_0, \psi)$ are convex. Moreover, the spaces $\mathcal{E}(X, \omega_0, \psi)$ also satisfy a convexity condition when we change the reference functions ψ . Namely, assume that $\psi \in \text{PSH}(X, \omega_0)$ and $\psi' \in \text{PSH}(X, \omega'_0)$. Under the assumption that

$$\int_X \langle (\text{dd}^c \psi + \omega_0)^{\wedge d} \rangle > 0 \quad \text{and} \quad \int_X \langle (\text{dd}^c \psi' + \omega'_0)^{\wedge d} \rangle > 0,$$

one can prove that given $\varphi \in \mathcal{E}(X, \omega_0, \psi)$ and $\varphi' \in \mathcal{E}(X, \omega'_0, \psi')$, then the sum $\varphi + \varphi' \in \mathcal{E}(X, \omega_0 + \omega'_0, \psi + \psi')$. Nevertheless, this convexity property is no longer true for the spaces $\mathcal{E}^1(X, \omega_0, \psi)$. That is, given $\varphi \in \mathcal{E}^1(X, \omega_0, \psi)$ and $\varphi' \in \mathcal{E}^1(X, \omega'_0, \psi')$, then the sum $\varphi + \varphi'$ need not necessarily belong to $\mathcal{E}^1(X, \omega_0 + \omega'_0, \psi + \psi')$. See [DN15, Example 4.5] for an example.

Before we introduce next the mixed version of the relative energy, we need some additional notation.

2.45. Notation. Given smooth closed positive $(1,1)$ -forms ω_j on X for $j = 0, \dots, d$, we write $\boldsymbol{\omega} = (\omega_0, \dots, \omega_d)$ and $\text{PSH}(X, \boldsymbol{\omega}) = \text{PSH}(X, \omega_0) \times \dots \times \text{PSH}(X, \omega_d)$. Next, given functions $\psi_j \in \text{PSH}(X, \omega_j)$ of small unbounded locus, we write $\boldsymbol{\psi} = (\psi_0, \dots, \psi_d)$ to obtain $\boldsymbol{\psi} \in \text{PSH}(X, \boldsymbol{\omega})$ of small unbounded locus; we also write $\mathcal{E}(X, \boldsymbol{\omega}, \boldsymbol{\psi}) = \mathcal{E}(X, \omega_0, \psi_0) \times \dots \times \mathcal{E}(X, \omega_d, \psi_d)$. Finally, given functions $\varphi_j \in \mathcal{E}(X, \omega_j, \psi_j)$, we write $\boldsymbol{\varphi} = (\varphi_0, \dots, \varphi_d)$ to obtain $\boldsymbol{\varphi} \in \mathcal{E}(X, \boldsymbol{\omega}, \boldsymbol{\psi})$. The notation $\boldsymbol{\varphi} \leq \boldsymbol{\psi}$ means $\varphi_j \leq \psi_j$; similarly, the notation $[\boldsymbol{\varphi}] = [\boldsymbol{\psi}]$ means $[\varphi_j] = [\psi_j]$.

2.46. Definition. Let $\boldsymbol{\omega} = (\omega_0, \dots, \omega_d)$ be smooth closed positive $(1,1)$ -forms on X and let $\boldsymbol{\psi} \in \text{PSH}(X, \boldsymbol{\omega})$ be of small unbounded locus. For $\boldsymbol{\varphi} \in \mathcal{E}(X, \boldsymbol{\omega}, \boldsymbol{\psi})$ with $[\boldsymbol{\varphi}] = [\boldsymbol{\psi}]$, we define the *mixed relative energy of $\boldsymbol{\varphi}$ (with respect to $\boldsymbol{\psi}$)* by

$$(2.4) \quad I_{\boldsymbol{\psi}}(\boldsymbol{\varphi}) := \sum_{k=0}^d \int_X (\varphi_k - \psi_k) \langle (\text{dd}^c \varphi_0 + \omega_0 \wedge \dots \wedge (\text{dd}^c \varphi_{k-1} + \omega_{k-1}) \wedge \dots \wedge (\text{dd}^c \psi_{k+1} + \omega_{k+1}) \wedge \dots \wedge (\text{dd}^c \psi_d + \omega_d) \rangle.$$

When $\boldsymbol{\psi}$ is clear from the context, we will call $I_{\boldsymbol{\psi}}(\boldsymbol{\varphi})$ simply the mixed relative energy of $\boldsymbol{\varphi}$.

2.47. **Lemma.** *With the notations and assumptions of Definition 2.46, the following statements hold:*

- (i) *The mixed relative energy $I_{\psi}(\varphi)$ is symmetric with respect to the simultaneous permutation of $\varphi_0, \dots, \varphi_d$ and ψ_0, \dots, ψ_d as well as of $\omega_0, \dots, \omega_d$, i. e., we have*

$$(2.5) \quad I_{\sigma(\psi)}(\sigma(\varphi)) = I_{\psi}(\varphi)$$

for every permutation σ of the symmetric group S_{d+1} of the set $\{0, \dots, d\}$, where $\sigma(\varphi)$ means $(\varphi_{\sigma(0)}, \dots, \varphi_{\sigma(d)})$.

- (ii) *The mixed relative energy $I_{\psi}(\varphi)$ is multilinear in the following sense: If, for example, ω'_0 is another smooth closed positive $(1, 1)$ -form on X , $\psi'_0 \in \text{PSH}(X, \omega'_0)$ is of small unbounded locus, $\varphi'_0 \in \mathcal{E}(X, \omega'_0, \psi'_0)$ with $[\varphi'_0] = [\psi'_0]$, and $\lambda_0, \lambda'_0 \in \mathbb{R}_{>0}$, we have*

$$I_{\lambda_0\psi_0 + \lambda'_0\psi'_0, \dots, \psi_d}(\lambda_0\varphi_0 + \lambda'_0\varphi'_0, \dots, \varphi_d) = \lambda_0 I_{\psi_0, \dots, \psi_d}(\varphi_0, \dots, \varphi_d) + \lambda'_0 I_{\psi'_0, \dots, \psi_d}(\varphi'_0, \dots, \varphi_d).$$

- (iii) *The mixed relative energy $I_{\psi}(\varphi)$ is monotonous with respect to the functions φ . That is, if $\varphi' \in \mathcal{E}(X, \omega, \psi)$ with $[\varphi'] = [\psi]$ and $\varphi' \leq \varphi$, then*

$$I_{\psi}(\varphi') \leq I_{\psi}(\varphi).$$

- (iv) *Given $\varphi \in \mathcal{E}(X, \omega, \psi)$ with $[\varphi] = [\psi]$, let $(\varphi_n)_{n \in \mathbb{N}^{d+1}}$ be sequences of $(d+1)$ -tuples of functions in $\mathcal{E}(X, \omega, \psi)$ with $[\varphi_n] = [\psi]$, converging componentwise from above to φ . Then, the net of mixed relative energies $(I_{\psi}(\varphi_n))_{n \in \mathbb{N}^{d+1}}$ converges from above to the mixed relative energy $I_{\psi}(\varphi)$, i. e., we have*

$$\lim_{n \rightarrow \infty} I_{\psi}(\varphi_n) = I_{\psi}(\varphi).$$

Proof. We start with the proof of (i). We note that it is enough to prove that the claimed equality (2.5) holds for the transpositions $\tau_{j, j+1} \in S_{d+1}$ interchanging $\varphi_j, \psi_j, \omega_j$ with $\varphi_{j+1}, \psi_{j+1}, \omega_{j+1}$ for $j = 0, \dots, d-1$ and leaving the remaining entries fixed, since these transpositions generate the whole symmetric group. For this, we recall from [BEGZ10, Theorem 1.14] applied to the bounded functions $\varphi_j - \psi_j$ and $\varphi_{j+1} - \psi_{j+1}$, and the symmetry of the non-pluripolar product, the equality

$$\int_X (\varphi_j - \psi_j) \langle \text{dd}^c(\varphi_{j+1} - \psi_{j+1}) \wedge \Theta \rangle = \int_X (\varphi_{j+1} - \psi_{j+1}) \langle \text{dd}^c(\varphi_j - \psi_j) \wedge \Theta \rangle,$$

where Θ stands for the product of currents

$$\Theta = (\text{dd}^c \varphi_0 + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_{j-1} + \omega_{j-1}) \wedge (\text{dd}^c \psi_{j+2} + \omega_{j+2}) \wedge \dots \wedge (\text{dd}^c \psi_d + \omega_d).$$

Rearranging terms, this leads to

$$\begin{aligned} & \int_X (\varphi_j - \psi_j) \langle (\text{dd}^c \psi_{j+1} + \omega_{j+1}) \wedge \Theta \rangle + (\varphi_{j+1} - \psi_{j+1}) \langle (\text{dd}^c \varphi_j + \omega_j) \wedge \Theta \rangle \\ &= \int_X (\varphi_{j+1} - \psi_{j+1}) \langle (\text{dd}^c \psi_j + \omega_j) \wedge \Theta \rangle + (\varphi_j - \psi_j) \langle (\text{dd}^c \varphi_{j+1} + \omega_{j+1}) \wedge \Theta \rangle. \end{aligned}$$

The latter equality immediately implies the claim for the transposition $\tau_{j, j+1}$.

The proof of (ii) follows immediately from the multilinearity of the non-pluripolar product of $(1, 1)$ -currents stated in part (ii) of Theorem 2.32.

We next prove (iii). By the symmetry of the mixed relative energy, it is enough to prove that

$$I_{\psi_0, \dots, \psi_d}(\varphi_0, \dots, \varphi_d) - I_{\psi_0, \dots, \psi_d}(\varphi'_0, \dots, \varphi_d) \geq 0.$$

A repeated use of the integration by parts [BEGZ10, Theorem 1.14] shows that

$$(2.6) \quad \begin{aligned} & I_{\psi_0, \dots, \psi_d}(\varphi_0, \dots, \varphi_d) - I_{\psi_0, \dots, \psi_d}(\varphi'_0, \dots, \varphi_d) \\ &= \int_X (\varphi_0 - \varphi'_0) \langle (\text{dd}^c \varphi_1 + \omega_1) \wedge \dots \wedge (\text{dd}^c \varphi_d + \omega_d) \rangle. \end{aligned}$$

Since $\varphi_0 - \varphi'_0 \geq 0$ and the non-pluripolar product defines a positive measure, we deduce the claim.

We finally prove (iv). In order to prove convergence in the sense of nets, by the monotonicity property (iii), it is enough to prove iterated convergence. This can be done using one function at the time. Again by the symmetry property (i), it is thus enough to show that

$$(2.7) \quad \lim_{n_0 \rightarrow \infty} I_{\psi_0, \dots, \psi_d}(\varphi_{0, n_0}, \dots, \varphi_d) = I_{\psi_0, \dots, \psi_d}(\varphi_0, \dots, \varphi_d).$$

As in (2.6), we have that

$$\begin{aligned} I_{\psi_0, \dots, \psi_d}(\varphi_{0, n_0}, \dots, \varphi_d) - I_{\psi_0, \dots, \psi_d}(\varphi_0, \dots, \varphi_d) \\ = \int_X (\varphi_{0, n_0} - \varphi_0) \langle (\text{dd}^c \varphi_1 + \omega_1) \wedge \dots \wedge (\text{dd}^c \varphi_d + \omega_d) \rangle. \end{aligned}$$

The claimed equality (2.7) now follows from this and the theorem of dominated convergence. \square

2.48. Definition. Let $\omega = (\omega_0, \dots, \omega_d)$ be smooth closed positive $(1, 1)$ -forms on X and let $\psi \in \text{PSH}(X, \omega)$ be of small unbounded locus. For arbitrary $\varphi \in \mathcal{E}(X, \omega, \psi)$, we define the *mixed relative energy of φ (with respect to ψ)* by

$$I_\psi(\varphi) := \inf \{ I_\psi(\mathbf{u}) \mid \mathbf{u} \in \mathcal{E}(X, \omega, \psi), [\mathbf{u}] = [\psi], \varphi \leq \mathbf{u} \}.$$

The space $\mathcal{E}^1(X, \omega, \psi)$ of *functions of finite mixed relative energy* is then defined as the subspace of $\mathcal{E}(X, \omega, \psi)$ consisting of those tuples of functions φ whose mixed relative energy is finite.

2.49. Remark. Given functions $\varphi_j \in \mathcal{E}^1(X, \omega_j, \psi_j)$ for $j = 0, \dots, d$, Remark 2.44 then shows that it is not necessarily true that $\varphi = (\varphi_0, \dots, \varphi_d)$ belongs to $\mathcal{E}^1(X, \omega, \psi)$. Therefore, the properties of the mixed relative energy have to be proven anew. Anyhow, the proofs are based on the corresponding properties of the usual relative energy.

For every $\varphi \in \mathcal{E}(X, \omega, \psi)$ and real number $C \geq 0$, the *canonical approximants* are defined as

$$\varphi^{(C)} := \max(\varphi, \psi - C).$$

We claim that $\varphi^{(C)} \in \mathcal{E}(X, \omega, \psi)$ and that $[\varphi^{(C)}] = [\psi]$. Indeed, since $\varphi^{(C)}$ is a maximum of two plurisubharmonic functions, it is also plurisubharmonic, whence $\varphi^{(C)} \in \text{PSH}(X, \omega)$. Moreover, since $\varphi \in \mathcal{E}(X, \omega, \psi)$, it is more singular than ψ , from which we derive that $\varphi^{(C)}$ has also to be more singular than ψ . Since, by definition $\psi \leq \varphi^{(C)} + C$, we deduce that ψ and $\varphi^{(C)}$ have equivalent singularities, that is, $\varphi^{(C)} - \psi$ is bounded. Whence, taking also into account Theorem 2.33, we deduce that $\varphi^{(C)} \in \mathcal{E}(X, \omega, \psi)$ and that $[\varphi^{(C)}] = [\psi]$.

2.50. Proposition. Let $\omega = (\omega_0, \dots, \omega_d)$ be smooth closed positive $(1, 1)$ -forms on X and let $\psi \in \text{PSH}(X, \omega)$ be of small unbounded locus. Then, all the properties of Lemma 2.47 can be extended to functions of finite mixed relative energy. In particular, if $\varphi \in \mathcal{E}^1(X, \omega, \psi)$, then $I_\psi(\varphi)$ is symmetric with respect to the simultaneous permutation of $\varphi_0, \dots, \varphi_d$ and ψ_0, \dots, ψ_d as well as of $\omega_0, \dots, \omega_d$. Moreover, if $(\varphi_n)_{n \in \mathbb{N}^{d+1}}$ is a sequence of $(d+1)$ -tuples of functions in $\mathcal{E}^1(X, \omega, \psi)$, converging componentwise from above to φ , then, the net of mixed relative energies $(I_\psi(\varphi_n))_{n \in \mathbb{N}^{d+1}}$ converges from above to the mixed relative energy $I_\psi(\varphi)$, i. e., we have

$$\lim_{\mathbf{n} \rightarrow \infty} I_\psi(\varphi_{\mathbf{n}}) = I_\psi(\varphi).$$

Proof. The proof is based on [BEGZ10, Proposition 2.10]. We start by proving that

$$(2.8) \quad I_\psi(\varphi) = \lim_{C \rightarrow \infty} I_\psi(\varphi^{(C)}),$$

where $\varphi^{(C)} = (\varphi_0^{(C)}, \dots, \varphi_d^{(C)})$. For this, let $u_j \in \mathcal{E}(X, \omega_j, \psi_j)$ satisfying $[u_j] = [\psi_j]$ and $\varphi_j \leq u_j$ for $j = 0, \dots, d$. Introducing the functions $\tilde{u}_{j,C} = \max(\varphi_j^{(C)}, u_j)$, we observe that $\tilde{u}_{j,C}$ converge from above to u_j as $C \rightarrow \infty$. Then, by Lemma 2.47, parts (iv) and (iii), we find that

$$I_\psi(\mathbf{u}) = \lim_{C \rightarrow \infty} I_\psi(\tilde{\mathbf{u}}_C) \geq \lim_{C \rightarrow \infty} I_\psi(\varphi^{(C)}),$$

where $\tilde{\mathbf{u}}_C = (\tilde{u}_{0,C}, \dots, \tilde{u}_{d,C})$. Applying the definition of the mixed relative energy as an infimum and using Lemma 2.47 (iii) once again, we deduce

$$I_\psi(\varphi) \geq \lim_{C \rightarrow \infty} I_\psi(\varphi^{(C)}) \geq I_\psi(\varphi),$$

from which equation (2.8) follows. This convergence result allows us to easily transfer the properties of the mixed relative energy $I_\psi(\varphi)$ from functions $\varphi \in \mathcal{E}(X, \omega, \psi)$ with $[\varphi] = [\psi]$ to arbitrary functions $\varphi \in \mathcal{E}^1(X, \omega, \psi)$. For instance, we show the continuity of the mixed relative energy with respect to monotone sequences. Let $(\varphi_n)_{n \in \mathbb{N}^{d+1}}$ be a sequence of $(d+1)$ -tuples of functions in $\mathcal{E}^1(X, \omega, \psi)$, converging componentwise from above to φ . Then, we have by Lemma 2.47 (iv)

$$\lim_{n \rightarrow \infty} I_\psi(\varphi_n^{(C)}) = I_\psi(\varphi^{(C)}),$$

while the previous discussion gives

$$\lim_{C \rightarrow \infty} I_\psi(\varphi_n^{(C)}) = I_\psi(\varphi_n) \quad \text{and} \quad \lim_{C \rightarrow \infty} I_\psi(\varphi^{(C)}) = I_\psi(\varphi).$$

By monotone convergence, we deduce

$$\lim_{n \rightarrow \infty} I_\psi(\varphi_n) = I_\psi(\varphi).$$

□

2.51. Lemma. *Let $\omega = (\omega_0, \dots, \omega_d)$ be smooth closed positive $(1, 1)$ -forms on X , let $\psi \in \text{PSH}(X, \omega)$ be of small unbounded locus, and let $\varphi \in \mathcal{E}^1(X, \omega, \psi)$. Then, for $j = 0, \dots, d$, the integral*

$$\int_X (\varphi_j - \psi_j) \langle (\text{dd}^c \varphi_0 + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_{j-1} + \omega_{j-1}) \wedge (\text{dd}^c \psi_{j+1} + \omega_{j+1}) \wedge \dots \wedge (\text{dd}^c \psi_d + \omega_d) \rangle$$

is finite.

Proof. The proof is based on the proof of [BEGZ10, Proposition 2.11 (ii)]. Since each function φ_j is more singular than the corresponding ψ_j , i. e., $\varphi_j \leq \psi_j + O(1)$, we can assume, after adding a constant to each φ_j , that $\varphi_j \leq \psi_j$. Then, for any $k \in \mathbb{N}$, we consider the sets

$$U_k = \{x \in X \mid \varphi_j > \psi_j - k, \text{ for } j = 0, \dots, d\},$$

and the canonical approximants $\varphi_j^{(k)} = \max(\varphi_j, \psi_j - k)$ giving rise to the measure

$$\tilde{\Theta}_k = \langle (\text{dd}^c \varphi_0^{(k)} + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_{j-1}^{(k)} + \omega_{j-1}) \wedge (\text{dd}^c \psi_{j+1} + \omega_{j+1}) \wedge \dots \wedge (\text{dd}^c \psi_d + \omega_d) \rangle$$

on X . By the very construction, it turns out that the measure

$$\Theta = \langle (\text{dd}^c \varphi_0 + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_{j-1} + \omega_{j-1}) \wedge (\text{dd}^c \psi_{j+1} + \omega_{j+1}) \wedge \dots \wedge (\text{dd}^c \psi_d + \omega_d) \rangle$$

is the increasing limit of the measures $\Theta_k = \mathbf{1}_{U_k} \tilde{\Theta}_k$, where $\mathbf{1}_{U_k}$ denotes the characteristic function for U_k , in other words

$$\Theta = \lim_{k \rightarrow \infty} \Theta_k.$$

Furthermore, we note that the measure $\tilde{\Theta}_k - \Theta_k$ is a positive measure on X . For each $\ell \in \mathbb{N}$, the functions $\varphi_j^{(\ell)} - \psi_j$ are bounded, measurable, and non-positive on X . Therefore, we arrive at the estimates

$$\begin{aligned} \int_X (\varphi_j^{(\ell)} - \psi_j) \Theta &= \lim_{k \rightarrow \infty} \int_X (\varphi_j^{(\ell)} - \psi_j) \Theta_k \geq \lim_{k \rightarrow \infty} \int_X (\varphi_j^{(k)} - \psi_j) \Theta_k \\ &\geq \lim_{k \rightarrow \infty} \int_X (\varphi_j^{(k)} - \psi_j) \tilde{\Theta}_k \geq \lim_{k \rightarrow \infty} I_\psi(\varphi^{(k)}) = I_\psi(\varphi). \end{aligned}$$

Here, we note that we have used in the third inequality above that the mixed relative energy $I_\psi(\varphi^{(k)})$ is a sum of non-positive terms, one of them being the one in the preceding inequality. By monotone convergence, we finally deduce that

$$\int_X (\varphi_j - \psi_j) \Theta \geq I_\psi(\varphi) > -\infty,$$

which proves the claim of the lemma. □

2.52. Theorem. *Let $\omega = (\omega_0, \dots, \omega_d)$ be smooth closed positive $(1, 1)$ -forms on X , let $\psi \in \text{PSH}(X, \omega)$ be of small unbounded locus, and let $\varphi \in \mathcal{E}^1(X, \omega, \psi)$. Then, formula (2.4) holds, that is*

$$I_\psi(\varphi) = \sum_{j=0}^d \int_X (\varphi_j - \psi_j) \langle (\text{dd}^c \varphi_0 + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_{j-1} + \omega_{j-1}) \wedge \\ \wedge (\text{dd}^c \varphi_{j+1} + \omega_{j+1}) \wedge \dots \wedge (\text{dd}^c \varphi_d + \omega_d) \rangle.$$

Proof. The proof of this result is based on the proof of [DDNL18a, Proposition 2.5]. As in the proof of Lemma 2.51, we can assume that $\varphi_j \leq \psi_j$ for $j = 0, \dots, d$. We note that, in the notation of Lemma 2.51, it is enough to prove that

$$(2.9) \quad \lim_{k \rightarrow \infty} \int_X (\varphi_0^{(k)} - \psi_0) \langle (\text{dd}^c \varphi_1^{(k)} + \omega_1) \wedge \dots \wedge (\text{dd}^c \varphi_d^{(k)} + \omega_d) \rangle \\ = \int_X (\varphi_0 - \psi_0) \langle (\text{dd}^c \varphi_1 + \omega_1) \wedge \dots \wedge (\text{dd}^c \varphi_d + \omega_d) \rangle,$$

as the mixed relative energy of the canonical approximants $I_\psi(\varphi^{(k)})$ converges to the mixed relative energy $I_\psi(\varphi)$, and the mixed relative energy is a sum of integrals of the above shape, after reordering and replacing some of the φ_j 's by the respective ψ_j 's. For each $k \in \mathbb{N}$ and each subset $I \subseteq \{0, \dots, d\}$, we now write

$$V_{I,k} = \left\{ x \in X \mid \begin{array}{ll} \varphi_j > \psi_j - k, & \text{if } j \notin I \\ \varphi_j \leq \psi_j - k, & \text{if } j \in I \end{array} \right\}.$$

In particular, we note that the set $V_{\emptyset,k}$ is the open set U_k introduced in the proof of Lemma 2.51. Since we trivially have that

$$\lim_{k \rightarrow \infty} \int_{V_{\emptyset,k}} (\varphi_0^{(k)} - \psi_0) \langle (\text{dd}^c \varphi_1^{(k)} + \omega_1) \wedge \dots \wedge (\text{dd}^c \varphi_d^{(k)} + \omega_d) \rangle \\ = \lim_{k \rightarrow \infty} \int_{V_{\emptyset,k}} (\varphi_0 - \psi_0) \langle (\text{dd}^c \varphi_1 + \omega_1) \wedge \dots \wedge (\text{dd}^c \varphi_d + \omega_d) \rangle \\ = \int_X (\varphi_0 - \psi_0) \langle (\text{dd}^c \varphi_1 + \omega_1) \wedge \dots \wedge (\text{dd}^c \varphi_d + \omega_d) \rangle,$$

we are reduced to show that, for $I \neq \emptyset$, we have

$$(2.10) \quad \lim_{k \rightarrow \infty} \int_{V_{I,k}} (\varphi_0^{(k)} - \psi_0) \langle (\text{dd}^c \varphi_1^{(k)} + \omega_1) \wedge \dots \wedge (\text{dd}^c \varphi_d^{(k)} + \omega_d) \rangle = 0.$$

To do so, we note that on $V_{I,k}$, the function $\varphi_j^{(k)}$ is equal to φ_j if $j \notin I$, and agrees with $\psi_j - k$ if $j \in I$. Since $\text{dd}^c(\psi_j - k) = \text{dd}^c \psi_j$ and $0 \geq \varphi_0^{(k)} - \psi_0 \geq \varphi_0 - \psi_0$, we can assume that the integrand is independent of k and thus corresponds to a non-positive function multiplied by a measure that does not charge any pluripolar set of the shape considered in Lemma 2.51. Thus, the integral over the whole of X is finite. Since the sets $V_{I,k}$ for $I \neq \emptyset$ converge to a pluripolar set, we deduce that the limit (2.10) is zero, completing the proof of the theorem. \square

The following criterion together with Lemma 2.43 will be useful to check that a $(d+1)$ -tuple of functions has finite mixed relative energy. To state the result we use the following notation: If $I \subseteq \{0, \dots, d\}$ and $\varphi = (\varphi_0, \dots, \varphi_d)$, we write $\varphi_I = \sum_{j \in I} \varphi_j$ and similarly $\omega_I = \sum_{j \in I} \omega_j$ as well as $\psi_I = \sum_{j \in I} \psi_j$.

2.53. Proposition. *Let $\omega = (\omega_0, \dots, \omega_d)$ be smooth closed positive $(1, 1)$ -forms on X , let $\psi \in \text{PSH}(X, \omega)$ be of small unbounded locus, and let $\varphi \in \mathcal{E}(X, \omega, \psi)$. If for all subsets $I \subseteq \{0, \dots, d\}$ the condition $\varphi_I \in \mathcal{E}^1(X, \omega_I, \psi_I)$ holds, then $\varphi \in \mathcal{E}^1(X, \omega, \psi)$ and the usual polarization formula*

$$(2.11) \quad (d+1)! I_\psi(\varphi) = \sum_{I \subseteq \{0, \dots, d\}} (-1)^{d+1-|I|} I_\psi(\varphi_I)$$

is satisfied.

Proof. By the multilinearity of the non-pluripolar product and the definition of the mixed relative energy, formula (2.11) is satisfied when we have minimal relative singularities, that is, when $[\varphi_j] = [\psi_j]$ for $j = 0, \dots, d$. In particular, we find

$$(d+1)! I_{\psi}(\varphi^{(C)}) = \sum_{I \subseteq \{0, \dots, d\}} (-1)^{d+1-|I|} I_{\psi_I}((\varphi^{(C)})_I).$$

If $\varphi_I \in \mathcal{E}^1(X, \omega_I, \psi_I)$ for all $I \subseteq \{0, \dots, d\}$, then the right-hand side has a finite limit as $C \rightarrow \infty$. Therefore, the left-hand side also has a finite limit, so $\varphi \in \mathcal{E}^1(X, \omega, \psi)$ and equation (2.11) is satisfied. \square

2.8. Algebraic and almost algebraic singularities. In this subsection, we fix a smooth closed positive $(1, 1)$ -form ω_0 .

2.54. Definition. An ω_0 -plurisubharmonic function φ on X is said to *have algebraic singularities*, if the function φ can locally be written as

$$(2.12) \quad \varphi = \frac{c}{2} \log(|f_1|^2 + \dots + |f_N|^2) + \lambda,$$

where $c \in \mathbb{Q}_{\geq 0}$ is a constant, f_1, \dots, f_N are non-zero algebraic rational functions, and λ is a bounded function.

2.55. Definition. An ω_0 -plurisubharmonic function φ on X is said to *have almost asymptotically algebraic singularities*, if there is a sequence of ω_0 -plurisubharmonic functions $(\psi_k)_{k \geq 1}$, an ω_0 -plurisubharmonic function f , having algebraic singularities, and a constant $C > 0$ such that the inequalities

$$\psi_k + \frac{1}{k} f - C \leq \varphi \leq \psi_k + C$$

hold.

It is easy to see that, if a function satisfies this definition, then it also satisfies [BBGHdJ21, Definition 3.2].

3. REVIEW OF THE THEORY OF ADELIC ARITHMETIC LINE BUNDLES

In this section we summarize some concepts and results from [YZ21], although the presentation is slightly different. For instance, instead of working with mixed (\mathbb{Q}, \mathbb{Z}) -divisors, we will work with divisors with real coefficients.

3.1. The geometric case, adelic divisors and b-divisors. Let X be either a projective normal variety over a field k or a projective flat normal scheme, separated and of finite type over a Dedekind domain, B an effective Cartier divisor on X (the “boundary” divisor) with support $|B|$, and $U := X \setminus |B|$. We denote by $R(X, U)$ the category of all proper normal modifications of X which are isomorphisms over U . Given $\pi \in R(X, U)$, i. e., $\pi: X_\pi \rightarrow X$, we denote by $\text{Div}(X_\pi)$ the group of Cartier divisors on X_π and we write $\text{Div}_{\mathbb{R}}(X_\pi) := \text{Div}(X_\pi) \otimes_{\mathbb{Z}} \mathbb{R}$.

3.1. Definition. The space of *model divisors on U* is defined as the limit

$$\text{Div}_{\mathbb{R}}(U)^{\text{mod}} := \varinjlim_{\pi \in R(X, U)} \text{Div}_{\mathbb{R}}(X_\pi).$$

On $\text{Div}_{\mathbb{R}}(U)^{\text{mod}}$ there is a B -adic norm (possibly taking the value ∞) defined by

$$(3.1) \quad \|D\| := \inf\{\varepsilon \in \mathbb{R}_{>0} \mid -\varepsilon B \leq D \leq \varepsilon B\};$$

here $D \in \text{Div}_{\mathbb{R}}(U)^{\text{mod}}$, which means that there is a $\pi \in R(X, U)$ such that $D \in \text{Div}_{\mathbb{R}}(X_\pi)$. The condition $D \leq \varepsilon B$, means that the divisor $\varepsilon \pi^* B - D$ is effective on X_π . Finally, the infimum of an empty set is assumed to be ∞ .

3.2. Definition. The space $\text{Div}_{\mathbb{R}}(U)^{\text{adel}}$ of *adelic divisors on U* is the completion of $\text{Div}_{\mathbb{R}}(U)^{\text{mod}}$ with respect to the B -adic topology induced by the B -adic norm defined above.

In other words, an element $D \in \text{Div}_{\mathbb{R}}(U)^{\text{adel}}$ is represented by a sequence of Cartier divisors $(D_n)_{n \in \mathbb{N}}$ that is Cauchy with respect to the B -adic norm. Two Cauchy sequences $(D_n)_{n \in \mathbb{N}}$ and $(D'_n)_{n \in \mathbb{N}}$ are equivalent if the sequence $(D''_n)_{n \in \mathbb{N}}$ with $D''_{2n} = D_n$ and $D''_{2n+1} = D'_n$ is still Cauchy.

3.3. Remark. If B' is another boundary divisor with $|B| = |B'|$, then there is a positive constant $c > 0$ such that $(1/c)B \leq B' \leq cB$. Therefore, the B -adic norm and the B' -adic norm are equivalent. Thus, the definition of $\text{Div}_{\mathbb{R}}(U)^{\text{adel}}$ only depends on the open set U and not on a particular choice of boundary divisor B .

3.4. Definition. The cone $\text{Div}_{\mathbb{R}}(U)^{\text{nef}}$ of *nef adelic divisors on U* is the closure of the cone of nef model divisors on U inside of $\text{Div}_{\mathbb{R}}(U)^{\text{adel}}$. The space of *integrable adelic divisors on U* is the vector space

$$\text{Div}_{\mathbb{R}}(U)^{\text{int}} := \text{Div}_{\mathbb{R}}(U)^{\text{nef}} - \text{Div}_{\mathbb{R}}(U)^{\text{nef}}.$$

3.5. Remark. If $V \subseteq U \subseteq X$ is an inclusion of open subsets, then there are injective maps

$$\text{Div}_{\mathbb{R}}(U)^{\text{adel}} \longrightarrow \text{Div}_{\mathbb{R}}(V)^{\text{adel}}, \quad \text{Div}_{\mathbb{R}}(U)^{\text{nef}} \longrightarrow \text{Div}_{\mathbb{R}}(V)^{\text{nef}}.$$

The notion of nef adelic divisor we give here corresponds more precisely to the notion of strongly nef adelic divisor in [YZ21]. Nevertheless, if one is willing to go to a smaller open subset, there is no essential difference between nef and strongly nef adelic divisors in [YZ21]. Namely, if D is a nef adelic divisor in an open subset U , then there is a subset $V \subseteq U$ such that the image of D in $\text{Div}_{\mathbb{R}}(V)^{\text{adel}}$ is strongly nef in the sense of [YZ21].

From now on we assume that we are in the case when k is a field. The following results are proven in [YZ21].

3.6. Proposition. *If $\pi \in R(X, U)$ and B' is an effective divisor on X_{π} with $|B'| = X_{\pi} \setminus U$, then the B' -adic norm is equivalent to the B -adic norm. Therefore, the definitions of the spaces $\text{Div}_{\mathbb{R}}(U)^{\text{adel}}$, $\text{Div}_{\mathbb{R}}(U)^{\text{nef}}$, and $\text{Div}_{\mathbb{R}}(U)^{\text{int}}$ only depend on U and not on a particular choice of compactification X and divisor B .*

3.7. Theorem. *The intersection pairing of nef Cartier divisors extends by continuity to a unique symmetric multilinear pairing*

$$\underbrace{\text{Div}_{\mathbb{R}}(U)^{\text{nef}} \otimes \dots \otimes \text{Div}_{\mathbb{R}}(U)^{\text{nef}}}_{\dim(U)\text{-times}} \longrightarrow \mathbb{R}.$$

This pairing can be extended by linearity to a pairing

$$\underbrace{\text{Div}_{\mathbb{R}}(U)^{\text{int}} \otimes \dots \otimes \text{Div}_{\mathbb{R}}(U)^{\text{int}}}_{\dim(U)\text{-times}} \longrightarrow \mathbb{R}.$$

Theorem 3.7 means that, if $D_j \in \text{Div}_{\mathbb{R}}(U)^{\text{nef}}$ with $j = 1, \dots, d = \dim(U)$ are nef adelic divisors on U represented by the Cauchy sequences $(D_{j,n})_{n \in \mathbb{N}}$ of nef model divisors on U , then the limit

$$(3.2) \quad D_1 \cdots \cdots D_d := \lim_{(n_1, \dots, n_d) \rightarrow (\infty, \dots, \infty)} D_{1, n_1} \cdots \cdots D_{d, n_d}$$

exists and is independent of the choice of approximating sequences.

3.8. Remark. In [YZ21] it is only stated that the diagonal limit

$$\lim_{n \rightarrow \infty} D_{1, n} \cdots \cdots D_{d, n}$$

exists and is independent of the chosen Cauchy sequences. But their proof actually shows that the multi-limit exists. In particular, we can compute it using iterated limits.

We next want to relate the notions of adelic divisors and of b-divisors. Recall that the spaces of Cartier b-divisors and Weil b-divisors on X are defined as

$$\text{C-b-Div}_{\mathbb{R}}(X) = \varinjlim_{\pi \in R(X)} \text{Div}_{\mathbb{R}}(X_{\pi}) \quad \text{and} \quad \text{W-b-Div}_{\mathbb{R}}(X) = \varprojlim_{\pi \in R(X)} \text{Div}_{\mathbb{R}}(X_{\pi}),$$

respectively, where the limits are taken with respect to all possible proper modifications of X , i. e., $R(X) = \{\pi: X_\pi \rightarrow X \mid \pi \text{ is a proper modification}\}$. Both limits are defined in the category of topological vector spaces and thus they have a topology.

There is a map $b: \text{Div}_{\mathbb{R}}(U)^{\text{mod}} \rightarrow \text{C-b-Div}_{\mathbb{R}}(X)$, since the directed set used to define the first one is a subset of the one used to define the second one. Since there is a canonical inclusion $\text{C-b-Div}_{\mathbb{R}}(X) \rightarrow \text{W-b-Div}_{\mathbb{R}}(X)$, we obtain a map, also denoted by b , from $\text{Div}_{\mathbb{R}}(U)^{\text{mod}}$ to $\text{W-b-Div}_{\mathbb{R}}(X)$.

3.9. Lemma. *Let $(D_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\text{Div}_{\mathbb{R}}(U)^{\text{mod}}$ with respect to the B -adic topology. Then, the sequence $(b(D_n))_{n \in \mathbb{N}}$ is convergent in $\text{W-b-Div}_{\mathbb{R}}(X)$.*

Proof. We start by recalling the topology of $\text{W-b-Div}_{\mathbb{R}}(X)$. Since it is defined as an inverse limit, an element of $\text{W-b-Div}_{\mathbb{R}}(X)$ is a collection $(D_\pi)_{\pi \in R(X)}$ of divisors, one on each proper modification X_π of X , that are compatible by push-forward. A sequence $((D_{n,\pi})_{\pi \in R(X)})_{n \in \mathbb{N}}$ in $\text{W-b-Div}_{\mathbb{R}}(X)$ converges to $(D_\pi)_{\pi \in R(X)}$, if and only if the sequence $(D_{n,\pi})_{n \in \mathbb{N}}$ converges to D_π for all $\pi \in R(X)$. This means that there is a divisor E_π on X_π such that $|D_{n,\pi}| \subseteq |E_\pi|$, that is, the supports of all the divisors $D_{n,\pi}$ are contained in a common divisor for all $n \in \mathbb{N}$, and the divisors $D_{n,\pi}$ converge to D_π componentwise.

Now assume that $(D_n)_{n \in \mathbb{N}}$ is a Cauchy sequence of model divisors on U . This means that, for all $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ such that the inequality

$$(3.3) \quad -\varepsilon B \leq D_n - D_m \leq \varepsilon B$$

holds for all $n, m \geq n_\varepsilon$. Here, D_n lives on some modification X_n , D_m on some modification X_m , B is the boundary divisor on X defining U , and the inequality is to be understood on a suitable modification of X that dominates both. For each modification $\pi: X_\pi \rightarrow X$, we define $B_\pi = \pi^*B$, and we choose a proper modification $X_{\pi'}$ that dominates at the same time X_π , X_n , and X_m . Then, we define $D_{n,\pi}$ and $D_{m,\pi}$ by first pulling back D_n and D_m to $X_{\pi'}$ and then pushing forward the resulting divisors to X_π , respectively. The obtained divisors do not depend on the choice of π' . Condition (3.3) implies that, for all $n \geq n_\varepsilon$, the support of $D_{n,\pi}$ is contained in $|D_{n_\varepsilon}| \cup |B|$ and that the multiplicities of $(D_{n,\pi})_{n \in \mathbb{N}}$ form a Cauchy sequence. Therefore, the sequence $(D_{n,\pi})_{n \in \mathbb{N}}$ converges to a divisor D_π in X_π . It is clear that the various D_π 's obtained in this way are compatible with respect to push-forward and hence define an element of $\text{W-b-Div}_{\mathbb{R}}(X)$. The sequence $(b(D_n))_{n \in \mathbb{N}}$ by construction converges to this element. \square

Lemma 3.9 implies that the map b can be extended to a continuous map

$$b: \text{Div}_{\mathbb{R}}(U)^{\text{adel}} \longrightarrow \text{W-b-Div}_{\mathbb{R}}(X).$$

In order to link adelic nef divisors with approximable nef b-divisors in the sense of [BBGHdJ21, Definition 4.8], it is convenient to have a monotonous approximation to a nef adelic divisor. This can always be achieved as soon as the boundary divisor is nef.

3.10. Lemma. *Let X be a complex projective manifold with boundary divisor B and $U = X \setminus |B|$ such that B is a nef divisor. Let D be a nef adelic divisor on U . Then, there is a sequence $(D_n)_{n \in \mathbb{N}}$ of nef model divisors on U converging monotonically decreasingly to D in the B -adic topology. That is, for all $n \in \mathbb{N}$, we have $D_{n+1} \leq D_n$.*

Proof. Since D is a nef adelic divisor on U , there is a sequence of nef model divisors $(E_n)_{n \in \mathbb{N}}$ on U converging to D in the B -adic topology. After extracting a subsequence, we can assume that

$$D - \frac{1}{2^n} B \leq E_n \leq D + \frac{1}{2^n} B.$$

Writing $D_n = E_n + \frac{4}{2^n} B$, we find

$$D_{n+1} = E_{n+1} + \frac{4}{2^{n+1}} B \leq D + \frac{5}{2^{n+1}} B \leq E_n + \frac{7}{2^{n+1}} B \leq E_n + \frac{4}{2^n} B = D_n.$$

This completes the proof of the lemma. \square

3.11. Corollary. *If D is a nef adelic divisor on U , then $b(D)$ is an approximable nef b -divisor in the sense of [BBGHdJ21, Definition 4.8]. Moreover, the intersection product of nef adelic divisors agrees through the map b with the intersection product of nef b -divisors introduced in [DF20].*

Proof. If D is a nef adelic divisor, after shrinking U if necessary, we can assume that the boundary divisor B is nef. Therefore, by Lemma 3.10, it can be approximated monotonically by nef Cartier b -divisors, so $b(D)$ is approximable nef. The intersection products of [YZ21] and of [DF20] now agree because both are continuous extensions of the intersection product of nef Cartier b -divisors. \square

3.2. The local arithmetic case over an archimedean place, Green functions. In this subsection, let X be a complex projective manifold of dimension d . Let B be an effective Cartier divisor on X with support $|B|$ and let $U := X \setminus |B|$. Let g_B be a Green function of continuous type for B such that there is an $\eta > 0$ with $g_B(x) > \eta$ for all $x \in U$. Such Green functions exist because B being effective, any Green function of continuous type for B is bounded below, and we can add to it a big enough constant to make it bigger than η . We call $\overline{B} = (B, g_B)$ an *arithmetic boundary divisor* for X .

We now repeat the steps of Subsection 3.1 for the local arithmetic case. For this, let $R(X, U)$ be as in the previous subsection.

3.12. Definition. An *arithmetic divisor* \overline{D} on X_π of *smooth type* is a pair (D, g_D) , where $D \in \text{Div}_{\mathbb{R}}(X_\pi)$ is a Cartier divisor on X_π and g_D is a Green function for D of smooth type; here $\pi \in R(X, U)$. The vector space of arithmetic divisors on X_π of smooth type is denoted by $\widehat{\text{Div}}_{\mathbb{R}}(X_\pi)$.

The space of *model arithmetic divisors on U of smooth type* is defined as the limit

$$\widehat{\text{Div}}_{\mathbb{R}}(U)^{\text{mod}} := \varinjlim_{\pi \in R(X, U)} \widehat{\text{Div}}_{\mathbb{R}}(X_\pi).$$

Similarly we can define *model arithmetic divisors on U of continuous type* by asking that the Green functions are of continuous type. To shorten the notation, by a model arithmetic divisor on U , we will mean a model arithmetic divisor on U of smooth type, and when we consider a model arithmetic divisor on U of continuous type, we will say so explicitly.

On $\widehat{\text{Div}}_{\mathbb{R}}(U)^{\text{mod}}$ there is a \overline{B} -adic norm (possibly again taking the value ∞) defined by

$$\|\overline{D}\| = \|(D, g_D)\| := \inf\{\varepsilon \in \mathbb{R}_{>0} \mid -\varepsilon B \leq D \leq \varepsilon B, -\varepsilon g_B \leq g_D \leq \varepsilon g_B\};$$

here $D \in \text{Div}_{\mathbb{R}}(U)^{\text{mod}}$, which means that there is a $\pi \in R(X, U)$ such that $D \in \text{Div}_{\mathbb{R}}(X_\pi)$, and g_D is a Green function for D of smooth type.

3.13. Definition. The space $\widehat{\text{Div}}_{\mathbb{R}}(U)^{\text{adel}}$ of *adelic arithmetic divisors on U* is the completion of $\widehat{\text{Div}}_{\mathbb{R}}(U)^{\text{mod}}$ with respect to the \overline{B} -adic topology induced by the \overline{B} -adic norm defined above. As before, elements of $\widehat{\text{Div}}_{\mathbb{R}}(U)^{\text{adel}}$ are represented by Cauchy sequences of model arithmetic divisors on U .

3.14. Remark. Since the set of smooth functions is dense inside of the set of continuous functions with respect to the topology of uniform convergence on compacta, the space $\widehat{\text{Div}}_{\mathbb{R}}(U)^{\text{adel}}$ can also be seen as the completion of the space of model arithmetic divisors on U of continuous type with respect to the \overline{B} -adic topology.

3.15. Definition. A model arithmetic divisor $\overline{D} = (D, g_D) \in \widehat{\text{Div}}_{\mathbb{R}}(U)^{\text{mod}}$ is called *nef*, if D is a nef divisor on X_π for some $\pi \in R(X, U)$ and g_D is a Green function for D that is at the same time of smooth type and of plurisubharmonic type.

The cone $\widehat{\text{Div}}_{\mathbb{R}}(U)^{\text{nef}}$ of *nef adelic arithmetic divisors on U* is the closure of the cone of nef model arithmetic divisors on U inside of $\widehat{\text{Div}}_{\mathbb{R}}(U)^{\text{adel}}$. That is, the divisors that can be represented by a Cauchy sequence of nef model arithmetic divisors on U . An adelic arithmetic divisor on U is called *integrable*, if it can be written as the difference of two nef adelic arithmetic divisors on U . The space of integrable adelic arithmetic divisors on U is denoted by $\widehat{\text{Div}}_{\mathbb{R}}(U)^{\text{int}}$.

3.16. Remark. The discussion of Remark 3.5 is also valid in this case.

3.17. Remark. By the global Richberg's regularization of continuous plurisubharmonic functions (see [Dem92a, Lemma 2.15]), an adelic arithmetic divisor on U is nef if and only if it can be written as the limit of a sequence of model arithmetic divisors on U with Green functions that are of continuous and of plurisubharmonic type, and which is Cauchy with respect to the \overline{B} -adic norm.

The analogue of the following result in the non-archimedean case is contained in [YZ21, Theorem 3.6.4]. The proof in the archimedean case is much simpler.

3.18. Proposition. *Let $(D_n, g_{D_n})_{n \in \mathbb{N}}$ be a Cauchy sequence of model arithmetic divisors on U representing an element in $\widehat{\text{Div}}_{\mathbb{R}}(U)^{\text{adel}}$. Then, the sequence $(D_n)_{n \in \mathbb{N}}$ converges to an element $D \in \text{Div}_{\mathbb{R}}(U)^{\text{adel}}$ and the sequence of functions $(g_{D_n})_{n \in \mathbb{N}}$ converges pointwise on U to a function g_D . Moreover, the convergence is uniform on compact subsets $K \subseteq U$. In particular, there is a short exact sequence*

$$0 \longrightarrow C^0(U)_0 \longrightarrow \widehat{\text{Div}}_{\mathbb{R}}(U)^{\text{adel}} \longrightarrow \text{Div}_{\mathbb{R}}(U)^{\text{adel}} \longrightarrow 0,$$

where $C^0(U)_0$ denotes the set of continuous functions f on U satisfying that the quotient f/g_B can be extended to a continuous function in the whole of X , taking the value zero in $|B|$. In other words,

$$|f(x)| = o(g_B(x)),$$

as $x \rightarrow |B|$.

As a consequence of Proposition 3.18, every element of $\widehat{\text{Div}}_{\mathbb{R}}(U)^{\text{adel}}$ will subsequently be written as (D, g_D) with $D \in \text{Div}_{\mathbb{R}}(U)^{\text{adel}}$ and g_D a function on U giving rise to a function on X up to a set of measure zero.

3.19. Definition. Let $D \in \text{Div}_{\mathbb{R}}(U)^{\text{adel}}$. A function g_D on U is called an *admissible Green function for D* , if the pair (D, g_D) belongs to $\widehat{\text{Div}}_{\mathbb{R}}(U)^{\text{nef}}$. In particular, if such functions exist, $D \in \text{Div}_{\mathbb{R}}(U)^{\text{nef}}$.

We next show that, after choosing a sufficiently large arithmetic reference divisor, admissible Green functions can be related to plurisubharmonic functions whose singularities are at most asymptotically algebraic.

3.20. Proposition. *Let $D \in \text{Div}_{\mathbb{R}}(U)^{\text{adel}}$ and let g_D be an admissible Green function for D . Choose a sufficiently large arithmetic reference divisor $\overline{E} = (E, g_E)$ on X with g_E a Green function of smooth type for E such that $\overline{E} \geq \overline{B}$ and $\overline{E} \geq \overline{D} + 2\overline{B}$. Letting $\omega_0 = \omega_E(g_E)$, the following statements hold:*

- (i) *The function $\varphi = g_D - g_E$ is ω_0 -plurisubharmonic. In particular, g_D belongs to $L_{\text{loc}}^1(X)$ and is a Green function for E of plurisubharmonic type.*
- (ii) *The function φ has almost asymptotically algebraic singularities in the sense of Definition 2.55.*

Proof. To prove (i), let $(D_n, g_{D_n})_{n \in \mathbb{N}}$ be a sequence of nef model arithmetic divisors on U converging to (D, g_D) in the \overline{B} -adic topology. Since $\overline{E} \geq \overline{D} + 2\overline{B}$, there is an $n_0 \in \mathbb{N}$ such that $E \geq D_n$ and $g_E \geq g_{D_n}$ for all $n \geq n_0$. By taking a suitable subsequence, we can assume that $E \geq D_n$ and $g_E \geq g_{D_n}$ for all $n \in \mathbb{N}$. Then, by Lemma 2.20, the functions g_{D_n} are Green functions for E of plurisubharmonic type for all $n \in \mathbb{N}$. Therefore, the functions $\varphi_n = g_{D_n} - g_E \leq 0$ are ω_0 -plurisubharmonic for all $n \in \mathbb{N}$. Thus, the sequence $(\varphi_n)_{n \in \mathbb{N}}$ is bounded above by 0 and converges to the function φ . By [Hör94, Theorem 3.2.12] (see the comment before Theorem 4.1.8), the function φ is ω_0 -plurisubharmonic. This proves (i).

To prove (ii), choose a sequence $(D_n, g_{D_n})_{n \in \mathbb{N}}$ of nef model arithmetic divisors on U converging to (D, g_D) formed by nef divisors with rational coefficients and a strictly increasing sequence $(n_k)_{k \geq 1}$ such that

$$-\frac{1}{k}g_E \leq -\frac{1}{k}g_B \leq g_D - g_{D_{n_k}} \leq \frac{1}{k}g_B \leq \frac{1}{k}g_E.$$

The functions $\psi_k = g_{D_{n_k}} - g_E$ are now ω_0 -plurisubharmonic functions with algebraic singularities. Similarly, the function $f = -g_E \leq 0$ is also a ω_0 -plurisubharmonic function with algebraic singularities. Since we have

$$\psi_k + \frac{1}{k}f \leq \varphi \leq \psi_k - \frac{1}{k}f,$$

we deduce that φ has almost asymptotically algebraic singularities. \square

3.21. Remark. Note that the inequalities $\overline{E} \geq \overline{B}$ and $\overline{E} \geq \overline{D} + 2\overline{B}$ chosen for the arithmetic reference divisor \overline{E} in Proposition 3.20 could be relaxed as the above proof shows. However, for later purposes and the sake of uniformity, we have decided to stick to these inequalities from now on.

We next show that the adelic convergence has strong consequences with respect to the convergence of the associated non-pluripolar Monge–Ampère measures.

3.22. Proposition. *For $j = 1, \dots, d$, let $\overline{D}_j = (D_j, g_j)$ be nef adelic arithmetic divisors on U and let $(D_{j,n}, g_{j,n})_{n \in \mathbb{N}}$ be sequences of nef model arithmetic divisors on U converging in the \overline{B} -adic topology to \overline{D}_j . Choose a sufficiently large arithmetic reference divisor $\overline{E} = (E, g_E)$ on X with g_E a Green function of smooth type for E such that $\overline{E} \geq \overline{B}$, $\overline{E} \geq \overline{D}_j + 2\overline{B}$, and hence, without loss of generality, $\overline{E} \geq \overline{D}_{j,n} + \overline{B}$ for all j, n . Letting $\omega_0 = \omega_E(g_E)$, the functions $\varphi_j = g_j - g_E$ and $\varphi_{j,n} = g_{j,n} - g_E$ become ω_0 -plurisubharmonic on X by Proposition 3.20 and Lemma 2.20, respectively. Then, the following statements hold:*

- (i) *For $j = 1, \dots, d$, the sequences of functions $(\varphi_{j,n})_{n \in \mathbb{N}}$ converge in capacity to the functions φ_j .*
- (ii) *There is an equality of integrals*

$$\int_X \langle (\text{dd}^c \varphi_1 + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_d + \omega_0) \rangle = \lim_{n \rightarrow \infty} \int_X \langle (\text{dd}^c \varphi_{1,n} + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_{d,n} + \omega_0) \rangle.$$

- (iii) *The sequence of measures $\langle (\text{dd}^c \varphi_{1,n} + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_{d,n} + \omega_0) \rangle$ converges weakly to the measure $\langle (\text{dd}^c \varphi_1 + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_d + \omega_0) \rangle$.*

- (iv) *Let $(0, f) \in \widehat{\text{Div}}_{\mathbb{R}}(U)^{\text{int}}$ with $|f| \leq C$. Let $(D_{0,m}, g_{0,m})_{m \in \mathbb{N}}$ and $(D_{0,m}, g'_{0,m})_{m \in \mathbb{N}}$ be sequences of nef model arithmetic divisors on U converging in the \overline{B} -adic topology to (D_0, g) and (D_0, g') , respectively, such that $f = g - g'$ and $|g_{0,m} - g'_{0,m}| \leq C$. Write $f_m = g_{0,m} - g'_{0,m}$. Then, the net of signed measures $f_m \langle (\text{dd}^c \varphi_{1,n} + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_{d,n} + \omega_0) \rangle$ converges weakly to the measure $f \langle (\text{dd}^c \varphi_1 + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_d + \omega_0) \rangle$. In particular, we have*

$$\lim_{m, n \rightarrow \infty} \int_X f_m \langle (\text{dd}^c \varphi_{1,n} + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_{d,n} + \omega_0) \rangle = \int_X f \langle (\text{dd}^c \varphi_1 + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_d + \omega_0) \rangle.$$

Proof. By means of Theorem 2.40, statement (iii) follows immediately from statements (i) and (ii).

We start by proving (i). For every open subset $W \subseteq X$ containing B , the set $X \setminus W$ is compact. Therefore, the Green function g_B is continuous on $X \setminus W$ and there is a positive constant M such that $0 < g_B|_{X \setminus W} < M$. Let now $\delta > 0$. Since the sequences of nef model arithmetic divisors $(D_{j,n}, g_{j,n})_{n \in \mathbb{N}}$ on U converge in the \overline{B} -adic topology to the nef adelic arithmetic divisors (D_j, g_j) for $j = 1, \dots, d$, there is an $n_0 \in \mathbb{N}$ such that

$$|g_{j,n} - g_j| \leq \frac{\delta}{M} g_B$$

for $n \geq n_0$. Thus, the set

$$\{x \in X \mid |g_{j,n} - g_j| > \delta\}$$

must be contained in W . Therefore, we find that

$$\limsup_{n \rightarrow \infty} \text{Cap}_{\omega}^* \{|\varphi_{j,n} - \varphi_j| > \delta\} \leq \text{Cap}_{\omega}(W).$$

Since B is a proper analytic subset, its outer capacity is zero. Therefore, we can find open sets $W \supseteq B$ with arbitrarily small capacity. We thus deduce that

$$\lim_{n \rightarrow \infty} \text{Cap}_\omega^* \{|\varphi_{j,n} - \varphi_j| > \delta\} = 0,$$

which proves part (i).

We next prove (ii). For this, let $0 < \varepsilon < 1$. Then, the convergence in the \overline{B} -adic topology implies that there is an $n_\varepsilon \in \mathbb{N}$ such that

$$(3.4) \quad \varphi_j \leq \varphi_{j,n} + \varepsilon g_B \quad \text{and} \quad \varphi_{j,n} \leq \varphi_j + \varepsilon g_B$$

for all $n \geq n_\varepsilon$. Since we have

$$\begin{aligned} \text{dd}^c(\varphi_{j,n} + \varepsilon g_B) + \omega_0 &= \text{dd}^c(g_{j,n} - g_E + \varepsilon g_B) + \omega_0 \\ &= \omega_{D_{j,n}}(g_{j,n}) - \delta_{D_{j,n}} - \omega_0 + \delta_E + \varepsilon \omega_B(g_B) - \varepsilon \delta_B + \omega_0 \\ &= \omega_{D_{j,n}}(g_{j,n}) + \varepsilon \omega_B(g_B) + \delta_{E - D_{j,n} - \varepsilon B} \geq 0, \end{aligned}$$

the functions $\varphi_{j,n} + \varepsilon g_B$ belong to $\text{PSH}(X, \omega_0)$ by Corollary 2.7. In a similar way, one shows that the functions $\varphi_j + \varepsilon g_B$ also belong to $\text{PSH}(X, \omega_0)$.

The inequalities (3.4) in conjunction with Theorem 2.33 imply

$$(3.5) \quad \begin{aligned} &\int_X \langle (\text{dd}^c \varphi_1 + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_d + \omega_0) \rangle \\ &\leq \int_X \langle (\text{dd}^c(\varphi_{1,n} + \varepsilon g_B) + \omega_0) \wedge \dots \wedge (\text{dd}^c(\varphi_{d,n} + \varepsilon g_B) + \omega_0) \rangle, \end{aligned}$$

$$(3.6) \quad \begin{aligned} &\int_X \langle (\text{dd}^c \varphi_{1,n} + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_{d,n} + \omega_0) \rangle \\ &\leq \int_X \langle (\text{dd}^c(\varphi_1 + \varepsilon g_B) + \omega_0) \wedge \dots \wedge (\text{dd}^c(\varphi_d + \varepsilon g_B) + \omega_0) \rangle. \end{aligned}$$

Recalling that

$$\text{dd}^c(g_B - g_E) + \omega_0 = \text{dd}^c g_B + \delta_E,$$

we find from the effectivity of E and the properties of the non-pluripolar product that the right-hand side of (3.5) can be rewritten as

$$\begin{aligned} &\int_X \langle (\text{dd}^c(\varphi_{1,n} + \varepsilon g_B) + \omega_0) \wedge \dots \wedge (\text{dd}^c(\varphi_{d,n} + \varepsilon g_B) + \omega_0) \rangle \\ &= \int_X \langle ((\text{dd}^c \varphi_{1,n} + \omega_0) + \varepsilon(\text{dd}^c g_B + \delta_E)) \wedge \dots \wedge ((\text{dd}^c \varphi_{d,n} + \omega_0) + \varepsilon(\text{dd}^c g_B + \delta_E)) \rangle \\ &= \int_X \langle ((\text{dd}^c \varphi_{1,n} + \omega_0) + \varepsilon(\text{dd}^c(g_B - g_E) + \omega_0)) \wedge \dots \wedge ((\text{dd}^c \varphi_{d,n} + \omega_0) + \varepsilon(\text{dd}^c(g_B - g_E) + \omega_0)) \rangle. \end{aligned}$$

Now, we can apply the multilinearity of the non-pluripolar product stated in Theorem 2.32 (ii) to the right-hand side of the above inequality to deduce, with a constant $C > 0$ independent of ε , from (3.5) that

$$(3.7) \quad \int_X \langle (\text{dd}^c \varphi_1 + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_d + \omega_0) \rangle \leq \int_X \langle (\text{dd}^c \varphi_{1,n} + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_{d,n} + \omega_0) \rangle + \varepsilon C$$

for all $n \geq n_\varepsilon$. In a similar way, we derive from (3.6) the inequality

$$(3.8) \quad \int_X \langle (\text{dd}^c \varphi_{1,n} + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_{d,n} + \omega_0) \rangle \leq \int_X \langle (\text{dd}^c \varphi_1 + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_d + \omega_0) \rangle + \varepsilon C$$

for all $n \geq n_\varepsilon$. It is now evident that the inequalities (3.7) and (3.8) imply the claimed statement (ii) of the proposition.

We are finally left to prove (iv). To do this, we follow the proof of [DDNL18b, Lemma 4.1]. For brevity, we set

$$\mu_n := \langle (\text{dd}^c \varphi_{1,n} + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_{d,n} + \omega_0) \rangle \quad \text{and} \quad \mu := \langle (\text{dd}^c \varphi_1 + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_d + \omega_0) \rangle.$$

Since, by (ii), we can freely add a constant to the function f , we can assume that f and the functions f_m are all non-negative. Furthermore, we need the following data: We fix $\varepsilon > 0$ and let $W \Subset V \Subset X \setminus |E|$ be relatively compact subsets such that $\mu(X \setminus W) < \varepsilon$. We also fix a continuous function χ on X and a non-negative continuous function ρ on X , which is identically 1 on W and 0 on $X \setminus V$. Since the sequence of functions $(\varphi_{j,n})_{n \in \mathbb{N}}$ converges uniformly to φ_j on compact subsets contained in U , the functions $\varphi_{j,n}$ and φ_j are uniformly bounded on V for $j = 1, \dots, d$ and $n \in \mathbb{N}$. From [GZ17, Theorem 4.26], we deduce that $\chi f_m \mu_n$ converges weakly to $\chi f \mu$ on V . The usual Bedford–Taylor theory now implies that also μ_n converges weakly to μ on V ; of course, we could also use the above statement (iii) restricted to V for that. Hence, we find

$$\liminf_{n \rightarrow \infty} \mu_n(W) \geq \mu(W),$$

and thus

$$\limsup_{n \rightarrow \infty} \mu_n(X \setminus W) \leq \mu(X \setminus W) \leq \varepsilon.$$

Since the functions χ , ρ , f_m , and f are uniformly bounded on X , there is a constant $C > 0$ such that the quantities

$$\limsup_{m,n \rightarrow \infty} \int_{X \setminus W} \rho |\chi f_m| \mu_n, \quad \limsup_{m,n \rightarrow \infty} \int_{X \setminus W} |\chi f_m| \mu_n, \quad \int_{X \setminus W} \rho |\chi f| \mu, \quad \int_{X \setminus W} |\chi f| \mu$$

are all bounded by εC . Since the $\chi f_m \mu_n$ converges weakly to $\chi f \mu$ on V and ρ is 0 outside of V , we deduce

$$\lim_{m,n \rightarrow \infty} \int_X \rho \chi f_m \mu_n = \int_X \rho \chi f \mu.$$

Thus, we arrive at

$$\begin{aligned} & \limsup_{m,n \rightarrow \infty} \left| \int_X \chi f_m \mu_n - \int_X \chi f \mu \right| \\ & \leq \limsup_{m,n \rightarrow \infty} \left| \int_X \rho \chi f_m \mu_n - \int_X \rho \chi f \mu \right| + \limsup_{m,n \rightarrow \infty} \left| \int_{X \setminus W} (1 - \rho) \chi f_m \mu_n - \int_{X \setminus W} (1 - \rho) \chi f \mu \right| \\ & \leq \limsup_{m,n \rightarrow \infty} \left| \int_X \rho \chi f_m \mu_n - \int_X \rho \chi f \mu \right| + 4 \cdot \varepsilon C = 4\varepsilon C. \end{aligned}$$

Letting ε tend to 0, we prove the claim. \square

We now link the intersection product of nef adelic arithmetic divisors to the non-pluripolar product of currents. For this, it is convenient to have the analogue of Lemma 3.10 in the current setting.

3.23. Lemma. *Let X be a complex projective manifold with boundary divisor B and $U = X \setminus |B|$ such that \overline{B} is a nef arithmetic divisor. Let \overline{D} be a nef adelic arithmetic divisor on U . Then, there is a sequence $(\overline{D}_n)_{n \in \mathbb{N}} = (D_n, g_{D_n})_{n \in \mathbb{N}}$ of nef model arithmetic divisors on U converging monotonically decreasingly to \overline{D} in the \overline{B} -adic topology. That is, for all $n \in \mathbb{N}$, we have $D_{n+1} \leq D_n$ and $g_{D_{n+1}} \leq g_{D_n}$.*

Proof. Repeat the proof of Lemma 3.10 by putting bars on all the divisors used there. \square

Let $\overline{D} = (D, g_D)$ be a nef adelic arithmetic divisor on U . Choose a sufficiently large ample reference divisor E on X such that $E \geq B$ and $E \geq D + 2B$. Consider the line bundle $L = \mathcal{O}(E)$ and a section s with $\text{div}(s) = E$. The Green function g_D defines a singular semipositive metric $\|\cdot\|_{g_D}$ on L by the rule

$$-\log \|s\|_{g_D}^2 = g_D.$$

We consider the b-divisor $D(L, \|\cdot\|_{g_D}, s)$ introduced in [BBGHdJ21, § 5.1].

3.24. Proposition. *With the above notations, the equality $D(L, \|\cdot\|_{g_D}, s) = b(D)$ holds.*

Proof. By [BBGHdJ21, Example 5.5], the result is true if $(D, g_D) \in \widehat{\text{Div}}_{\mathbb{R}}(U)^{\text{mod}}$. Moreover, the monotonicity of Lelong numbers implies that, if $(D_1, g_{D_1}) \leq (D_2, g_{D_2})$ and $E \geq D_1, D_2$, then $D(L, \|\cdot\|_{g_{D_1}}, s) \leq D(L, \|\cdot\|_{g_{D_2}}, s)$. Let $(D_n, g_{D_n})_{n \in \mathbb{N}}$ be a Cauchy sequence of nef model arithmetic divisors on U converging to (D, g_D) . Let $0 < \varepsilon < 1$ and let $n_\varepsilon \in \mathbb{N}$ be such that

$$-\varepsilon \overline{B} \leq \overline{D} - \overline{D}_n \leq \varepsilon \overline{B}$$

for all $n \geq n_\varepsilon$. By the monotonicity discussed above, we have

$$D(L, \|\cdot\|_{g_D}, s) \leq D(L, \|\cdot\|_{g_{D_n + \varepsilon g_B}}, s) \quad \text{and} \quad D(L, \|\cdot\|_{g_{D_n}}, s) \leq D(L, \|\cdot\|_{g_D + \varepsilon g_B}, s).$$

In turn, by the case of model arithmetic divisors and the additivity of Lelong numbers, this implies that

$$D(L, \|\cdot\|_{g_D}, s) \leq D_n + \varepsilon B \quad \text{and} \quad D_n \leq D(L, \|\cdot\|_{g_D}, s) + \varepsilon B.$$

Hence the sequence $(D_n)_{n \in \mathbb{N}}$ converges to $D(L, \|\cdot\|_{g_D}, s)$, which proves the claimed equality. \square

3.25. Corollary. *For $j = 1, \dots, d$, let $\overline{D}_j = (D_j, g_j)$ be nef adelic arithmetic divisors on U . Choose a sufficiently large arithmetic reference divisor $\overline{E} = (E, g_E)$ on X with g_E a Green function of smooth type for E such that $\overline{E} \geq \overline{B}$ and $\overline{E} \geq \overline{D}_j + 2\overline{B}$. Letting $\omega_0 = \omega_E(g_E)$, the functions $\varphi_j = g_j - g_E$ become ω_0 -plurisubharmonic on X by Proposition 3.20. Then, the equality*

$$D_1 \cdot \dots \cdot D_d = \int_X \langle (\text{dd}^c \varphi_1 + \omega_0) \wedge \dots \wedge (\text{dd}^c \varphi_d + \omega_0) \rangle$$

holds.

Proof. This is a consequence of Corollary 3.11, Proposition 3.24, and [BBGHdJ21, Theorem 5.20]. \square

3.3. The global arithmetic case. We recall the definition of an arithmetic variety. Our definition is a particular case of an arithmetic variety in the sense of [GS90].

3.26. Definition. Let K be a number field and \mathcal{O}_K its ring of integers. An *arithmetic variety* \mathcal{X} over $\text{Spec}(\mathcal{O}_K)$ is a flat quasi-projective normal scheme over $\text{Spec}(\mathcal{O}_K)$ such that its generic fiber $\mathcal{X}_K = \mathcal{X} \times_{\text{Spec}(\mathcal{O}_K)} \text{Spec}(K)$ is smooth. The arithmetic variety \mathcal{X} is called *projective*, if \mathcal{X} is projective over $\text{Spec}(\mathcal{O}_K)$.

If \mathcal{X} is an arithmetic variety over $\text{Spec}(\mathcal{O}_K)$ and Σ is the set of complex embeddings of K , we set

$$\mathcal{X}_\Sigma := \bigcup_{\sigma \in \Sigma} \mathcal{X}_K \times_{\sigma} \text{Spec}(\mathbb{C}),$$

which gives rise to the complex manifold $X = \mathcal{X}_\Sigma(\mathbb{C})$.

3.27. Notation. Given an arithmetic variety \mathcal{X} over $\text{Spec}(\mathcal{O}_K)$ together with its complex manifold $X = \mathcal{X}_\Sigma(\mathbb{C})$ as above, we adopt in the sequel the following notation: \mathbb{R} -Cartier divisors on \mathcal{X} will be denoted by calligraphic letters, the corresponding divisors induced by base change on X will be denoted by the corresponding latin letter. More concretely, if $\mathcal{D} \in \text{Div}_{\mathbb{R}}(\mathcal{X})$ is an \mathbb{R} -Cartier divisor on \mathcal{X} , it determines by base change a divisor \mathcal{D}_Σ on \mathcal{X}_Σ , which gives rise to an \mathbb{R} -Cartier divisor on the complex manifold X , which is consequently denoted by D .

3.28. Definition. Let \mathcal{X} be an arithmetic variety over $\text{Spec}(\mathcal{O}_K)$ and X its associated complex manifold as above. An *arithmetic divisor (with \mathbb{R} -coefficients)* on \mathcal{X} is a pair $\overline{\mathcal{D}} = (\mathcal{D}, g_D)$, where $\mathcal{D} \in \text{Div}_{\mathbb{R}}(\mathcal{X})$ and g_D is a Green function on X of smooth type for D . We let $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{X})$ denote the group of arithmetic divisors (with \mathbb{R} -coefficients) on \mathcal{X} .

We recall the definitions of nef and ample arithmetic divisors.

3.29. Definition. Let \mathcal{X} be a projective arithmetic variety over $\text{Spec}(\mathcal{O}_K)$. An arithmetic divisor $\overline{\mathcal{D}} = (\mathcal{D}, g_D) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{X})$ is called *nef*, if \mathcal{D} is relatively nef, the Green function g_D for D is of plurisubharmonic type, and the height $h_{\overline{\mathcal{D}}}(x) \geq 0$ for every point $x \in \mathcal{X}(\overline{K})$. It is called *ample*, if \mathcal{D} is relatively ample, the Green function g_D for D is of plurisubharmonic type, and the height $h_{\overline{\mathcal{D}}}(x) > 0$ for every point $x \in \mathcal{X}(\overline{K})$.

We now let \mathcal{X} be a projective arithmetic variety over $\text{Spec}(\mathcal{O}_K)$ and let X be its associated complex manifold as above. Let \mathcal{B} be an effective Cartier divisor on \mathcal{X} with support $|\mathcal{B}|$ and let $\mathcal{U} := \mathcal{X} \setminus |\mathcal{B}|$, as well as $U := X \setminus B$; we note that $|\mathcal{B}|$ may contain whole fibers. Let g_B be a Green function of smooth type for B such that there is an $\eta > 0$ with $g_B(x) > \eta$ for all $x \in U$. Such Green functions exist because B being effective, any Green function of smooth type for B is bounded below, and we can add to it a big enough constant to make it bigger than η . We call $\overline{\mathcal{B}} = (\mathcal{B}, g_B)$ an *arithmetic boundary divisor* for \mathcal{X} .

We now repeat the preceding subsections in the arithmetic setting. For this, we denote by $R(\mathcal{X}, \mathcal{U})$ the category of all proper normal modifications of \mathcal{X} which are isomorphisms over \mathcal{U} .

3.30. Definition. The space of *model arithmetic divisors on \mathcal{U} of smooth type* is defined as the limit

$$\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{mod}} := \varinjlim_{\pi \in R(\mathcal{X}, \mathcal{U})} \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{X}_{\pi}).$$

Similarly, one can define *model arithmetic divisors on \mathcal{U} of continuous type*, by asking that the Green functions in the definition of $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{X}_{\pi})$ are only of continuous type.

3.31. Definition. A model arithmetic divisor on \mathcal{U} , given by $\overline{\mathcal{D}} = (\mathcal{D}, g_{\mathcal{D}}) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{X}_{\pi})$ for some $\pi \in R(\mathcal{X}, \mathcal{U})$, is called a *nef model arithmetic divisor on \mathcal{U}* , if $\overline{\mathcal{D}}$ is nef in the sense of Definition 3.29. The set of these divisors defines a convex cone.

On $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{mod}}$ there is a $\overline{\mathcal{B}}$ -adic norm defined as in Subsection 3.2.

3.32. Definition. The space $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{adel}}$ of *adelic arithmetic divisors on \mathcal{U}* is the completion of $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{mod}}$ with respect to the $\overline{\mathcal{B}}$ -adic topology induced by the $\overline{\mathcal{B}}$ -adic norm. Again, elements of $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{adel}}$ are represented by Cauchy sequences of model arithmetic divisors on \mathcal{U} of smooth type. As in Remark 3.14, a Cauchy sequence of model arithmetic divisors on \mathcal{U} of continuous type also defines an adelic arithmetic divisor on \mathcal{U} .

Clearly, if $(\mathcal{D}_n, g_n)_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to the $\overline{\mathcal{B}}$ -adic topology, then the sequence $(\mathcal{D}_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the \mathcal{B} -adic topology, hence defines an adelic divisor $\mathcal{D} \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{adel}}$. Moreover, the sequence of functions $(g_n)_{n \in \mathbb{N}}$ is Cauchy in the g_B -adic topology. This implies that the sequence of functions $(g_n)_{n \in \mathbb{N}}$ converges pointwise to a function g on U . Moreover, this convergence is uniform on compact subsets of U . The global analogue of Proposition 3.18 is given by the following statement.

3.33. Corollary. *There is a short exact sequence*

$$0 \longrightarrow C^0(U)_0 \longrightarrow \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{adel}} \longrightarrow \text{Div}_{\mathbb{R}}(\mathcal{U})^{\text{adel}} \longrightarrow 0,$$

where $\text{Div}_{\mathbb{R}}(\mathcal{U})^{\text{adel}}$ are the geometric adelic divisors on \mathcal{U} as in Subsection 3.1.

We next recall the notions of nef adelic arithmetic divisors and integrable adelic arithmetic divisors. These are adaptations of [YZ21, §2.1.5] and [Zha95, Corollary 5.7] to divisors with real coefficients.

3.34. Definition. The cone $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}$ of *nef adelic arithmetic divisors on \mathcal{U}* is the closure of the convex cone of nef model arithmetic divisors on \mathcal{U} inside of $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{adel}}$, i. e., any nef adelic arithmetic divisor $\overline{\mathcal{D}}$ on \mathcal{U} can be represented by a Cauchy sequence $(\overline{\mathcal{D}}_n)_{n \in \mathbb{N}}$ of nef model arithmetic divisors on \mathcal{U} . The space of *integrable adelic arithmetic divisors on \mathcal{U}* is the vector space

$$\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{int}} := \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}} - \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}.$$

We make the following easy observation.

3.35. Lemma. *Let $(\mathcal{D}, g_1), (\mathcal{D}, g_2) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{adel}}$. Assume that $(\mathcal{D}, g_1) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}$ and that $(\mathcal{D}, g_2) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}$. If $g_2 \geq g_1$, then $(\mathcal{D}, g_2) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}$.*

The following results are proven in [YZ21].

3.36. Proposition. *If $\pi \in R(\mathcal{X}, \mathcal{U})$ and $\overline{\mathcal{B}}' = (\mathcal{B}', g_{\mathcal{B}'})$ is an effective arithmetic divisor on \mathcal{X}_π with $|\mathcal{B}'| = \mathcal{X}_\pi \setminus \mathcal{U}$, then the $\overline{\mathcal{B}}'$ -adic norm is equivalent to the $\overline{\mathcal{B}}$ -adic norm. Therefore, the definitions of the spaces $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{adel}}$, $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}$, and $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{int}}$ only depend on \mathcal{U} and not on a particular choice of arithmetic boundary divisor $\overline{\mathcal{B}}$ for \mathcal{X} .*

3.37. Theorem. *The intersection pairing of nef arithmetic divisors on \mathcal{X} extends by continuity to a unique symmetric multilinear pairing*

$$\underbrace{\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}} \otimes \dots \otimes \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}}_{\dim(\mathcal{U})\text{-times}} \longrightarrow \mathbb{R}.$$

This pairing can be extended by linearity to a pairing

$$\underbrace{\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{int}} \otimes \dots \otimes \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{int}}}_{\dim(\mathcal{U})\text{-times}} \longrightarrow \mathbb{R}.$$

3.38. Remark. As in the geometric case, Theorem 3.37 means that, if $\overline{\mathcal{D}}_j$ with $j = 0, \dots, d$ are nef adelic arithmetic divisors on \mathcal{U} represented by Cauchy sequences $(\overline{\mathcal{D}}_{j,n})_{n \in \mathbb{N}}$ of nef model arithmetic divisors on \mathcal{U} , then the limit

$$(3.9) \quad \overline{\mathcal{D}}_0 \cdots \cdots \overline{\mathcal{D}}_d := \lim_{(n_0, \dots, n_d) \rightarrow (\infty, \dots, \infty)} \overline{\mathcal{D}}_{0, n_0} \cdots \cdots \overline{\mathcal{D}}_{d, n_d}$$

exists and is independent of the choice of approximating sequences.

3.39. Remark. If $\mathcal{V} \subseteq \mathcal{U}$ is a smaller open subset, then there is a map $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U}) \rightarrow \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{V})$ and the intersection product is the same computed in both spaces. Therefore, for the purpose of computing arithmetic intersection products it is harmless to enlarge the boundary divisor, for example in order to ask it to be ample or nef.

The analogue of Lemma 3.10 in the current context is the following.

3.40. Lemma. *Let \mathcal{X} be a projective arithmetic variety over $\text{Spec}(\mathcal{O}_K)$ with boundary divisor \mathcal{B} and $\mathcal{U} = \mathcal{X} \setminus \mathcal{B}$ such that $\overline{\mathcal{B}}$ is a nef arithmetic divisor. Let $\overline{\mathcal{D}}$ be a nef adelic arithmetic divisor on \mathcal{U} . Then, there is a sequence $(\overline{\mathcal{D}}_n)_{n \in \mathbb{N}} = (\mathcal{D}_n, g_n)_{n \in \mathbb{N}}$ of nef model arithmetic divisors on \mathcal{U} converging monotonically decreasingly to $\overline{\mathcal{D}}$ in the $\overline{\mathcal{B}}$ -adic topology. That is, for all $n \in \mathbb{N}$, we have $\mathcal{D}_{n+1} \leq \mathcal{D}_n$ and $g_{n+1} \leq g_n$.*

Proof. Copy the proof of Lemma 3.23. □

The next result deals with integrable adelic arithmetic divisors on \mathcal{U} whose geometric part is trivial.

3.41. Lemma. *Let \mathcal{X} be an arithmetic variety over $\text{Spec}(\mathcal{O}_K)$, $\overline{\mathcal{B}}$ a nef arithmetic boundary divisor for \mathcal{X} , and $(0, f) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{int}}$. Then, the following statements hold:*

- (i) *We can choose sequences of nef model arithmetic divisors $(\mathcal{D}_n, g_n)_{n \in \mathbb{N}}$ and $(\mathcal{D}_n, g'_n)_{n \in \mathbb{N}}$ on \mathcal{U} with the same divisorial part, converging to (\mathcal{D}, g) and to (\mathcal{D}, g') , respectively, and such that $f = g - g'$.*
- (ii) *If the function f is globally bounded, i. e., $|f| \leq C$, then we can choose the above sequences in such a way that $|g_n - g'_n| \leq C$ for all $n \in \mathbb{N}$.*

Proof. To prove (i), we note that since $(0, f)$ is an integrable adelic arithmetic divisor on \mathcal{U} , there exist sequences of nef model arithmetic divisors $(\mathcal{D}_{0,n}, g_{0,n})_{n \in \mathbb{N}}$ and $(\mathcal{D}'_{0,n}, g'_{0,n})_{n \in \mathbb{N}}$ on \mathcal{U} converging to (\mathcal{D}, g) and (\mathcal{D}, g') , respectively, such that $f = g - g'$. After adding to $(\mathcal{D}_{0,n}, g_{0,n})$ a small positive multiple of $\overline{\mathcal{B}}$ (which continues to be a nef model arithmetic divisor on \mathcal{U} by the nefness of $\overline{\mathcal{B}}$), we can further assume that $\mathcal{D}_{0,n} \geq \mathcal{D}'_{0,n}$. For $n \in \mathbb{N}$, we now set

$$\mathcal{D}_n := \mathcal{D}_{0,n}, \quad g_n := \max(g_{0,n}, g'_{0,n} - n), \quad g'_n := \max(g'_{0,n}, g_{0,n} - n).$$

Then, the functions g_n and g'_n are Green functions for \mathcal{D}_n of continuous and of plurisubharmonic type. We claim that the sequence $(\mathcal{D}_n, g_n)_{n \in \mathbb{N}}$ converges in the $\overline{\mathcal{B}}$ -adic topology to (\mathcal{D}, g) and the

sequence $(\mathcal{D}_n, g'_n)_{n \in \mathbb{N}}$ converges in the $\overline{\mathcal{B}}$ -adic topology to (\mathcal{D}, g') . To prove the claim we start by showing that for each $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that the inequalities

$$(3.10) \quad g - \varepsilon g_B \leq g_n \leq g + \varepsilon g_B$$

hold for all $n \geq n_0$. By Corollary 3.33, the function f is continuous on U , and we have that $|f| = o(g_B)$ when approaching the boundary B . Furthermore, since X is compact, there is a constant $C > 0$ such that

$$|g - g'| = |f| \leq C + \frac{\varepsilon}{2} g_B$$

for given $\varepsilon > 0$. Now, choose $n_0 \geq C$ such that the inequalities

$$|g_{0,n} - g| \leq \varepsilon g_B \quad \text{and} \quad |g'_{0,n} - g'| \leq \frac{\varepsilon}{2} g_B$$

hold for all $n \geq n_0$. Then, for $n \geq n_0$, we derive the bounds

$$g'_{0,n} - n \leq g' - n + \frac{\varepsilon}{2} g_B \leq g + C - n + \varepsilon g_B \leq g + \varepsilon g_B.$$

Since we have that $g - \varepsilon g_B \leq g_{0,n} \leq g + \varepsilon g_B$, we deduce for $n \geq n_0$ that

$$g - \varepsilon g_B \leq g_{0,n} \leq g_n = \max(g_{0,n}, g'_{0,n} - n) \leq g + \varepsilon g_B,$$

which proves the claimed inequality 3.10 and thus the convergence of the sequence $(g_n)_{n \in \mathbb{N}}$ to g . Similarly, one shows that the sequence $(g'_n)_{n \in \mathbb{N}}$ converges to g' . We note, if needed, using global regularization of plurisubharmonic functions that we can change the Cauchy sequences in question so that the Green functions are of smooth and of plurisubharmonic type.

To prove (ii), in case that $|f| \leq C$, it is enough to define

$$g_n := \max(g_{0,n}, g'_{0,n} - C) \quad \text{and} \quad g'_n := \max(g'_{0,n}, g_{0,n} - C).$$

The same argument as above now completes the proof in this case. \square

The last technical result we need to prove states that, given two adelic arithmetic divisors on \mathcal{U} with the same divisorial part, we can approximate the more singular one by Green functions that have the same singularity type as the less singular one.

3.42. Proposition. *Let \mathcal{X} be an arithmetic variety over $\text{Spec}(\mathcal{O}_K)$ and $\overline{\mathcal{B}}$ a nef arithmetic boundary divisor for \mathcal{X} . Let (\mathcal{D}, g) and (\mathcal{D}, g') be elements of $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{adel}}$ sharing the same divisorial part. Furthermore, assume that $g' \leq g$ at every point of U . For $n \in \mathbb{N}$, we set*

$$\gamma_n = \max(g - n, g').$$

Then, for each $n \in \mathbb{N}$, the arithmetic divisor (\mathcal{D}, γ_n) belongs to $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{adel}}$ and the sequence $(\mathcal{D}, \gamma_n)_{n \in \mathbb{N}}$ converges to (\mathcal{D}, g') in the $\overline{\mathcal{B}}$ -adic topology. Moreover, if (\mathcal{D}, g) and (\mathcal{D}, g') belong to $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}$, the same is true for (\mathcal{D}, γ_n) .

Proof. By Corollary 3.33, the difference $g - g'$ belongs to $C^0(U)_0$. Since $0 \leq \gamma_n - g' \leq g - g'$, we deduce that $\gamma_n - g'$ also belongs to $C^0(U)_0$, and by the same corollary, the pair (\mathcal{D}, γ_n) thus belongs to $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{adel}}$.

We next prove that the sequence $(\mathcal{D}, \gamma_n)_{n \in \mathbb{N}}$ converges in the $\overline{\mathcal{B}}$ -adic topology to (\mathcal{D}, g') . Since $g - g'$ belongs to $C^0(U)_0$, there is a continuous function h that vanishes on $|B|$ and such that $g - g' = h \cdot g_B$ on U . For every $\varepsilon > 0$, let

$$K_\varepsilon = \{x \in X \mid h(x) \geq \varepsilon\},$$

which is a compact subset of U . Since $h \cdot g_B$ is continuous on U , there is an integer $n_\varepsilon \geq 1$ such that $h \cdot g_B \leq n_\varepsilon$ in K_ε . For $n \geq n_\varepsilon$, we claim that

$$(3.11) \quad 0 \leq \gamma_n - g' \leq \varepsilon g_B.$$

Obviously, the left-hand inequality is clear. For the right-hand inequality, let $x \in U$. If $g(x) - n \leq g'(x)$, then $\gamma_n(x) = g'(x)$, and the right-hand inequality follows from the positivity of g_B . If, on

the other hand, $g(x) - n > g'(x)$, then $h(x) \cdot g_B(x) = g(x) - g'(x) > n \geq n_\varepsilon$. Therefore, $x \notin K_\varepsilon$, and hence $h(x) < \varepsilon$. From this, we derive

$$\gamma_n(x) - g'(x) = g(x) - g'(x) - n \leq g(x) - g'(x) = h(x) \cdot g_B(x) < \varepsilon g_B(x),$$

which proves the right-hand inequality in (3.11). The claimed convergence now follows directly from (3.11).

Assume finally that (\mathcal{D}, g) and (\mathcal{D}, g') belong to $\text{Div}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}$. By Lemma 3.41, we can find sequences $(\mathcal{D}_m, g_m)_{m \in \mathbb{N}}$ and $(\mathcal{D}_m, g'_m)_{m \in \mathbb{N}}$ of nef model arithmetic divisors on \mathcal{U} sharing the same divisorial part and converging to (\mathcal{D}, g) and (\mathcal{D}, g') , respectively. Writing $\gamma_{n,m} = \max(g_m - n, g'_m)$, we obtain a sequence $(\mathcal{D}_m, \gamma_{n,m})_{m \in \mathbb{N}}$ of nef continuous model arithmetic divisors on \mathcal{U} that converges to (\mathcal{D}, γ_n) showing that $(\mathcal{D}, \gamma_n) \in \text{Div}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}$. \square

3.4. Adelic arithmetic line bundles. We start this subsection by recalling the notion of \mathbb{Q} -line bundles. Let X be a scheme. Denote by $\mathbf{Pic}(X)$ the category of locally free sheaves of rank one and by $\text{Pic}(X)$ the group of isomorphism classes of objects in $\mathbf{Pic}(X)$ with group law given by the tensor product. The category $\mathbf{Pic}_{\mathbb{Q}}(X)$ of \mathbb{Q} -line bundles on X is the category whose objects are pairs (n, L) , where $n \in \mathbb{N}_{>0}$ and L is a locally free sheaf of rank one on X , and whose morphisms are

$$\text{Hom}_{\mathbf{Pic}_{\mathbb{Q}}(X)}((n, L), (n', L')) = \varinjlim_{a \in \mathbb{N}_{>0}} \text{Hom}_{\mathbf{Pic}(X)}(L^{\otimes an'}, L'^{\otimes an});$$

The group structure on $\mathbf{Pic}_{\mathbb{Q}}(X)$ is given by the tensor product $(n, L) \otimes (n', L') = (nn', L^{\otimes n'} \otimes L'^{\otimes n})$. The group of isomorphism classes of $\mathbf{Pic}_{\mathbb{Q}}(X)$ is denoted by $\text{Pic}_{\mathbb{Q}}(X)$ and is nothing else than

$$\text{Pic}_{\mathbb{Q}}(X) = \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The global \mathbb{Q} -sections of a \mathbb{Q} -line bundle are defined as

$$\Gamma(X, (n, L)) = \varinjlim_{a \in \mathbb{N}_{>0}} \Gamma(X, L^{\otimes a}),$$

where the maps $\Gamma(X, L^{\otimes a}) \rightarrow \Gamma(X, L^{\otimes ab})$ are given by $s \mapsto s^{\otimes b}$ for $b \in \mathbb{N}_{>0}$. A global \mathbb{Q} -section of (n, L) , represented by a global section s of L^a , is formally given by $s^{\otimes 1/an}$. We note that, for $n, n', a, a' \in \mathbb{N}_{>0}$, the groups $\Gamma(X, (n, L^{\otimes a}))$ and $\Gamma(X, (n', L^{\otimes a'}))$ are canonically isomorphic and that we have the identification

$$\Gamma(X, (n, L)) = \text{Hom}_{\mathbf{Pic}_{\mathbb{Q}}(X)}((1, \mathcal{O}_X), (n, L)).$$

The divisor of a global \mathbb{Q} -section is defined as

$$\text{div}(s^{\otimes 1/an}) = \frac{1}{an} \text{div}(s) \in \text{Div}_{\mathbb{Q}}(X).$$

Let now \mathcal{X} be an arithmetic variety over $\text{Spec}(\mathcal{O}_K)$. The category $\widehat{\mathbf{Pic}}(\mathcal{X})$ of hermitian line bundles on \mathcal{X} is the category whose objects are pairs $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$, where \mathcal{L} is a locally free sheaf of rank one on \mathcal{X} and $\|\cdot\|$ is a hermitian metric on the base change \mathcal{L}_{Σ} over \mathcal{X}_{Σ} , and whose morphisms are

$$\text{Hom}_{\widehat{\mathbf{Pic}}(\mathcal{X})}(\overline{\mathcal{L}}, \overline{\mathcal{L}}') = \{\varphi \in \text{Hom}_{\mathbf{Pic}(\mathcal{X})}(\mathcal{L}, \mathcal{L}') \mid \|\varphi(v)\| \leq \|v\|\}.$$

For instance, $\overline{\mathcal{O}}_{\mathcal{X}}$ will denote the trivial line bundle $\mathcal{O}_{\mathcal{X}}$ with the metric $\|1\| = 1$. We note that $\text{Hom}_{\widehat{\mathbf{Pic}}(\mathcal{X})}(\overline{\mathcal{O}}, \overline{\mathcal{L}})$ can be identified with the set of *small sections*

$$\widehat{\Gamma}(\mathcal{X}, \overline{\mathcal{L}}) = \{s \in \Gamma(\mathcal{X}, \overline{\mathcal{L}}) \mid \|s\| \leq 1\}.$$

The group $\widehat{\text{Pic}}(\mathcal{X})$ is the set of isomorphism classes of objects in $\widehat{\mathbf{Pic}}(\mathcal{X})$ with group law given by the tensor product.

The category $\widehat{\mathbf{Pic}}_{\mathbb{Q}}(\mathcal{X})$ of hermitian \mathbb{Q} -line bundles on \mathcal{X} is the category whose objects are pairs $(n, \overline{\mathcal{L}})$, where $n \in \mathbb{N}_{>0}$ and $\overline{\mathcal{L}} \in \widehat{\mathbf{Pic}}(\mathcal{X})$, and whose morphisms are

$$\text{Hom}_{\widehat{\mathbf{Pic}}_{\mathbb{Q}}(\mathcal{X})}((n, \overline{\mathcal{L}}), (n', \overline{\mathcal{L}}')) = \varinjlim_{a \in \mathbb{N}_{>0}} \text{Hom}_{\widehat{\mathbf{Pic}}(\mathcal{X})}(\overline{\mathcal{L}}^{\otimes an'}, \overline{\mathcal{L}}'^{\otimes an}).$$

If $(n, \overline{\mathcal{L}}_0) \in \widehat{\mathbf{Pic}}_{\mathbb{Q}}(\mathcal{X})$ and s_0 is a rational section of $\mathcal{L}_0^{\otimes a}$, a power of the underlying line bundle \mathcal{L}_0 , then s_0 defines a section $s = s_0^{\otimes 1/an}$, whose associated arithmetic divisor is defined as

$$\widehat{\mathrm{div}}(s) = \frac{1}{an}(\mathrm{div}(s_0), -\log \|s_0\|^2).$$

Following [YZ21], we introduce the category of adelic arithmetic line bundles. To this end, we assume that the arithmetic variety \mathcal{X} is projective with arithmetic boundary divisor $\overline{\mathcal{B}}$ and $\mathcal{U} = \mathcal{X} \setminus |\mathcal{B}|$. Note that \mathcal{U} is a quasi-projective arithmetic variety.

Let $\mathcal{L} \in \mathbf{Pic}_{\mathbb{Q}}(\mathcal{U})$. A sequence of triples $(\mathcal{X}_{\pi_n}, \overline{\mathcal{L}}_n, \ell_n)_{n \in \mathbb{N}}$, where $\pi_n \in R(\mathcal{X}, \mathcal{U})$, $\overline{\mathcal{L}}_n \in \widehat{\mathbf{Pic}}_{\mathbb{Q}}(\mathcal{X}_{\pi_n})$, and $\ell_n: \mathcal{L} \rightarrow \mathcal{L}_n|_{\mathcal{U}}$ is an isomorphism, is called a Cauchy sequence (in the $\overline{\mathcal{B}}$ -adic topology), if for any non-zero rational section s of \mathcal{L} , the sequence $(\widehat{\mathrm{div}}(\ell_n(s)))_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to the $\overline{\mathcal{B}}$ -adic norm of model arithmetic divisors.

The category of adelic arithmetic line bundles $\widehat{\mathbf{Pic}}_{\mathbb{Q}}(\mathcal{U})^{\mathrm{adel}}$ is the category of pairs $(\mathcal{L}, (\mathcal{X}_{\pi_n}, \overline{\mathcal{L}}_n, \ell_n)_{n \in \mathbb{N}})$, where $\mathcal{L} \in \mathbf{Pic}_{\mathbb{Q}}(\mathcal{U})$ and $(\mathcal{X}_{\pi_n}, \overline{\mathcal{L}}_n, \ell_n)_{n \in \mathbb{N}}$ is a Cauchy sequence as before. A morphism between $(\mathcal{L}, (\mathcal{X}_{\pi_n}, \overline{\mathcal{L}}_n, \ell_n)_{n \in \mathbb{N}})$ and $(\mathcal{L}', (\mathcal{X}'_{\pi_n}, \overline{\mathcal{L}}'_n, \ell'_n)_{n \in \mathbb{N}})$ is an isomorphism $\lambda: \mathcal{L} \rightarrow \mathcal{L}'$ over \mathcal{U} such that, if we write

$$(\mathcal{Y}_{\pi_k}, \overline{\mathcal{M}}_k, m_k) = \begin{cases} (\mathcal{X}_{\pi_n}, \overline{\mathcal{L}}_n, \ell_n), & \text{if } k = 2n, \\ (\mathcal{X}'_{\pi_n}, \overline{\mathcal{L}}'_n, \ell'_n \circ \lambda), & \text{if } k = 2n - 1, \end{cases}$$

the sequence $(\mathcal{Y}_{\pi_k}, \overline{\mathcal{M}}_k, m_k)_{k \in \mathbb{N}}$ is Cauchy. As we have defined $\widehat{\mathbf{Pic}}_{\mathbb{Q}}(\mathcal{U})^{\mathrm{adel}}$, it is a groupoid, that is, all the morphisms are isomorphisms. The tensor product of hermitian line bundles induces a tensor product in $\widehat{\mathbf{Pic}}_{\mathbb{Q}}(\mathcal{U})^{\mathrm{adel}}$. The group of adelic arithmetic line bundles $\widehat{\mathbf{Pic}}_{\mathbb{Q}}(\mathcal{U})^{\mathrm{adel}}$ is the set of isomorphism classes in $\widehat{\mathbf{Pic}}_{\mathbb{Q}}(\mathcal{U})^{\mathrm{adel}}$ with group law given by the tensor product.

3.43. Remark. Let $\overline{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_{\pi_n}, \overline{\mathcal{L}}_n, \ell_n)_{n \in \mathbb{N}})$ be an adelic arithmetic line bundle on \mathcal{U} and let $U = \mathcal{U}(\mathbb{C})$. All the line bundles \mathcal{L}_n , when restricted to U , agree with $L = \mathcal{L}|_U$. The metric of each $\overline{\mathcal{L}}_n$ induces a metric $\|\cdot\|_n$ on L . These metrics converge (uniformly on compact subsets of U) to a metric $\|\cdot\|$ on L over U . We call this metric *the archimedean component* of $\overline{\mathcal{L}}$.

3.44. Remark. Let $\overline{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_{\pi_n}, \overline{\mathcal{L}}_n, \ell_n)_{n \in \mathbb{N}})$ be an adelic arithmetic line bundle on \mathcal{U} and s a non-zero rational section of \mathcal{L} . Then, the sequence $(\widehat{\mathrm{div}}(\ell_n(s)))_{n \in \mathbb{N}}$ determines an adelic arithmetic divisor in $\widehat{\mathrm{Div}}_{\mathbb{R}}(\mathcal{U})^{\mathrm{adel}}$ that we will denote simply by $\widehat{\mathrm{div}}(s)$. We will say that $\overline{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_{\pi_n}, \overline{\mathcal{L}}_n, \ell_n)_{n \in \mathbb{N}})$ is nef if and only if $\widehat{\mathrm{div}}(s)$ is nef for any rational section s of \mathcal{L} .

3.5. Functorial properties of the adelic arithmetic intersection product. We will only discuss the functoriality in the global arithmetic case. The first functorial property is the restriction to a smaller open subset.

Let \mathcal{X} be an arithmetic variety over $\mathrm{Spec}(\mathcal{O}_K)$. Let $\mathcal{U}' \subseteq \mathcal{U}$ be two open subsets. If $\pi: \mathcal{X}_1 \rightarrow \mathcal{X}$ belongs to $R(\mathcal{X}, \mathcal{U})$, it also belongs to $R(\mathcal{X}, \mathcal{U}')$. Therefore, there is a map

$$\widehat{\mathrm{Div}}_{\mathbb{R}}(\mathcal{U})^{\mathrm{mod}} \longrightarrow \widehat{\mathrm{Div}}_{\mathbb{R}}(\mathcal{U}')^{\mathrm{mod}}.$$

Let $\overline{\mathcal{B}} = (\mathcal{B}, g_{\mathcal{B}})$ and $\overline{\mathcal{B}}' = (\mathcal{B}', g_{\mathcal{B}'})$ be boundary divisors with $\mathcal{U} = \mathcal{X} \setminus |\mathcal{B}|$ and $\mathcal{U}' = \mathcal{X} \setminus |\mathcal{B}'|$. Then, there is a constant $c > 0$ such that $\overline{\mathcal{B}} \leq c\overline{\mathcal{B}}'$. Therefore, the $\overline{\mathcal{B}}$ -adic and $\overline{\mathcal{B}}'$ -adic norms satisfy

$$\|\cdot\|_{\overline{\mathcal{B}}} \leq c \|\cdot\|_{\overline{\mathcal{B}}}.$$

This implies that there is a continuous restriction map

$$(3.12) \quad \widehat{\mathrm{Div}}_{\mathbb{R}}(\mathcal{U})^{\mathrm{adel}} \longrightarrow \widehat{\mathrm{Div}}_{\mathbb{R}}(\mathcal{U}')^{\mathrm{adel}}.$$

This map sends $\widehat{\mathrm{Div}}_{\mathbb{R}}(\mathcal{U})^{\mathrm{nef}}$ to $\widehat{\mathrm{Div}}_{\mathbb{R}}(\mathcal{U}')^{\mathrm{nef}}$.

More generally, let \mathcal{X} and \mathcal{X}' be arithmetic varieties over $\mathrm{Spec}(\mathcal{O}_K)$, and let $\overline{\mathcal{B}}$ and $\overline{\mathcal{B}}'$ be boundary divisors on \mathcal{X} and \mathcal{X}' , respectively. Write $\mathcal{U} = \mathcal{X} \setminus |\mathcal{B}|$ and $\mathcal{U}' = \mathcal{X}' \setminus |\mathcal{B}'|$. For every dominant morphism $f: \mathcal{U}' \rightarrow \mathcal{U}$, there is a pull-back map

$$f_{\mathrm{adel}}^*: \widehat{\mathrm{Div}}_{\mathbb{R}}(\mathcal{U})^{\mathrm{adel}} \longrightarrow \widehat{\mathrm{Div}}_{\mathbb{R}}(\mathcal{U}')^{\mathrm{adel}}$$

that sends $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}$ to $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U}')^{\text{nef}}$. This pull-back map is constructed as follows. Given $\pi \in R(\mathcal{X}, \mathcal{U})$, there exist $\pi' \in R(\mathcal{X}', \mathcal{U}')$ and $f_{\pi}: \mathcal{X}'_{\pi'} \rightarrow \mathcal{X}_{\pi}$ such that the diagram

$$\begin{array}{ccc} \mathcal{U}' & \xrightarrow{f} & \mathcal{U} \\ \downarrow & & \downarrow \\ \mathcal{X}'_{\pi'} & \xrightarrow{f_{\pi}} & \mathcal{X}_{\pi} \end{array}$$

commutes. From this we obtain a pull-back map

$$f_{\text{mod}}^*: \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{mod}} \longrightarrow \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U}')^{\text{mod}}$$

induced by mapping $(\mathcal{D}, g_{\mathcal{D}}) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{X}_{\pi})$ to $(f_{\pi}^* \mathcal{D}, f_{\pi}^* g_{\mathcal{D}}) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{X}'_{\pi'})$. After taking the respective $\overline{\mathcal{B}}$ -adic and $\overline{\mathcal{B}'}$ -adic completions, we obtain the pull-back maps

$$f_{\text{adel}}^*: \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{adel}} \longrightarrow \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U}')^{\text{adel}} \quad \text{and} \quad f_{\text{adel}}^*: \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}} \longrightarrow \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U}')^{\text{nef}}.$$

For more details see [YZ21, § 2.5.5].

3.45. Remark. We have defined pull-back maps of adelic divisors only for dominant morphisms. By contrast, for adelic arithmetic line bundles, given an arbitrary morphism $f: \mathcal{U}' \rightarrow \mathcal{U}$, there is a well defined pull-back functor $f^*: \widehat{\mathbf{Pic}}_{\mathbb{Q}}(\mathcal{U})^{\text{adel}} \rightarrow \widehat{\mathbf{Pic}}_{\mathbb{Q}}(\mathcal{U}')^{\text{adel}}$ that preserves nefness and tensor products.

The intersection product and the pull-back are related by the following projection formula.

3.46. Proposition. *Let $f: \mathcal{U}' \rightarrow \mathcal{U}$ be a dominant morphism as before. Furthermore, assume that f is projective and flat of relative dimension e , and let d be the relative dimension of \mathcal{U} over $\text{Spec}(\mathcal{O}_K)$. Let $\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_d$ be integrable adelic arithmetic divisors on \mathcal{U} and $\overline{\mathcal{E}}_1, \dots, \overline{\mathcal{E}}_e$ be integrable adelic arithmetic divisors on \mathcal{U}' . Then, we have the formula*

$$\overline{\mathcal{E}}_1 \cdots \overline{\mathcal{E}}_e \cdot f_{\text{adel}}^* \overline{\mathcal{D}}_0 \cdots f_{\text{adel}}^* \overline{\mathcal{D}}_d = \deg(\mathcal{E}_1|_F, \dots, \mathcal{E}_e|_F) \cdot \overline{\mathcal{D}}_0 \cdots \overline{\mathcal{D}}_d,$$

where F is a generic fiber of f and $\deg(\mathcal{E}_1|_F, \dots, \mathcal{E}_e|_F)$ is the degree of the generic fiber with respect to the underlying geometric divisors.

Proof. See [YZ21, Lemma 4.6.1]. □

3.47. Proposition. *Let $f: \mathcal{U}' \rightarrow \mathcal{U}$ be a finite flat dominant morphism of degree n and let d be the relative dimension of \mathcal{U} over $\text{Spec}(\mathcal{O}_K)$. Let $\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_d$ be integrable adelic arithmetic divisors on \mathcal{U} . Then, we have the formula*

$$f_{\text{adel}}^* \overline{\mathcal{D}}_0 \cdots f_{\text{adel}}^* \overline{\mathcal{D}}_d = n \overline{\mathcal{D}}_0 \cdots \overline{\mathcal{D}}_d.$$

Proof. This result also is covered in [YZ21, Lemma 4.6.1]. □

4. EXTENSION OF YUAN–ZHANG’S ARITHMETIC INTERSECTION PRODUCT

4.1. Comparison of log-log singular and adelic arithmetic line bundles. In the study of the arithmetic of pure Shimura varieties, log-log singular line bundles appear in a natural way. The paradigmatic example is the Hodge bundle on the moduli space of principally polarized abelian varieties with a level structure. In [YZ21, § 5.5], it is proven that the Hodge bundle equipped with the Faltings metric is an example of an adelic arithmetic line bundle. However, it is not proven that it is integrable and we suspect that, in general, this is not the case. In this subsection, we give an example of a toric line bundle that has the same kind of singularities as the Faltings metric on the Hodge bundle over the modular curve and give the sketch of a proof that this line bundle is not integrable, at least when the boundary divisor is small enough to be able to apply toric methods. The details will appear in the PhD thesis of Gari Peralta.

4.1. Construction. We consider the arithmetic variety $\mathcal{X} = \mathbb{P}_{\mathbb{Z}}^1$ over the integers \mathbb{Z} with homogeneous coordinates $(x : y)$ and affine coordinate $t = x/y$. On $\mathbb{P}_{\mathbb{Z}}^1$, we consider the line bundle $\mathcal{O}(1)$ with the section $s = y$, so $\text{div}(s) = [(1 : 0)]$. At the archimedean place $v = \infty$, we equip $\mathcal{O}(1)_{\infty}$ with a hermitian metric $\|\cdot\|$ defined as follows: We define the function $\varphi_{\infty} : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\varphi_{\infty}(u) = \begin{cases} 1 + \log(u), & \text{for } u \geq 1, \\ u, & \text{for } u \leq 1. \end{cases}$$

On $\mathbb{P}_{\mathbb{C}}^1 \setminus \{0, \infty\}$, we then set

$$\log \|s\| = \varphi_{\infty}(-\log |t|),$$

which becomes an \mathbb{S}^1 -invariant hermitian metric on $\mathcal{O}(1)_{\infty}$, where \mathbb{S}^1 is the compact torus $\{z \in \mathbb{C} \mid |z| = 1\}$. This metric is singular at the point $(0 : 1)$, where it has a singularity similar to the singularity of the Faltings metric when approaching a cusp of a modular curve.

For all non-archimedean places v , we equip the line bundle $\mathcal{O}(1)_v$ with the canonical metric $\|\cdot\|_{v,\text{can}}$ induced by the function $\varphi_v(u) = \min(u, 0)$. This is the same as the metric induced by the model $\mathcal{O}(1)_{\mathbb{Z}}$ over $\mathbb{P}_{\mathbb{Z}}^1$. We denote by $\overline{\mathcal{O}(1)}_{\log \log}$ the line bundle $\mathcal{O}(1)$ with this metric. Write $g = -\log \|s\|^2$ and let $\overline{\mathcal{D}} = (\text{div}(s), g)$.

Let $\mathcal{U} \subseteq \mathcal{X}$ be an open subset such that

$$(4.1) \quad (0 : 1) \notin \mathcal{U}_{\mathbb{Q}} \quad \text{and} \quad \mathbb{P}_{\mathbb{Q}}^1 \setminus \{(0 : 1), (1 : 0)\} \subseteq \mathcal{U}.$$

A consequence of Corollary 3.33 is that $\overline{\mathcal{D}} \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{adel}}$.

4.2. Proposition. *For any choice of \mathcal{U} satisfying conditions (4.1), we have $\overline{\mathcal{D}} \notin \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{int}}$.*

Proof. For the proof we will use freely the theory of arithmetic intersection products on toric varieties developed in [BGPS14].

Assume that the adelic arithmetic divisor $\overline{\mathcal{D}}$ is integrable. Then, we have $\overline{\mathcal{D}} = \overline{\mathcal{D}}_1 - \overline{\mathcal{D}}_2$ with $\overline{\mathcal{D}}_1, \overline{\mathcal{D}}_2 \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}$. Thus, there are Cauchy sequences $(\overline{\mathcal{D}}_{1,n})_{n \in \mathbb{N}}$ and $(\overline{\mathcal{D}}_{2,n})_{n \in \mathbb{N}}$ of nef model arithmetic divisors on \mathcal{U} converging to $\overline{\mathcal{D}}_1$ and $\overline{\mathcal{D}}_2$, respectively.

By the conditions on \mathcal{U} , the divisors $\mathcal{D}_{i,n,\mathbb{Q}}$, for $i = 1, 2$ and $n \in \mathbb{N}$, are of the form

$$(4.2) \quad \mathcal{D}_{i,n,\mathbb{Q}} = a_{i,n}[(0 : 1)] + b_{i,n}[(1 : 0)] + E,$$

where $a_{i,n}, b_{i,n} \in \mathbb{R}$ and E is a divisor supported on $\mathbb{P}_{\mathbb{Q}}^1 \setminus \{(0 : 1), (1 : 0)\}$, which is independent of i and n . Since for large enough $c \in \mathbb{R}$, the divisor $c[(1 : 0)] - E$ is nef, after adding to all the divisors $\overline{\mathcal{D}}_{i,n}$ a suitable nef model arithmetic divisor, we can assume that the divisor E in (4.2) is zero. This means that the divisors $\mathcal{D}_{i,n,\mathbb{Q}}$ are toric. Therefore, using [BGPS14, Proposition 4.3.4] for the smooth metric over the archimedean place and [BGPS14, Proposition 4.3.4] for the model metrics over the non-archimedean places, we can average the initial nef model arithmetic divisors to obtain sequences of toric arithmetic divisors $(\overline{\mathcal{D}}_{1,n}^{\mathbb{S}})_{n \in \mathbb{N}}$ and $(\overline{\mathcal{D}}_{2,n}^{\mathbb{S}})_{n \in \mathbb{N}}$. These sequences are again Cauchy sequences of nef model arithmetic divisors and converge to nef adelic arithmetic divisors $\overline{\mathcal{D}}_1^{\mathbb{S}}$ and $\overline{\mathcal{D}}_2^{\mathbb{S}}$, respectively, satisfying $\overline{\mathcal{D}} = \overline{\mathcal{D}}_1^{\mathbb{S}} - \overline{\mathcal{D}}_2^{\mathbb{S}}$. In conclusion, we can assume that the Cauchy sequences $(\overline{\mathcal{D}}_{1,n})_{n \in \mathbb{N}}$ and $(\overline{\mathcal{D}}_{2,n})_{n \in \mathbb{N}}$ are made of *toric* nef model arithmetic divisors. Furthermore, by Lemma 3.40, we can assume that they are monotonically decreasing.

For each $i = 1, 2$ and $n \in \mathbb{N}$, the toric nef model arithmetic divisors $\overline{\mathcal{D}}_{i,n}$ are encoded by a family of concave functions $\varphi_{v,i,n}$, one for each place v of \mathbb{Q} . The Legendre–Fenchel dual of $\varphi_{v,i,n}$ is the function

$$\vartheta_{v,i,n}(x) = \inf_{u \in \mathbb{R}} (\langle x, u \rangle - \varphi_{v,i,n}(u))$$

and is called the local roof function of $\overline{\mathcal{D}}_{i,n}$. The global roof function of $\overline{\mathcal{D}}_{i,n}$ is given by

$$\vartheta_{i,n} = \sum_v \vartheta_{v,i,n}.$$

The function $\vartheta_{i,n}$ and the functions $\vartheta_{v,i,n}$ are concave and they all have support on a common polytope $\Delta_{i,n}$. The fact that the toric model arithmetic divisors $\overline{\mathcal{D}}_{i,n}$ are nef is reflected by the

condition

$$(4.3) \quad \vartheta_{i,n}(x) \geq 0$$

for all $x \in \Delta_{i,n}$ (see [BGMPS16, Theorem 6.1]).

The fact that the Cauchy sequences $(\overline{\mathcal{D}}_{i,n})_{n \in \mathbb{N}}$ are monotonically decreasing implies that the sequences of polytopes $(\Delta_{i,n})_{n \in \mathbb{N}}$ are nested for $i = 1, 2$. We define Δ_i as the intersection of all the $\Delta_{i,n}$'s for $n \in \mathbb{N}$. This is a closed bounded convex subset and the roof functions $\vartheta_{i,n}$ converge monotonically to a concave function ϑ_i on Δ_i . Condition (4.3) implies that $\vartheta_i \geq 0$. By concavity, this implies that the functions ϑ_i are bounded for $i = 1, 2$. We define the functions φ_i as the Legendre–Fenchel dual of the functions ϑ_i . Note that one could have gone directly from the functions $\varphi_{v,i,n}$ by taking a sup-convolution (see [BGPS14, § 2.3]) and a limit. We also introduce the functions Ψ_i as the support functions (see [BGPS14, Example 2.2.1]) of the convex sets Δ_i . Since the functions ϑ_i are bounded, the functions $\varphi_i - \Psi_i$ are also bounded.

We finally come back to the adelic arithmetic divisor $\overline{\mathcal{D}}$, which we assumed to be integrable. For the non-archimedean places v , the Legendre–Fenchel dual ϑ_v of φ_v becomes the function zero over the interval $[0, 1]$. However, for the archimedean place, the Legendre–Fenchel dual ϑ_∞ of φ_∞ becomes

$$\vartheta_\infty(x) = \inf_{u \in \mathbb{R}} (\langle x, u \rangle - \varphi_\infty(u)).$$

This infimum is achieved for $u_x \in \mathbb{R}$ satisfying $\varphi'_\infty(u_x) = x$, i. e.,

$$\frac{1}{u_x} = x \quad \iff \quad u_x = \frac{1}{x}.$$

This gives

$$\vartheta_\infty(x) = x \cdot \frac{1}{x} - \left(1 + \log\left(\frac{1}{x}\right)\right) = \log(x).$$

Thus, the total roof function ϑ is given by

$$\vartheta(x) = \sum_v \vartheta_v(x) = \log(x),$$

which is unbounded as $x \rightarrow 0$. The Legendre–Fenchel dual of ϑ is the function $\varphi = \varphi_\infty$. Let Ψ be the support function of the interval $[0, 1]$. Then, $\Psi = \varphi_v$ for any non-archimedean place v .

The equation $\overline{\mathcal{D}}_1 = \overline{\mathcal{D}} + \overline{\mathcal{D}}_2$ implies the equations

$$\varphi_1 = \varphi + \varphi_2 \quad \text{and} \quad \Psi_1 = \Psi + \Psi_2.$$

The contradiction now comes from the fact that $(\varphi_1 - \Psi_1) - (\varphi_2 - \Psi_2)$ is bounded, while $\varphi - \Psi$ is not. \square

4.2. Adelic arithmetic intersections and mixed relative energy. In this subsection, we relate the adelic arithmetic intersection product with the non-pluripolar product of currents and the mixed relative energy.

As in Subsection 3.3, let K be a number field with ring of integers \mathcal{O}_K , let \mathcal{X} be a projective arithmetic variety of dimension d over $\text{Spec}(\mathcal{O}_K)$, and let X be its associated complex manifold. Let $\overline{\mathcal{B}} = (\mathcal{B}, g_B)$ be an arithmetic boundary divisor for \mathcal{X} and write as before $\mathcal{U} = \mathcal{X} \setminus |\mathcal{B}|$, as well as $U = X \setminus B$. Furthermore, we assume that $\overline{\mathcal{B}}$ is nef to be able to use Lemma 3.41.

4.3. Lemma. *For $j = 1, \dots, d$, let $\overline{\mathcal{D}}_j = (\mathcal{D}_j, g_j) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}$ be nef adelic arithmetic divisors on \mathcal{U} . Moreover, let $\overline{\mathcal{D}}_0 = (\mathcal{D}_0, g_0)$, $\overline{\mathcal{D}}'_0 = (\mathcal{D}_0, g'_0) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{int}}$ be two integrable adelic arithmetic divisors on \mathcal{U} sharing the same divisorial part and such that $f = g_0 - g'_0$ is globally bounded. Choose sufficiently large reference divisors E_j on X such that $E_j \geq B$ and $E_j \geq \mathcal{D}_j + 2B$. Then, the equality*

$$\overline{\mathcal{D}}_0 \cdot \overline{\mathcal{D}}_1 \cdot \dots \cdot \overline{\mathcal{D}}_d - \overline{\mathcal{D}}'_0 \cdot \overline{\mathcal{D}}_1 \cdot \dots \cdot \overline{\mathcal{D}}_d = \int_X f \langle \omega_{E_1}(g_1) \wedge \dots \wedge \omega_{E_d}(g_d) \rangle$$

holds.

Proof. By multilinearity, the arithmetic intersection product $(0, f) \cdot \overline{\mathcal{D}}_1 \cdot \dots \cdot \overline{\mathcal{D}}_d$ does not depend on how we write $(0, f)$ as a difference of adelic arithmetic divisors on \mathcal{U} sharing the same divisorial part. Therefore, we can assume that (\mathcal{D}_0, g_0) and (\mathcal{D}_0, g'_0) are nef adelic arithmetic divisors on \mathcal{U} . Thus, there are Cauchy sequences $(\overline{\mathcal{D}}_{0,n})_{n \in \mathbb{N}}$, $(\overline{\mathcal{D}}'_{0,n})_{n \in \mathbb{N}}$, and $(\overline{\mathcal{D}}_{j,n})_{n \in \mathbb{N}}$ of nef model arithmetic divisors on \mathcal{U} converging in the $\overline{\mathcal{B}}$ -adic topology to $\overline{\mathcal{D}}_0$, $\overline{\mathcal{D}}'_0$, and $\overline{\mathcal{D}}_j$ for $j = 1, \dots, d$, respectively. By Lemma 3.41, we can assume that $\overline{\mathcal{D}}_{0,n} = (\mathcal{D}_{0,n}, g_{0,n})$ and $\overline{\mathcal{D}}'_{0,n} = (\mathcal{D}_{0,n}, g'_{0,n})$ share the same divisorial part and that the functions $g_{0,n} - g'_{0,n}$ are globally bounded by the same bound as the one for f . By assumption, we have $E_j \geq D_j + 2B$, which implies that $E_j \geq D_{j,n}$ for all $n \in \mathbb{N}$ after possibly taking suitable subsequences. By Lemma 2.28, the Green functions $g_{j,n}$ thus become Green functions for E_j of plurisubharmonic type for all $j = 1, \dots, d$ and for all $n \in \mathbb{N}$. From this we obtain

$$\begin{aligned} \overline{\mathcal{D}}_0 \cdot \overline{\mathcal{D}}_1 \cdot \dots \cdot \overline{\mathcal{D}}_d - \overline{\mathcal{D}}'_0 \cdot \overline{\mathcal{D}}_1 \cdot \dots \cdot \overline{\mathcal{D}}_d &= \lim_{n \rightarrow \infty} (\overline{\mathcal{D}}_{0,n} \cdot \overline{\mathcal{D}}_{1,n} \cdot \dots \cdot \overline{\mathcal{D}}_{d,n} - \overline{\mathcal{D}}'_{0,n} \cdot \overline{\mathcal{D}}_{1,n} \cdot \dots \cdot \overline{\mathcal{D}}_{d,n}) \\ &= \lim_{n \rightarrow \infty} \int_X (g_{0,n} - g'_{0,n}) \omega_{D_{1,n}}(g_{1,n}) \wedge \dots \wedge \omega_{D_{d,n}}(g_{d,n}) \\ &= \lim_{n \rightarrow \infty} \int_X (g_{0,n} - g'_{0,n}) \langle \omega_{E_1}(g_{1,n}) \wedge \dots \wedge \omega_{E_d}(g_{d,n}) \rangle, \end{aligned}$$

where, for the last equality, we have used Proposition 2.35. Proposition 3.22 (iv) now shows that

$$\lim_{n \rightarrow \infty} \int_X (g_{0,n} - g'_{0,n}) \langle \omega_{E_1}(g_{1,n}) \wedge \dots \wedge \omega_{E_d}(g_{d,n}) \rangle = \int_X (g_0 - g'_0) \langle \omega_{E_1}(g_1) \wedge \dots \wedge \omega_{E_d}(g_d) \rangle,$$

which proves the claim. \square

Now we are able to relate the adelic arithmetic intersection product with the mixed relative energy.

4.4. Theorem. *For $j = 0, \dots, d$, let $\overline{\mathcal{D}}_j = (\mathcal{D}_j, g_j) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}$ and $\overline{\mathcal{D}}'_j = (\mathcal{D}_j, g'_j) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}$ be two nef adelic arithmetic divisors on \mathcal{U} sharing the same divisorial part. Choose sufficiently large arithmetic reference divisors $\overline{\mathcal{E}}_j = (\mathcal{E}_j, g_{E_j})$ on \mathcal{X} such that $\overline{\mathcal{E}}_j \geq \overline{\mathcal{B}}$ and $\overline{\mathcal{E}}_j \geq \overline{\mathcal{D}}_j + 2\overline{\mathcal{B}}$ with g_{E_j} Green functions of smooth type for E_j , so that the functions $\varphi_j := g_j - g_{E_j}$ and $\varphi'_j := g'_j - g_{E_j}$ become ω_j -plurisubharmonic, where $\omega_j = \omega_{E_j}(g_{E_j})$. Assume that the Green functions g_j 's are more singular than the g'_j 's, that is, $[\varphi_j] \leq [\varphi'_j]$. As in Subsection 2.7, we write $\varphi = (\varphi_0, \dots, \varphi_d)$, $\varphi' = (\varphi'_0, \dots, \varphi'_d)$, and $\omega = (\omega_0, \dots, \omega_d)$. Then, we have*

- (i) $\varphi \in \mathcal{E}(X, \omega, \varphi')$.
- (ii) $\varphi \in \mathcal{E}^1(X, \omega, \varphi')$ and the formula

$$\overline{\mathcal{D}}_0 \cdot \dots \cdot \overline{\mathcal{D}}_d = \overline{\mathcal{D}}'_0 \cdot \dots \cdot \overline{\mathcal{D}}'_d + I_{\varphi'}(\varphi)$$

holds, where $I_{\varphi'}(\varphi)$ is the mixed relative energy of φ with respect to φ' defined in Definition 2.46.

Proof. To prove (i), we just note that for $j = 0, \dots, d$, we have by Corollary 3.25

$$\int_X \langle (\text{dd}^c \varphi_j + \omega_j)^{\wedge d} \rangle = D_j^d = \int_X \langle (\text{dd}^c \varphi'_j + \omega_j)^{\wedge d} \rangle.$$

Together with the assumption $[\varphi_j] \leq [\varphi'_j]$, this implies that $\varphi_j \in \mathcal{E}(X, \omega_j, \varphi'_j)$.

To prove (ii), assume for the moment that $|g_j - g'_j|$ is bounded for $j = 0, \dots, d$. Since the nef adelic arithmetic divisors $\overline{\mathcal{D}}_j$ and $\overline{\mathcal{D}}'_j$ have the same divisorial part for $j = 0, \dots, d$, we compute,

using Proposition 2.35 and Lemma 4.3 that

$$\begin{aligned} & \overline{\mathcal{D}}_0 \cdot \dots \cdot \overline{\mathcal{D}}_d - \overline{\mathcal{D}}'_0 \cdot \dots \cdot \overline{\mathcal{D}}'_d \\ &= \sum_{k=0}^d \left(\overline{\mathcal{D}}_0 \cdot \dots \cdot \overline{\mathcal{D}}_{k-1} \cdot \overline{\mathcal{D}}_k \cdot \overline{\mathcal{D}}'_{k+1} \cdot \dots \cdot \overline{\mathcal{D}}'_d - \overline{\mathcal{D}}_0 \cdot \dots \cdot \overline{\mathcal{D}}_{k-1} \cdot \overline{\mathcal{D}}'_k \cdot \overline{\mathcal{D}}'_{k+1} \cdot \dots \cdot \overline{\mathcal{D}}'_d \right) \\ &= \sum_{k=0}^d \int_X (g_k - g'_k) \langle \omega_{E_0}(g_0) \wedge \dots \wedge \omega_{E_{k-1}}(g_{k-1}) \wedge \omega_{E_{k+1}}(g'_{k+1}) \wedge \dots \wedge \omega_{E_d}(g'_d) \rangle. \end{aligned}$$

Since we now have for $j = 0, \dots, d$ that

$$\begin{aligned} \omega_{E_j}(g_j) &= \text{dd}^c g_j + \delta_{E_j} = \text{dd}^c(g_j - g_{E_j}) + \delta_{E_j} + \text{dd}^c g_{E_j} \\ &= \text{dd}^c \varphi_j + \delta_{E_j} + \omega_j - \delta_{E_j} = \text{dd}^c \varphi_j + \omega_j, \end{aligned}$$

and similarly $\omega_{E_j}(g'_j) = \text{dd}^c \varphi'_j + \omega_j$, we find, using $g_k - g'_k = \varphi_k - \varphi'_k$, that

$$\overline{\mathcal{D}}_0 \cdot \dots \cdot \overline{\mathcal{D}}_d - \overline{\mathcal{D}}'_0 \cdot \dots \cdot \overline{\mathcal{D}}'_d = I_{\varphi'}(\varphi),$$

after recalling formula (2.4).

Next, we consider the general case when $|g_j - g'_j|$ is not necessarily bounded. For each $n \in \mathbb{N}$, we write

$$\varphi_{j,n} = \max(\varphi'_j - n, \varphi_j) \quad \text{and} \quad g_{j,n} = g_{E_j} + \varphi_{j,n}.$$

Since the functions φ_j are more singular than the functions φ'_j , we deduce that $g_j - n \leq g_{j,n} \leq g'_j$. Thus, the functions $|g_{j,n} - g'_j|$ are bounded. Writing $\overline{\mathcal{D}}_{j,n} = (\mathcal{D}_j, g_{j,n})$, we deduce from the preceding case that

$$\overline{\mathcal{D}}_{0,n} \cdot \dots \cdot \overline{\mathcal{D}}_{d,n} - \overline{\mathcal{D}}'_0 \cdot \dots \cdot \overline{\mathcal{D}}'_d = I_{\varphi'}(\varphi_n),$$

where $\varphi_n = (\varphi_{0,n}, \dots, \varphi_{d,n})$. By the continuity of the arithmetic intersection pairing and Proposition 3.42, we have that

$$\lim_{n \rightarrow \infty} \overline{\mathcal{D}}_{0,n} \cdot \dots \cdot \overline{\mathcal{D}}_{d,n} = \overline{\mathcal{D}}_0 \cdot \dots \cdot \overline{\mathcal{D}}_d.$$

Moreover, since the sequence $(\varphi_{j,n})_{n \in \mathbb{N}}$ converges monotonically to φ_j for $j = 0, \dots, d$, we know from Proposition 2.50 that

$$\lim_{n \rightarrow \infty} I_{\varphi'}(\varphi_n) = I_{\varphi'}(\varphi).$$

This completes the proof of the theorem. \square

Before being able to define generalized arithmetic intersection products, we need the following definition.

4.5. Definition. Let $\overline{\mathcal{D}} = (\mathcal{D}, g') \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}$ be a nef adelic arithmetic divisor on \mathcal{U} . A function g is said to be a *Green function for \mathcal{D} of plurisubharmonic type and relative full mass with respect to g'* , if the following three conditions are satisfied:

- (i) The function $g - g'$ is bounded above, that is, g is more singular than g' .
- (ii) The function $g|_U$ is plurisubharmonic.
- (iii) If $\overline{\mathcal{E}} = (\mathcal{E}, g_E)$ is a model arithmetic divisor with $\mathcal{E} \geq \mathcal{D}$ on \mathcal{X} and g_E a Green function of smooth type for E with $\omega_0 = \omega_E(g_E)$, the function $\varphi' = g' - g_E$ is ω_0 -plurisubharmonic by the nefness of $\overline{\mathcal{D}}$ and conditions (i) and (ii) imply that the function $\varphi = g - g_E$ is also ω_0 -plurisubharmonic, and we further require the relative full mass condition

$$\int_X \langle (\text{dd}^c \varphi + \omega_0)^{\wedge d} \rangle = \int_X \langle (\text{dd}^c \varphi' + \omega_0)^{\wedge d} \rangle;$$

in other words, $\varphi \in \mathcal{E}(X, \omega_0, \varphi')$.

Theorem 4.4 together with the above definition allows us to make the following definition for generalized arithmetic intersection numbers.

4.6. Definition. For $j = 0, \dots, d$, let $\overline{\mathcal{D}}'_j = (\mathcal{D}_j, g'_j) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}$ be nef adelic arithmetic divisors on \mathcal{U} and let $\overline{\mathcal{D}}_j = (\mathcal{D}_j, g_j) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})$ be adelic arithmetic divisors on \mathcal{U} so that g_j are Green functions for D_j of plurisubharmonic type and relative full mass with respect to g'_j . Choose sufficiently large arithmetic reference divisors $\overline{\mathcal{E}}_j = (\mathcal{E}_j, g_{E_j})$ on \mathcal{X} such that $\overline{\mathcal{E}}_j \geq \overline{\mathcal{B}}$ and $\overline{\mathcal{E}}_j \geq \overline{\mathcal{D}}_j + 2\overline{\mathcal{B}}$ with g_{E_j} Green functions of smooth type for E_j , so that the functions $\varphi_j := g_j - g_{E_j}$ and $\varphi'_j := g'_j - g_{E_j}$ become ω_j -plurisubharmonic, where $\omega_j = \omega_{E_j}(g_{E_j})$. Then, for $\boldsymbol{\varphi} = (\varphi_0, \dots, \varphi_d) \in \mathcal{E}(X, \boldsymbol{\omega}, \boldsymbol{\varphi}')$, the *generalized arithmetic intersection product* $\overline{\mathcal{D}}_0 \cdots \overline{\mathcal{D}}_d$ is defined as

$$(4.4) \quad \overline{\mathcal{D}}_0 \cdots \overline{\mathcal{D}}_d := \overline{\mathcal{D}}'_0 \cdots \overline{\mathcal{D}}'_d + I_{\boldsymbol{\varphi}'}(\boldsymbol{\varphi}).$$

We will say that $(\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_d)$ has *finite energy with respect to* $(\overline{\mathcal{D}}'_0, \dots, \overline{\mathcal{D}}'_d)$, if $I_{\boldsymbol{\varphi}'}(\boldsymbol{\varphi}) > -\infty$. In this case, the arithmetic intersection product (4.4) is a real number.

Similarly, if $\overline{\mathcal{D}}' = (\mathcal{D}, g') \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})^{\text{nef}}$ is a nef adelic arithmetic divisor on \mathcal{U} and $\overline{\mathcal{D}} = (\mathcal{D}, g) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{U})$ is an adelic arithmetic divisor on \mathcal{U} so that g is a Green function for D of plurisubharmonic type and relative full mass with respect to g' , we say that $\overline{\mathcal{D}}$ has *finite energy with respect to* $\overline{\mathcal{D}}'$, if $(\overline{\mathcal{D}}, \dots, \overline{\mathcal{D}})$ has finite energy with respect to $(\overline{\mathcal{D}}', \dots, \overline{\mathcal{D}}')$.

4.7. Lemma. *The generalized arithmetic intersection product of Definition 4.6 does not depend on the choice of the arithmetic reference divisors $\overline{\mathcal{E}}_j$ ($j = 0, \dots, d$).*

Proof. This follows easily from the definitions. \square

4.3. Functoriality. All the functorial properties of Subsection 3.5 carry over to the finite energy case. In particular, we have the following two propositions.

4.8. Proposition. *Let $f: \mathcal{U}' \rightarrow \mathcal{U}$ be a dominant morphism as in Subsection 3.5. Furthermore, assume that f is projective and flat of relative dimension e , and let d be the relative dimension of \mathcal{U} over $\text{Spec}(\mathcal{O}_K)$. Let $\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_d$ be adelic arithmetic divisors on \mathcal{U} and $\overline{\mathcal{E}}_1, \dots, \overline{\mathcal{E}}_e$ be adelic arithmetic divisors on \mathcal{U}' having finite energy with respect to some nef adelic arithmetic divisors. Then, we have the formula*

$$\overline{\mathcal{E}}_1 \cdots \overline{\mathcal{E}}_e \cdot f_{\text{adel}}^* \overline{\mathcal{D}}_0 \cdots f_{\text{adel}}^* \overline{\mathcal{D}}_d = \deg(\mathcal{E}_1|_F, \dots, \mathcal{E}_e|_F) \cdot \overline{\mathcal{D}}_0 \cdots \overline{\mathcal{D}}_d,$$

where F is a generic fiber of f and $\deg(\mathcal{E}_1|_F, \dots, \mathcal{E}_e|_F)$ is the degree of the generic fiber with respect to the underlying geometric divisors.

Similarly, we have

4.9. Proposition. *Let $f: \mathcal{U}' \rightarrow \mathcal{U}$ be a finite flat dominant morphism of degree n and let d be the relative dimension of \mathcal{U} over $\text{Spec}(\mathcal{O}_K)$. Let $\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_d$ be adelic arithmetic divisors on \mathcal{U} having finite energy with respect to some nef adelic arithmetic divisors. Then, we have the formula*

$$f_{\text{adel}}^* \overline{\mathcal{D}}_0 \cdots f_{\text{adel}}^* \overline{\mathcal{D}}_d = n \overline{\mathcal{D}}_0 \cdots \overline{\mathcal{D}}_d.$$

5. THE SELF-INTERSECTION OF THE LINE BUNDLE OF SIEGEL–JACOBI FORMS

5.1. The line bundle of Siegel–Jacobi forms. The basic reference for this subsection is the book [FC90] by G. Faltings and C-L. Chai. We let \mathcal{A}_g denote the moduli stack of principally polarized abelian schemes of dimension g over $\text{Spec}(\mathbb{Z})$ and we let $\pi: \mathcal{B}_g \rightarrow \mathcal{A}_g$ be the universal abelian scheme over $\text{Spec}(\mathbb{Z})$ with zero section $\varepsilon: \mathcal{A}_g \rightarrow \mathcal{B}_g$. Recall that $d' = g(g+1)/2$ is the relative dimension of \mathcal{A}_g over $\text{Spec}(\mathbb{Z})$ and that $d = d' + g$ is the relative dimension of \mathcal{B}_g over $\text{Spec}(\mathbb{Z})$.

The *Hodge bundle* ω on \mathcal{A}_g is defined as the line bundle

$$\omega := \varepsilon^* \det(\Omega_{\mathcal{B}_g/\mathcal{A}_g}^1).$$

For $g > 1$, the global sections of the line bundle $\omega^{\otimes k}$ are arithmetic Siegel modular forms of weight k , i. e., Siegel modular forms of weight k with integral Fourier coefficients. By identifying \mathcal{B}_g

with the dual abelian scheme \mathcal{B}_g^\vee via the principal polarization, the Poincaré bundle $\mathcal{P}_{\mathcal{B}_g}$ is the tautological invertible sheaf on the product $\mathcal{B}_g \times_{\mathcal{A}_g} \mathcal{B}_g$. By means of the diagonal morphism

$$\Delta: \mathcal{B}_g \longrightarrow \mathcal{B}_g \times_{\mathcal{A}_g} \mathcal{B}_g,$$

we obtain via pull-back the distinguished line bundle $\mathcal{L} := \Delta^* \mathcal{P}_{\mathcal{B}_g}$, which is π -ample and symmetric. The restriction of \mathcal{L} to any fiber of \mathcal{B}_g is the line bundle associated with twice the principal polarization of the fiber. The *line bundle of Siegel–Jacobi forms of weight k and index m on \mathcal{B}_g* is now defined as

$$\mathcal{I}_{k,m} := \pi^* \omega^{\otimes k} \otimes \mathcal{L}^{\otimes m}.$$

For $g > 1$, the global sections of the line bundle $\mathcal{I}_{k,m}$ are arithmetic Siegel–Jacobi forms of weight k and index m , i. e., Siegel–Jacobi forms of weight k and index m with integral Fourier coefficients. The arithmetic theory of Siegel–Jacobi forms has been investigated in detail in [Kra95].

Next, we fix an integer $N \geq 3$ and let $\mathcal{A}_{g,N}$ denote the fine moduli space of principally polarized abelian schemes of dimension g with full level N -structure over $\mathrm{Spec}(\mathbb{Z}[1/N, \zeta_N])$, where ζ_N is a primitive N -th root of unity, and we let $\pi_N: \mathcal{B}_{g,N} \rightarrow \mathcal{A}_{g,N}$ be the universal abelian scheme over $\mathrm{Spec}(\mathbb{Z}[1/N, \zeta_N])$ with zero section $\varepsilon_N: \mathcal{A}_{g,N} \rightarrow \mathcal{B}_{g,N}$. We recall that $\mathcal{A}_{g,N}$ and $\mathcal{B}_{g,N}$ are smooth, quasi-projective schemes over $\mathrm{Spec}(\mathbb{Z}[1/N, \zeta_N])$. We denote by $\rho'_N: \mathcal{A}_{g,N} \rightarrow \mathcal{A}_g$ and by $\rho_N: \mathcal{B}_{g,N} \rightarrow \mathcal{B}_g$ the respective morphisms of stacks forgetting the level N -structure.

As before, the *Hodge bundle ω_N of level N on $\mathcal{A}_{g,N}$* is defined as the line bundle

$$\omega_N := \varepsilon_N^* \det(\Omega_{\mathcal{B}_{g,N}/\mathcal{A}_{g,N}}^1).$$

For $g > 1$, the global sections of the line bundle $\omega_N^{\otimes k}$ are arithmetic Siegel modular forms of weight k and level N , i. e., Siegel modular forms of weight k and level N with Fourier coefficients lying in $\mathbb{Z}[1/N, \zeta_N]$. Recalling from [FC90, Chapter IV, Theorem 6.7] that $\mathcal{B}_{g,N}$ is the pull-back of \mathcal{B}_g to $\mathcal{A}_{g,N}$, we observe that $\omega_N \cong \rho_N^* \omega$. The pull-back $\mathcal{L}_N := \rho_N^* \mathcal{L}$ becomes a line bundle on the scheme $\mathcal{B}_{g,N}$, which is again π_N -ample and symmetric. For the sequel, we note that the symmetric line bundle \mathcal{L}_N satisfies the theorem of the cube with respect to the morphism $[2]: \mathcal{B}_{g,N} \rightarrow \mathcal{B}_{g,N}$ given by fiberwise multiplication by 2, i. e., there is an isomorphism

$$(5.1) \quad [2]^* \mathcal{L}_N \cong \mathcal{L}_N^{\otimes 4}$$

of line bundles over $\mathcal{B}_{g,N}$. The *line bundle of Siegel–Jacobi forms of weight k , index m , and level N on $\mathcal{B}_{g,N}$* is now defined as

$$\mathcal{I}_{k,m,N} := \pi_N^* \omega_N^{\otimes k} \otimes \mathcal{L}_N^{\otimes m}.$$

For $g > 1$, the global sections of the line bundle $\mathcal{I}_{k,m,N}$ are arithmetic Siegel–Jacobi forms of weight k , index m , and level N , i. e., Siegel–Jacobi forms of weight k , index m , and level N with Fourier coefficients lying in $\mathbb{Z}[1/N, \zeta_N]$.

5.2. Siegel–Jacobi forms over the complex numbers. In this subsection, we recall the classical definition of Siegel–Jacobi forms over the complex numbers. We start by recalling the construction of $\mathcal{A}_{g,N}(\mathbb{C})$ and $\mathcal{B}_{g,N}(\mathbb{C})$ as quotient spaces. Let $g \geq 1$ be an integer. The symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ is the group of real $(2g \times 2g)$ -matrices of the form

$$(5.2) \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

such that

$$A^t C = C^t A, \quad D^t B = B^t D, \quad A^t D - C^t B = \mathbb{1}_g,$$

where $\mathbb{1}_g$ is the $(g \times g)$ -identity matrix. There is an inclusion

$$\mathrm{Sp}(2g, \mathbb{R}) \hookrightarrow \mathrm{Sp}(2g+2, \mathbb{R}),$$

sending a matrix of the form (5.2) to the matrix

$$\begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For any commutative ring R , let $H_R^{(g,1)}$ be the Heisenberg group

$$H_R^{(g,1)} = \{[(\lambda, \mu), x] \mid \lambda, \mu \in R^{(1,g)}, x \in R\},$$

where $R^{(1,g)}$ denotes the set of row vectors of size g with coefficients in R , with composition law given by

$$[(\lambda, \mu), x] \circ [(\lambda', \mu'), x'] = [(\lambda + \lambda', \mu + \mu'), x + x' + \lambda\mu'^t - \mu\lambda'^t].$$

The real Heisenberg group $H_{\mathbb{R}}^{(g,1)}$ can be realized as the subgroup of $\mathrm{Sp}(2g+2, \mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} \mathbb{1}_g & 0 & 0 & \mu^t \\ \lambda & 1 & \mu & x \\ 0 & 0 & \mathbb{1}_g & -\lambda^t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The full Jacobi group $G_{\mathbb{R}}^{(g,1)} = \mathrm{Sp}(2g, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,1)}$ is the subgroup of $\mathrm{Sp}(2g+2, \mathbb{R})$ generated by $\mathrm{Sp}(2g, \mathbb{R})$ and $H_{\mathbb{R}}^{(g,1)}$.

Let

$$\mathbb{H}_g = \{Z = X + iY \mid X, Y \in \mathrm{Mat}_{g \times g}(\mathbb{R}), Z^t = Z, Y > 0\}$$

be the Siegel upper half-space. The group $\mathrm{Sp}(2g, \mathbb{R})$ acts transitively on \mathbb{H}_g , where for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{R})$ and $Z \in \mathbb{H}_g$ this action is given by

$$Z \mapsto M\langle Z \rangle = (AZ + B)(CZ + D)^{-1}.$$

On the other hand, the group $G_{\mathbb{R}}^{(g,1)}$ acts transitively on $\mathbb{H}_g \times \mathbb{C}^{(1,g)}$, where for $(M, [(\lambda, \mu), x]) \in G_{\mathbb{R}}^{(g,1)}$ and $(Z, W) \in \mathbb{H}_g \times \mathbb{C}^{(1,g)}$ this action is given by

$$(Z, W) \mapsto (M\langle Z \rangle, (W + \lambda Z + \mu)(CZ + D)^{-1}).$$

For a subgroup $\Gamma \subseteq \mathrm{Sp}(2g, \mathbb{Z})$ of finite index, we write $\tilde{\Gamma} = \Gamma \ltimes H_{\mathbb{Z}}^{(g,1)} \subseteq G_{\mathbb{Z}}^{(g,1)}$, and define the quotient spaces $\mathcal{A}_{g,\Gamma} := \Gamma \backslash \mathbb{H}_g$ and $\mathcal{B}_{g,\Gamma} := \tilde{\Gamma} \backslash \mathbb{H}_g \times \mathbb{C}^{(1,g)}$. In particular, these definitions apply to the principal congruence subgroup of level N of the symplectic group defined by

$$\Gamma(N) = \ker(\mathrm{Sp}(2g, \mathbb{Z}) \rightarrow \mathrm{Sp}(2g, \mathbb{Z}/N\mathbb{Z})),$$

for which we then find that

$$\mathcal{A}_{g,\Gamma(N)} = \mathcal{A}_{g,N}(\mathbb{C}) \quad \text{and} \quad \mathcal{B}_{g,\Gamma(N)} = \mathcal{B}_{g,N}(\mathbb{C}).$$

We next recall the classical definition of Siegel–Jacobi forms by introducing a suitable automorphy factor for the group $G_{\mathbb{R}}^{(g,1)}$ (see, e.g., [Zie89]). For $k, m \in \mathbb{Z}$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{R})$, $\zeta = [(\lambda, \mu), x] \in H_{\mathbb{R}}^{(g,1)}$, we define for a complex-valued function $f: \mathbb{H}_g \times \mathbb{C}^{(1,g)} \rightarrow \mathbb{C}$ the slash operators

$$(f|_{k,m}M)(Z, W) := \det(CZ + D)^{-k} e^{-2\pi i m W(CZ + D)^{-1} C W^t} f(M\langle Z \rangle, W(CZ + D)^{-1})$$

and

$$(f|_{k,m}\zeta)(Z, W) := e^{2\pi i m(\lambda Z \lambda^t + 2\lambda W^t + (x + \mu \lambda^t))} f(Z, W + \lambda Z + \mu).$$

A quadratic matrix T is called *half-integral*, if $2T$ has integral entries, while the diagonal entries of T are even. Note that if T is symmetric, then T is half-integral if and only if the associated quadratic form is integral.

5.1. Definition. A meromorphic function $f: \mathbb{H}_g \times \mathbb{C}^{(1,g)} \rightarrow \mathbb{C}$ is called a *meromorphic Siegel–Jacobi form of weight k and index m for the subgroup $\Gamma \subseteq \mathrm{Sp}(2g, \mathbb{Z})$ of finite index*, if the following conditions are satisfied:

- (i) $f|_{k,m}M = f$ for all $M \in \Gamma$.
- (ii) $f|_{k,m}\zeta = f$ for all $\zeta \in H_{\mathbb{Z}}^{(g,1)}$.

A meromorphic Siegel–Jacobi form f of weight k and index m for Γ is called a *Siegel–Jacobi form of weight k and index m for Γ* , if f is holomorphic and

(iii) for each $M \in \mathrm{Sp}(2g, \mathbb{Z})$, the function $f|_{k,m}M$ has a Fourier expansion of the form

$$(f|_{k,m}M)(Z, W) = \sum_{\substack{T=T^t \geq 0 \\ T \text{ half-integral}}} \sum_{R \in \mathbb{Z}^{(g,1)}} c(T, R) e^{2\pi i/n_M \mathrm{tr}(TZ)} e^{2\pi iWR}$$

for some suitable positive integer n_M depending only on Γ and such that $c(T, R) \neq 0$ implies

$$\begin{pmatrix} T/n_M & R/2 \\ R^t/2 & m \end{pmatrix} \geq 0.$$

The \mathbb{C} -vector space of Siegel–Jacobi forms of weight k and index m for Γ is denoted by $J_{k,m}(\Gamma)$.

A Siegel–Jacobi form $f \in J_{k,m}(\Gamma)$ is said to be a *Siegel–Jacobi cusp form of weight k and index m for Γ* , if $c(T, R) \neq 0$ implies

$$\begin{pmatrix} T/n_M & R/2 \\ R^t/2 & m \end{pmatrix} > 0.$$

We note that when $g > 1$, if f is holomorphic, condition (iii) is a consequence of conditions (i) and (ii) due to the Koecher principle (see [Zie89, Lemma 1.6]).

5.2. Remark. Fix $N \geq 3$. The meromorphic Siegel–Jacobi forms of weight k and index m for $\Gamma(N)$ are the meromorphic sections of the line bundle $\mathcal{J}_{k,m,N} \otimes_{\mathbb{Z}[1/N, \zeta_N]} \mathbb{C}$ over $\mathcal{B}_{g,N}(\mathbb{C})$. For $g > 1$, the Siegel–Jacobi forms of weight k and index m for $\Gamma(N)$ correspond to the global sections of the line bundle $\mathcal{J}_{k,m,N} \otimes_{\mathbb{Z}[1/N, \zeta_N]} \mathbb{C}$ over $\mathcal{B}_{g,N}(\mathbb{C})$, i. e., we have

$$J_{k,m}(\Gamma(N)) = \Gamma(\mathcal{B}_{g,N}(\mathbb{C}), \mathcal{J}_{k,m,N} \otimes_{\mathbb{Z}[1/N, \zeta_N]} \mathbb{C}).$$

For $g = 1$, Jacobi forms of weight k and index m for $\Gamma(N)$ correspond to the global sections of a certain extension of $\mathcal{J}_{k,m,N} \otimes_{\mathbb{Z}[1/N, \zeta_N]} \mathbb{C}$ to a suitable compactification of $\mathcal{B}_{g,N}(\mathbb{C})$.

5.3. Toroidal compactifications. We give a brief account of the theory of toroidal compactifications of the smooth, quasi-projective schemes $\mathcal{A}_{g,N}$ and $\mathcal{B}_{g,N}$ over $\mathrm{Spec}(\mathbb{Z}[1/N, \zeta_N])$ for $N \geq 3$. For more details we refer again to the book [FC90] by G. Faltings and C.-L. Chai.

Let \mathcal{C}_g be the open cone of real symmetric positive definite $(g \times g)$ -matrices and $\overline{\mathcal{C}}_g$ the cone of real symmetric positive semidefinite $(g \times g)$ -matrices with rational kernel. Let $\tilde{\mathcal{C}}_g \subseteq \overline{\mathcal{C}}_g \times \mathbb{R}^{(1,g)}$ be the cone defined by

$$\tilde{\mathcal{C}}_g = \{(Y, \beta) \in \overline{\mathcal{C}}_g \times \mathbb{R}^{(1,g)} \mid \exists \alpha \in \mathbb{R}^{(1,g)} : \beta = \alpha Y\}.$$

We will denote by $\overline{\mathcal{C}}_{g,\mathbb{Z}}$ the subset of half-integral matrices of $\overline{\mathcal{C}}_g$. Furthermore, we set $\tilde{\mathcal{C}}_{g,\mathbb{Z}} = \tilde{\mathcal{C}}_g \cap (\overline{\mathcal{C}}_{g,\mathbb{Z}} \times \mathbb{Z}^{(1,g)})$.

Let $P_g \subset \mathrm{Sp}(2g, \mathbb{Z})$ denote the parabolic subgroup consisting of the symplectic matrices $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$. Then, there is a group homomorphism

$$P_g \longrightarrow \mathrm{GL}(g, \mathbb{Z}),$$

given by the assignment $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mapsto A$. We denote the image of the intersection $\Gamma(N) \cap P_g$ by this homomorphism in $\mathrm{GL}(g, \mathbb{Z})$ by $\overline{\Gamma}(N)$. Similarly, we denote the image of the intersection $\tilde{\Gamma}(N) \cap P_{g+1}$ in $\mathrm{GL}(g+1, \mathbb{Z})$ by $\tilde{\overline{\Gamma}}(N)$. Then, the group $\tilde{\overline{\Gamma}}(N)$ is given as the semi-direct product $\overline{\Gamma}(N) \ltimes \mathbb{Z}^{(1,g)}$ and is therefore contained in the subgroup of matrices of the form $\begin{pmatrix} A & 0 \\ \lambda & 1 \end{pmatrix}$, where $A \in \mathrm{GL}(g, \mathbb{Z})$ and $\lambda \in \mathbb{Z}^{(1,g)}$; in the sequel, we denote the elements of $\tilde{\overline{\Gamma}}(N)$ by (A, λ) . The group $\tilde{\overline{\Gamma}}(N)$ acts on $\overline{\mathcal{C}}_g$ by the rule

$$A \cdot Y = AY A^t,$$

and the group $\tilde{\overline{\Gamma}}(N)$ acts on $\tilde{\mathcal{C}}_g$ by the rule

$$(5.3) \quad (A, \lambda) \cdot (Y, \beta) = (AY A^t, (\beta + \lambda Y) A^t).$$

5.3. Definition. An *admissible cone decomposition of $\overline{\mathcal{C}}_g$* is a set of cones Σ in $\overline{\mathcal{C}}_g$ such that the following conditions are satisfied:

- (i) Each $\sigma \in \Sigma$ is generated by a finite set of elements of $\overline{C}_{g,\mathbb{Z}}$ and contains no lines. In other words, it is a rational polyhedral strictly convex cone.
- (ii) If σ belongs to Σ , each face of σ belongs to Σ .
- (iii) If σ and σ' belong to Σ , their intersection is a common face.
- (iv) The union of all the cones of Σ is \overline{C}_g .
- (v) The group $\overline{\Gamma}(N)$ leaves Σ invariant with finitely many orbits.

5.4. Definition. Let Σ be an admissible cone decomposition of \overline{C}_g . An *admissible cone decomposition* of \tilde{C}_g over Σ is a set of cones Π in \tilde{C}_g such that the following conditions are satisfied:

- (i) Each $\tau \in \Pi$ is generated by a finite set of elements of $\tilde{C}_{g,\mathbb{Z}}$ and contains no lines.
- (ii) If τ belongs to Π , each face of τ belongs to Π .
- (iii) If τ and τ' belong to Π , their intersection is a common face.
- (iv) The union of all the cones of Π is \tilde{C}_g .
- (v) The group $\tilde{\Gamma}(N)$ leaves Π invariant with finitely many orbits.
- (vi) For each $\tau \in \Pi$, the projection of τ to \overline{C}_g is contained in a cone $\sigma \in \Sigma$.

We say that Σ and Π are *smooth*, if every cone of Σ and every cone of Π is generated by part of a \mathbb{Z} -basis of the abelian groups generated by $\overline{C}_{g,\mathbb{Z}}$ and $\tilde{C}_{g,\mathbb{Z}}$, respectively. We say that Π is *equidimensional (over Σ)*, if for every cone τ , the projection of τ to \overline{C}_g is a cone $\sigma \in \Sigma$.

5.5. Definition. Let Σ be an admissible cone decomposition of \overline{C}_g . An *admissible divisorial function* on Σ is a continuous and $\overline{\Gamma}(N)$ -invariant function $\phi: \overline{C}_g \rightarrow \mathbb{R}$ satisfying the following properties:

- (i) It is conical, in the sense that $\phi(tY) = t\phi(Y)$ for all $Y \in \overline{C}_g$ and $t \in \mathbb{R}_{\geq 0}$.
- (ii) It is linear on each cone $\sigma \in \Sigma$.
- (iii) It takes integral values on $\overline{C}_{g,\mathbb{Z}}$.

An admissible divisorial function ϕ on Σ is called *strictly anti-effective*, if, in addition, it satisfies

- (iv) $\phi(Y) > 0$ for all $Y \in \overline{C}_g \setminus \{0\}$.

A strictly anti-effective admissible divisorial function ϕ on Σ is called an *admissible polarization function* on Σ , if the following two additional properties are satisfied:

- (v) The function ϕ is concave.
- (vi) The function ϕ is strictly concave on Σ , in the sense that, if σ' is a cone in \overline{C}_g such that the restriction of ϕ is linear on σ' , then σ' is contained in a cone $\sigma \in \Sigma$; in other words, the maximal cones of Σ are the maximal cones of linearity of ϕ .

5.6. Remark. For a given admissible cone decomposition Σ of \overline{C}_g it may happen that there are no admissible polarization functions on Σ . An admissible cone decomposition Σ of \overline{C}_g that admits an admissible polarization function is called *projective*. As explained in [FC90, Chapter V, § 5], every admissible cone decomposition Σ of \overline{C}_g admits a smooth projective refinement.

5.7. Definition. Let Σ be an admissible cone decomposition of \overline{C}_g and Π an admissible cone decomposition of \tilde{C}_g over Σ . An *admissible divisorial function* on Π is a continuous and $\tilde{\Gamma}(N)$ -invariant function $\phi: \tilde{C}_g \rightarrow \mathbb{R}$ satisfying the following properties:

- (i) It is conical, in the sense that $\phi(t\tilde{Y}) = t\phi(\tilde{Y})$ for all $\tilde{Y} = (Y, \beta) \in \tilde{C}_g$ and $t \in \mathbb{R}_{\geq 0}$.
- (ii) It is linear on each cone $\tau \in \Pi$.
- (iii) It takes rational values on $\tilde{C}_{g,\mathbb{Z}}$ with bounded denominators.
- (iv) For $\lambda \in \mathbb{Z}^{(1,g)}$ and $\tilde{Y} = (Y, \beta) \in \tilde{C}_g$, the condition

$$\phi(\tilde{Y}) - \phi(\lambda \cdot \tilde{Y}) = \lambda Y \lambda^t + 2\beta \lambda^t$$

holds, where we recall from (5.3) that $\lambda \cdot \tilde{Y} := (\mathbb{1}_g, \lambda) \cdot (Y, \beta) = (Y, \beta + \lambda Y)$.

An admissible divisorial function ϕ on Π is called an *admissible polarization function* on Π , if the following two additional properties are satisfied:

- (v) The function ϕ is concave.

- (vi) The function ϕ is strictly concave over each cone $\sigma \in \Sigma$, that is, for each maximal cone $\tau \in \Pi$ over $\sigma \in \Sigma$, there is a linear function φ_τ such that $\varphi_\tau(\tilde{Y}) = \phi(\tilde{Y})$ for each $\tilde{Y} = (Y, \beta) \in \tau$, but $\varphi_\tau(\tilde{Y}) > \phi(\tilde{Y})$ for each $\tilde{Y} = (Y, \beta) \notin \tau$, but with $Y \in \sigma$.

5.8. Remark. For a given admissible cone decomposition Π of \tilde{C}_g , even smooth, it may be possible that there are no admissible polarization functions on Π . Nevertheless, there is always a refinement Π' of Π such that there exists an admissible polarization function on Π' . As in the case of the admissible cone decompositions Σ of \overline{C}_g , an admissible cone decomposition Π of \tilde{C}_g that admits an admissible polarization function is called *projective*.

The theory of toroidal embeddings allows us to compactify the smooth, quasi-projective schemes $\mathcal{A}_{g,N}$ and $\mathcal{B}_{g,N}$ over $\text{Spec}(\mathbb{Z}[1/N, \zeta_N])$ for $N \geq 3$. The precise statements are as follows.

5.9. Theorem. *Let Σ be a projective admissible cone decomposition of \overline{C}_g and Π a projective admissible cone decomposition of \tilde{C}_g over Σ .*

- (i) *Given the cone decomposition Σ , there is a projective scheme $\overline{\mathcal{A}}_{g,N,\Sigma}$ over $\text{Spec}(\mathbb{Z}[1/N, \zeta_N])$ that contains $\mathcal{A}_{g,N}$ as an open dense subscheme.*
(ii) *Given the cone decomposition Π , there is a projective scheme $\overline{\mathcal{B}}_{g,N,\Pi}$ over $\text{Spec}(\mathbb{Z}[1/N, \zeta_N])$ that contains $\mathcal{B}_{g,N}$ as an open dense subscheme and a projective morphism*

$$\pi_{\Sigma,\Pi} : \overline{\mathcal{B}}_{g,N,\Pi} \longrightarrow \overline{\mathcal{A}}_{g,N,\Sigma}$$

that extends the canonical projection $\pi_N : \mathcal{B}_{g,N} \rightarrow \mathcal{A}_{g,N}$.

- (iii) *If Σ or Π are smooth, then the corresponding schemes $\overline{\mathcal{A}}_{g,N,\Sigma}$ or $\overline{\mathcal{B}}_{g,N,\Pi}$ are smooth over $\text{Spec}(\mathbb{Z}[1/N, \zeta_N])$, respectively. If Π is equidimensional over Σ , then $\pi_{\Sigma,\Pi}$ is equidimensional.*
(iv) *The projective scheme $\overline{\mathcal{A}}_{g,N,\Sigma}$ admits a stratification by locally closed subschemes indexed by the $\overline{\Gamma}(N)$ -orbits of Σ , i. e., we have*

$$\overline{\mathcal{A}}_{g,N,\Sigma} = \bigcup_{\overline{\sigma} \in \Sigma/\overline{\Gamma}(N)} \overline{\mathcal{A}}_{g,N,\overline{\sigma}}.$$

The correspondence between cones $\overline{\sigma} \in \Sigma/\overline{\Gamma}(N)$ and strata $\overline{\mathcal{A}}_{g,N,\overline{\sigma}}$ reverses dimensions; a stratum $\overline{\mathcal{A}}_{g,N,\overline{\sigma}}$ lies in the closure of another stratum $\overline{\mathcal{A}}_{g,N,\overline{\sigma}'}$ if and only if there are representatives σ and σ' of $\overline{\sigma}$ and $\overline{\sigma}'$, respectively, such that σ' is a face of σ .

- (v) *There is an analogous stratification of $\overline{\mathcal{B}}_{g,N,\Pi}$ indexed by the $\overline{\Gamma}(N)$ -orbits of Π .*

5.10. Remark. As in Theorem 5.9, we assume that Σ and Π are smooth projective admissible cone decompositions of \overline{C}_g and \tilde{C}_g , respectively, so that we have a projective morphism of smooth projective schemes $\pi_{\Sigma,\Pi} : \overline{\mathcal{B}}_{g,N,\Pi} \rightarrow \overline{\mathcal{A}}_{g,N,\Sigma}$ over $\text{Spec}(\mathbb{Z}[1/N, \zeta_N])$. A $\overline{\Gamma}(N)$ -orbit $\overline{\tau}$ now determines a stratum $\overline{\mathcal{B}}_{g,N,\overline{\tau}}$ of $\overline{\mathcal{B}}_{g,N,\Pi}$. In the sequel, we will need local coordinates around a point $p_{\overline{\tau}} \in \overline{\mathcal{B}}_{g,N,\overline{\tau}}(\mathbb{C})$, which we recall from [FC90, Chapter V, p. 141]; we note that these local coordinates will depend on the choice of a representative τ of the orbit under consideration. Assume that $\dim(\tau) = n$ and recall that $\dim(\mathcal{B}_{g,N}(\mathbb{C})) = d$. The chosen cone τ is then generated by the set $\{(Y_1, \beta_1), \dots, (Y_n, \beta_n)\}$, where Y_j are half-integral symmetric positive semidefinite matrices and β_j are integral row vectors of the form $\beta_j = \alpha_j Y_j$, which is part of an integral basis

$$\{(Y_1, \beta_1), \dots, (Y_n, \beta_n), (Y_{n+1}, \beta_{n+1}), \dots, (Y_d, \beta_d)\}$$

of $\tilde{C}_{g,\mathbb{Z}}$. Then, there exists a pair (Y_0, β_0) with Y_0 a real symmetric positive definite $(g \times g)$ -matrix and a row vector $\beta_0 \in \mathbb{R}^{(1,g)}$ satisfying $\beta_0 = \alpha_0 Y_0$, and there are real numbers $0 < r_1, \dots, r_d < 1/e$ such that the set of pairs

$$V' := \left\{ (Z, W) = i(Y_0, \beta_0) + \sum_{j=1}^d z_j (Y_j, \beta_j) \mid z_j \in \mathbb{C} : \begin{array}{l} 2\pi \text{Im}(z_j) > -\log(r_j), \quad j = 1, \dots, n \\ |z_j| < r_j, \quad j = n+1, \dots, d \end{array} \right\}$$

constitutes an open subset of $\mathbb{H}_g \times \mathbb{C}^{(1,g)}$. We note that the set of pairs $\{(Y_1, \beta_1), \dots, (Y_d, \beta_d)\}$ can also be chosen to be an integral basis of any maximal cone τ' that has τ as a face. By

choosing V' small enough, the uniformization map $\mathbb{H}_g \times \mathbb{C}^{(1,g)} \rightarrow \mathcal{B}_{g,N}(\mathbb{C})$ maps the open subset $V' \subseteq \mathbb{H}_g \times \mathbb{C}^{(1,g)}$ to the coordinate neighbourhood $V \subseteq \mathcal{B}_{g,N,\Pi}(\mathbb{C})$ centered at $p_{\overline{\tau}}$ by means of the assignment

$$(5.4) \quad (z_1, \dots, z_d) \mapsto (q_1, \dots, q_d), \text{ where } q_j = \begin{cases} e^{2\pi iz_j}, & j = 1, \dots, n, \\ z_j, & j = n+1, \dots, d, \end{cases}$$

so that V' maps surjectively onto $V \cap \mathcal{B}_{g,N}(\mathbb{C})$. By construction, the coordinate neighbourhood V is of the form $\Delta_{r_1} \times \dots \times \Delta_{r_d}$, where $\Delta_{r_j} \subseteq \mathbb{C}$ is the open disk of radius r_j ($j = 1, \dots, d$) centered at the origin.

5.11. Remark. We end this subsection by mentioning that the projectivity assumption made in Theorem 5.9 is used to show that formal versions of the moduli schemes under consideration are algebraizable.

5.4. The adelic arithmetic line bundle of Siegel–Jacobi forms. The aim of this subsection is to upgrade the line bundle of Siegel–Jacobi forms $\mathcal{J}_{k,m,N}$ of weight k , index m , and level $N \geq 3$ in a canonical way to an adelic arithmetic line bundle on $\mathcal{B}_{g,N}$. However, from our construction it will not be clear if this adelic arithmetic line bundle is integrable. In this regard, we will show in the subsequent subsections that this adelic arithmetic line bundle has finite energy, so it will have a well-defined arithmetic self-intersection product that we will compute.

We start by showing that the Hodge bundle ω_N of level N can be upgraded to an adelic arithmetic line bundle on the fine moduli space $\mathcal{A}_{g,N}$. Our presentation will follow the argument given in [YZ21, § 5.5]. We start from the moduli stack \mathcal{A}_g over $\text{Spec}(\mathbb{Z})$ and its Hodge bundle ω . In analogy to Theorem 5.9, it is shown in [FC90, Chapter IV, Theorem 5.7] that after choosing a smooth admissible cone decomposition Σ of $\overline{\mathcal{C}}_g$, one obtains a smooth proper algebraic stack $\overline{\mathcal{A}}_{g,\Sigma}$ over $\text{Spec}(\mathbb{Z})$, which contains \mathcal{A}_g as an open dense algebraic substack. The universal abelian scheme $\pi: \mathcal{B}_g \rightarrow \mathcal{A}_g$ can now be extended to a semi-abelian scheme $\overline{\pi}: \mathcal{G}_g \rightarrow \overline{\mathcal{A}}_{g,\Sigma}$ with zero section $\overline{\varepsilon}: \overline{\mathcal{A}}_{g,\Sigma} \rightarrow \mathcal{G}_g$. The Hodge bundle ω on \mathcal{A}_g extends to a line bundle on $\overline{\mathcal{A}}_{g,\Sigma}$, again denoted by ω , defined by

$$\omega := \overline{\varepsilon}^* \det(\Omega_{\mathcal{G}_g/\overline{\mathcal{A}}_{g,\Sigma}}^1).$$

By [FC90, Chapter V, Theorem 2.3], after contracting the toroidal compactification $\overline{\mathcal{A}}_{g,\Sigma}$ through the linear systems associated to powers of ω , one obtains the minimal compactification \mathcal{A}_g^* , which is a normal proper scheme of finite type over $\text{Spec}(\mathbb{Z})$ and provides a compactification of the coarse moduli space \mathcal{A}'_g of principally polarized abelian schemes of dimension g over $\text{Spec}(\mathbb{Z})$; in particular, we have a natural morphism $\rho': \mathcal{A}_g \rightarrow \mathcal{A}'_g$. We note that the construction of \mathcal{A}_g^* turns out to be independent of the particular choice of toroidal compactification $\overline{\mathcal{A}}_{g,\Sigma}$. By the very construction the line bundle ω on $\overline{\mathcal{A}}_{g,\Sigma}$ descends as a \mathbb{Q} -line bundle on \mathcal{A}_g^* and, after restriction, gives rise to a \mathbb{Q} -line bundle on \mathcal{A}'_g , which we denote again by ω .

We will now equip the \mathbb{Q} -line bundle $\omega_{\otimes \mathbb{Z}} \mathbb{C}$ on $\mathcal{A}'_g(\mathbb{C})$ with a hermitian metric. For this, we first note that $\mathcal{A}'_g(\mathbb{C})$ can be identified with the quotient space $\text{Sp}(2g, \mathbb{Z}) \backslash \mathbb{H}_g$, so that the $\text{Sp}(2g, \mathbb{Z})$ -orbit of a point $Z \in \mathbb{H}_g$ corresponds to the isomorphism class of the abelian variety $A_Z = \mathbb{C}^{(1,g)} / (\mathbb{Z}^{(1,g)} Z \oplus \mathbb{Z}^{(1,g)})$. Given a holomorphic section s of $\omega_{\otimes \mathbb{Z}} \mathbb{C}$ over $\mathcal{A}'_g(\mathbb{C})$, the Faltings metric on $\omega_{\otimes \mathbb{Z}} \mathbb{C}$ is now defined by the formula

$$(5.5) \quad \|s(Z)\|_{\text{Fal}}^2 := \frac{i^{g^2}}{2^g} \int_{A_Z} s(Z) \wedge \overline{s(Z)}.$$

5.12. Lemma. *With the above notations, the hermitian line bundle $\overline{\omega} := (\omega, \|\cdot\|_{\text{Fal}})$ defines an adelic arithmetic line bundle on \mathcal{A}'_g , i. e., $\overline{\omega} \in \mathbf{Pic}_{\mathbb{Q}}(\mathcal{A}'_g)^{\text{adel}}$.*

Proof. Let \mathcal{B} denote the boundary divisor of the projective arithmetic variety \mathcal{A}'_g over $\text{Spec}(\mathbb{Z})$. Then, we obtain $\mathcal{A}'_g = \mathcal{A}_g^* \setminus |\mathcal{B}|$. As in Subsection 3.3, we upgrade \mathcal{B} to an arithmetic boundary divisor $\overline{\mathcal{B}} = (\mathcal{B}, g_B)$ by means of a suitable Green function g_B of continuous type for B . Denoting by \mathcal{A}_g^{**} the blow-up of \mathcal{A}'_g along \mathcal{B} , the Faltings metric will have logarithmic singularities along the boundary $\mathcal{A}_g^{**} \setminus \mathcal{A}'_g$.

In order to complete the proof of the lemma, we additionally equip the \mathbb{Q} -line bundle ω on \mathcal{A}'_g with a smooth hermitian metric $\|\cdot\|'$ to obtain the hermitian line bundle $\overline{\omega}'$. We now choose a rational \mathbb{Q} -section s_ω of ω to obtain the arithmetic divisor $\widehat{\text{div}}_{\overline{\omega}'}(s_\omega)$, which we can view as an adelic arithmetic divisor on \mathcal{A}'_g . Consider next the function $f = -\log(\|s_\omega\|_{\text{Fal}}^2 / \|s_\omega\|'^2)$ on $\mathcal{A}'_g(\mathbb{C})$. Since the hermitian metric $\|\cdot\|'$ extends smoothly to $\mathcal{A}'_g(\mathbb{C})$, but the Faltings metric $\|\cdot\|_{\text{Fal}}$ develops logarithmic singularities along the boundary B , we find that $f = o(g_B)$. In this way, we find that

$$\widehat{\text{div}}_{\overline{\omega}}(s_\omega) = \widehat{\text{div}}_{\overline{\omega}'}(s_\omega) + (0, f).$$

By Corollary 3.33, we conclude that $\widehat{\text{div}}_{\overline{\omega}}(s_\omega) \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{A}'_g)^{\text{adel}}$, which implies that $\overline{\omega} \in \widehat{\text{Pic}}_{\mathbb{Q}}(\mathcal{A}'_g)^{\text{adel}}$. \square

We are now able to upgrade the Hodge bundle ω_N of level N to an adelic arithmetic line bundle on $\mathcal{A}_{g,N}$. For this, we fix a projective toroidal compactification $\overline{\mathcal{A}}_{g,N,\Sigma}$ of $\mathcal{A}_{g,N}$ over $\text{Spec}(\mathbb{Z}[1/N, \zeta_N])$ as in Theorem 5.9. Of course, the scheme $\overline{\mathcal{A}}_{g,N,\Sigma}$ is not projective over $\text{Spec}(\mathbb{Z})$. However, since $\overline{\mathcal{A}}_{g,N,\Sigma}$ is projective over $\text{Spec}(\mathbb{Z}[1/N])$, we can find a projective scheme \mathcal{X}' over $\text{Spec}(\mathbb{Z})$ that contains $\overline{\mathcal{A}}_{g,N,\Sigma}$ as an open dense subscheme. By construction, we have that $\mathcal{X}' \setminus \mathcal{A}_{g,N}$ is the support of an effective divisor, namely the union of the fibers over the primes dividing N together with the union of the closure of the components of the boundary divisor of the toroidal compactification under consideration. In this way, we can view $\mathcal{A}_{g,N}$ as an open dense subset of the projective arithmetic variety \mathcal{X}' over $\text{Spec}(\mathbb{Z})$.

5.13. Lemma. *With the above notations, the hermitian line bundle $\overline{\omega}_N := (\omega_N, \|\cdot\|_{\text{Fal}})$ defines an adelic arithmetic line bundle on $\mathcal{A}_{g,N}$, i. e., $\overline{\omega}_N \in \widehat{\text{Pic}}_{\mathbb{Q}}(\mathcal{A}_{g,N})^{\text{adel}}$.*

Proof. By considering the pull-back of the line bundle ω on \mathcal{A}'_g by means of the composition of the two morphisms

$$(5.6) \quad \mathcal{A}_{g,N} \xrightarrow{\rho'_N} \mathcal{A}_g \xrightarrow{\rho'} \mathcal{A}'_g,$$

we obtain the Hodge bundle ω_N of level N on $\mathcal{A}_{g,N}$. Now, Remark 3.45 shows that the adelic arithmetic line bundle $\overline{\omega} = (\omega, \|\cdot\|_{\text{Fal}})$ on \mathcal{A}'_g pulls back to the adelic arithmetic line bundle $\overline{\omega}_N = (\omega_N, \|\cdot\|_{\text{Fal}})$ on $\mathcal{A}_{g,N}$, as claimed. \square

Next, we show that the line bundle \mathcal{L}_N can be upgraded to an adelic arithmetic line bundle on the universal abelian scheme $\mathcal{B}_{g,N}$. We introduce an analogous set-up as above. We fix a projective toroidal compactification $\overline{\mathcal{B}}_{g,N,\Pi}$ of $\mathcal{B}_{g,N}$ over $\text{Spec}(\mathbb{Z}[1/N, \zeta_N])$ as in Theorem 5.9. Of course, the scheme $\overline{\mathcal{B}}_{g,N,\Pi}$ is not projective over $\text{Spec}(\mathbb{Z})$. However, since $\overline{\mathcal{B}}_{g,N,\Pi}$ is projective over $\text{Spec}(\mathbb{Z}[1/N])$, we can find a projective scheme \mathcal{X} over $\text{Spec}(\mathbb{Z})$ that contains $\overline{\mathcal{B}}_{g,N,\Pi}$ as an open dense subscheme. By construction, we have that $\mathcal{X} \setminus \mathcal{B}_{g,N}$ is the support of an effective divisor, namely the union of the fibers over the primes dividing N together with the union of the closure of the components of the boundary divisor of the toroidal compactification under consideration. In this way, we can view $\mathcal{B}_{g,N}$ as an open dense subset of the projective arithmetic variety \mathcal{X} over $\text{Spec}(\mathbb{Z})$.

5.14. Lemma. *With the above notations, the symmetric line bundle \mathcal{L}_N has a unique extension to an adelic arithmetic line bundle $\overline{\mathcal{L}}_N := (\mathcal{L}_N, \|\cdot\|_{\text{cub}})$ on $\mathcal{B}_{g,N}$, i. e., $\overline{\mathcal{L}}_N \in \widehat{\text{Pic}}_{\mathbb{Q}}(\mathcal{B}_{g,N})^{\text{adel}}$, making the isomorphism (5.1) provided by the theorem of the cube into an isometry. Moreover, the adelic arithmetic line bundle $\overline{\mathcal{L}}_N$ is integrable.*

Proof. The claim follows immediately as an application of [YZ21, Theorem 6.1.3] to the symmetric line bundle \mathcal{L}_N on the universal abelian scheme $\pi_N: \mathcal{B}_{g,N} \rightarrow \mathcal{A}_{g,N}$. \square

Lemmas 5.13 and 5.14 now lead to the main result of this subsection.

5.15. Proposition. *With the above notations, the line bundle $\mathcal{I}_{k,m,N}$ of Siegel–Jacobi forms of weight k , index m , and level N has an extension to an adelic arithmetic line bundle $\overline{\mathcal{I}}_{k,m,N}$ on $\mathcal{B}_{g,N}$, i. e., $\overline{\mathcal{I}}_{k,m,N} \in \widehat{\text{Pic}}_{\mathbb{Q}}(\mathcal{B}_{g,N})^{\text{adel}}$.*

Proof. As we have $\mathcal{J}_{k,m,N} = \pi_N^* \omega_N^{\otimes k} \otimes \mathcal{L}_N^{\otimes m}$, the proof follows immediately by pulling back the k -th power of the adelic arithmetic line bundle $\overline{\omega}_N^{\otimes k} = (\omega_N^{\otimes k}, \|\cdot\|_{\text{Fal}}^k)$ of Lemma 5.13 from $\mathcal{A}_{g,N}$ via π_N to $\mathcal{B}_{g,N}$ and then tensor this pull-back with the m -th power of the adelic arithmetic line bundle $\overline{\mathcal{L}}_N^{\otimes m} = (\mathcal{L}_N^{\otimes m}, \|\cdot\|_{\text{cub}}^m)$ of Lemma 5.14 to obtain $\overline{\mathcal{J}}_{k,m,N} = (\mathcal{J}_{k,m,N}, \|\cdot\|_{\text{Fal}}^k \cdot \|\cdot\|_{\text{cub}}^m) \in \widehat{\text{Pic}}_{\mathbb{Q}}(\mathcal{B}_{g,N})^{\text{adel}}$. \square

5.16. Remark. Proposition 5.15 shows that the complex line bundle of Siegel–Jacobi forms $\mathcal{J}_{k,m,N} \otimes_{\mathbb{Z}[1/n, \zeta_N]} \mathbb{C}$ on $\mathcal{B}_{g,N}(\mathbb{C})$ is equipped with the hermitian metric $\|\cdot\|_{\text{Fal}}^k \cdot \|\cdot\|_{\text{cub}}^m$. On the other, it is well-known that it can also be equipped with the natural invariant metric $\|\cdot\|_{\text{inv}}$, which is defined for a meromorphic section f by

$$\|f(Z, W)\|_{\text{inv}}^2 := |f(Z, W)|^2 \det(Y)^k e^{-4\pi m \beta Y^{-1} \beta^t},$$

where $Z \in \mathbb{H}_g$, $Y = \text{Im}(Z)$ and $W \in \mathbb{C}^{(1,g)}$, $\beta = \text{Im}(W)$. We claim that

$$\|\cdot\|_{\text{inv}} = \|\cdot\|_{\text{Fal}}^k \cdot \|\cdot\|_{\text{cub}}^m.$$

In order to prove this claim, it suffices to consider the two cases $k = 1, m = 0$ and $k = 0, m = 1$. In the first case, the meromorphic section f becomes a meromorphic modular form of weight 1 for $\Gamma(N)$, which gives rise to the translation-invariant differential form $f(Z) dw_1 \wedge \dots \wedge dw_g$ on $\mathbb{C}^{(1,g)}$. Therefore, formula (5.5) defining the Faltings metric shows

$$\begin{aligned} \|f(Z)\|_{\text{Fal}}^2 &= \frac{i^{g^2}}{2^g} \int_{A_Z} f(Z) dw_1 \wedge \dots \wedge dw_g \wedge \overline{f(Z)} d\overline{w}_1 \wedge \dots \wedge d\overline{w}_g \\ &= \left(\frac{i}{2}\right)^g \int_{A_Z} |f(Z)|^2 dw_1 \wedge d\overline{w}_1 \wedge \dots \wedge dw_g \wedge d\overline{w}_g \\ &= |f(Z)|^2 \det(Y) = \|f(Z)\|_{\text{inv}}^2. \end{aligned}$$

In the second case, the meromorphic section f becomes a meromorphic Siegel–Jacobi form of weight 0 and index 1 for $\Gamma(N)$, i.e., a meromorphic section of the line bundle \mathcal{L}_N . Recalling that the isomorphism (5.1) becomes an isometry when \mathcal{L}_N is equipped with the hermitian metric $\|\cdot\|_{\text{cub}}$, we now verify that this is also the case when \mathcal{L}_N is equipped with the natural invariant metric $\|\cdot\|_{\text{inv}}$. By the unicity of the hermitian metric $\|\cdot\|_{\text{cub}}$, the second claimed equality then also follows.

The section $[2]^* f$ corresponds to the function $f_2(Z, W) := f(Z, 2W)$, which is easily checked to be a meromorphic Siegel–Jacobi form of weight 0 and index 4 for $\Gamma(N)$, and thus corresponds to a meromorphic section of the line bundle $\mathcal{L}_N^{\otimes 4}$. Therefore, the isomorphism (5.1) sends $[2]^* f$ to the section of $\mathcal{L}_N^{\otimes 4}$ corresponding to the function $f_2(Z, W)$. Since $f(Z, W)^4$ also corresponds to a section of $\mathcal{L}_N^{\otimes 4}$, there is a meromorphic function $h(Z, W)$ such that $f_2(Z, W) = h(Z, W) f(Z, W)^4$. By construction, the function h is $\widehat{\Gamma}(N)$ -invariant and thus descends to a meromorphic function on $\mathcal{B}_{g,N}(\mathbb{C})$, again denoted by h . The isomorphism (5.1) now becomes an isometry if and only if we have

$$\|f_2(Z, W)\|_{\text{inv}} = |h(Z, W)| \|f(Z, W)\|_{\text{inv}}^4.$$

Indeed, by the definition of the natural invariant metric, we compute

$$\begin{aligned} \|f_2(Z, W)\|_{\text{inv}} &= |f(Z, 2W)| e^{-2\pi(2\beta)Y^{-1}(2\beta)^t} = |h(Z, W)| |f(Z, W)^4| e^{-2\pi 4\beta Y^{-1} \beta^t} \\ &= |h(Z, W)| (|f(Z, W)| e^{-2\pi \beta Y^{-1} \beta^t})^4 = |h(Z, W)| \|f(Z, W)\|_{\text{inv}}^4, \end{aligned}$$

which completes the proof of the second claimed equality.

5.17. Remark. We note that the proof of Lemma 5.12 (and thus also of Lemma 5.13) and the proof of Lemma 5.14 are of very different nature. The adelic arithmetic line bundle $\overline{\omega} = (\omega, \|\cdot\|_{\text{Fal}})$ on \mathcal{A}'_g is constructed by means of the \mathbb{Q} -line bundle ω defined on the projective model \mathcal{A}'_g with the logarithmically singular metric $\|\cdot\|_{\text{Fal}}$. However, it is not clear from the construction that the adelic arithmetic line bundle $\overline{\omega}$ is integrable in the sense of [YZ21]. Therefore, we cannot apply

the techniques of that paper to define an arithmetic self-intersection product for $\overline{\omega}$, although this can be done using the techniques of [BGKK05].

On the other hand, with regard to the adelic arithmetic line bundle $\overline{\mathcal{L}}_N$, even the underlying geometric adelic line bundle is not a model line bundle but a true adelic line bundle. Thus, the techniques of [BGKK05] cannot be applied in this case. Nevertheless, it is an integrable adelic arithmetic line bundle on $\mathcal{B}_{g,N}$, therefore its arithmetic self-intersection product can be defined using the techniques developed in [YZ21].

Thus, our objective in the remaining two subsections is to use our extension of the formalism of [YZ21] developed in the previous sections to define the arithmetic self-intersection product for the adelic arithmetic line bundle $\overline{\mathcal{F}}_{k,m,N}$ of Siegel–Jacobi forms on $\mathcal{B}_{g,N}$ and compute it.

5.5. Some preparatory lemmas. For the proof of the finiteness of the arithmetic self-intersection product for the adelic arithmetic line bundle $\overline{\mathcal{F}}_{k,m,N}$ on $\mathcal{B}_{g,N}$, we need to study and estimate the quantity

$$\beta Y^{-1} \beta^t = \text{Im}(W) \text{Im}(Z)^{-1} \text{Im}(W)^t$$

and certain derivatives thereof emerging from the definition of the natural invariant metric introduced in Remark 5.16, as (Z, W) ranges through the open subset $V' \subseteq \mathbb{H}_g \times \mathbb{C}^{(1,g)}$ introduced in Remark 5.10. Recalling from Remark 5.10 that

$$(Z, W) = i(Y_0, \beta_0) + \sum_{j=1}^d z_j(Y_j, \beta_j)$$

for suitable $z_j = x_j + iy_j \in \mathbb{C}$ ($j = 1, \dots, d$), we find that

$$Y = Y(y_1, \dots, y_d) = Y_0 + \sum_{j=1}^d y_j Y_j \quad \text{and} \quad \beta = \beta(y_1, \dots, y_d) = \beta_0 + \sum_{j=1}^d y_j \beta_j.$$

Thus, we will have to investigate the function $\varphi: \mathbb{R}_{\geq 0}^d \rightarrow \mathbb{R}$ defined by

$$\varphi(y_1, \dots, y_d) = \beta Y^{-1} \beta^t$$

with $Y = Y(y_1, \dots, y_d)$ and $\beta = \beta(y_1, \dots, y_d)$ as above.

The first task is to compute the Hessian matrix of the function φ . To this end, we introduce for $j = 0, \dots, d$ the quantities $W_j = \beta_j - \beta Y^{-1} Y_j$; they satisfy the linear relation

$$(5.7) \quad W_0 + \sum_{j=1}^d y_j W_j = 0,$$

which follows by a direct computation.

5.18. Lemma. *With the above notations, the equality*

$$\frac{\partial^2 \varphi}{\partial y_i \partial y_j} = 2W_i Y^{-1} W_j^t$$

holds for $i, j = 1, \dots, d$. Therefore, the Hessian matrix of φ is a Gram matrix and, in particular, the rank of the Hessian of φ is at most g .

Proof. This is a simple computation. Indeed, for the first partial derivative of φ , we compute

$$\frac{\partial \varphi}{\partial y_i} = 2\beta_i Y^{-1} \beta^t - \beta Y^{-1} Y_i Y^{-1} \beta^t.$$

Therefore, the second partial derivative of φ becomes

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial y_i \partial y_j} &= 2\beta_i Y^{-1} \beta_j^t - 2\beta_i Y^{-1} Y_j Y^{-1} \beta^t - 2\beta Y^{-1} Y_i Y^{-1} \beta_j^t + 2\beta Y^{-1} Y_i Y^{-1} Y_j Y^{-1} \beta^t \\ &= 2(\beta_i - \beta Y^{-1} Y_i) Y^{-1} (\beta_j - \beta Y^{-1} Y_j)^t \\ &= 2W_i Y^{-1} W_j^t. \end{aligned}$$

This completes the proof of the lemma. □

In order to ease the bound of the entries of the Hessian matrix above, we introduce the following condition.

5.19. Definition. We say that *the matrices* Y_0, \dots, Y_d *satisfy the flag condition*, if $\ker(Y_j) \subseteq \ker(Y_{j+k})$ for $j = 0, \dots, d-1$ and $k = 0, \dots, d-j$.

From now on, we assume that the matrices Y_0, \dots, Y_d satisfy the flag condition. We write $\text{rk}_j = \text{rk}(Y_j)$ and choose a basis such that each matrix Y_j can be written in block form as

$$Y_j = \begin{pmatrix} Y'_j & 0 \\ 0 & 0 \end{pmatrix},$$

where Y'_j is a real symmetric positive definite $(\text{rk}_j \times \text{rk}_j)$ -matrix for $j = 0, \dots, d$. Note that, by hypothesis, $Y'_0 = Y_0$ and thus $\text{rk}_0 = g$. The following result is proven in [BGHdJ18, Lemma 3.6].

5.20. Lemma. *With the above notations, there is a constant $C > 0$ such that the upper bound*

$$(Y^{-1})_{k,\ell} \leq \frac{C}{1 + \sum_{j: \text{rk}_j \geq \min(k,\ell)} y_j}$$

holds for $k, \ell = 1, \dots, g$.

Note that the number 1 in the denominator above comes from the fact that Y_0 has maximal rank being positive definite and that Y_0 occurs with the coefficient 1 in the definition of $Y = Y(y_1, \dots, y_d)$. Next, we will bound the entries of W_j .

5.21. Lemma. *With the above notations, we have the following two results for the row vectors W_j for $j = 0, \dots, d$: If $g \geq \ell > \text{rk}_j$, the relation*

$$(W_j)_\ell = 0$$

holds. Moreover, if $0 \leq \ell \leq \text{rk}_j$, there is a constant $C > 0$ such that the upper bound

$$(W_j)_\ell \leq C$$

holds.

Proof. The first statement follows immediately from the choice of basis that puts each Y_j in the desired block form. In order to prove the second claim, observing that $W_j = \beta_j - \beta Y^{-1} Y_j$, it is enough to show that the entries of βY^{-1} are uniformly bounded. In this regard, by the definition of $\beta = \beta(y_1, \dots, y_d)$, there is a constant $C_1 > 0$ such that the k -th entry of β can be bounded as

$$(\beta)_k \leq C_1 \left(1 + \sum_{j: \text{rk}_j \geq k} y_j \right).$$

By means of Lemma 5.20 (with the constant C renamed as C_2), we then derive the bound

$$(\beta Y^{-1})_\ell = \sum_{k=1}^g (\beta)_k (Y^{-1})_{k,\ell} \leq \sum_{k=1}^g C_1 C_2 \frac{1 + \sum_{j: \text{rk}_j \geq k} y_j}{1 + \sum_{j: \text{rk}_j \geq \min(k,\ell)} y_j} \leq g C_1 C_2.$$

This completes the proof of the second claim. \square

As a consequence of the above two lemmas we deduce the following bound for the elements of the Hessian matrix.

5.22. Lemma. *With the above notations, there are constants $C, C' > 0$ such that the upper bounds*

$$W_i Y^{-1} W_j^t \leq \frac{C}{1 + \sum_{k: \text{rk}_k \geq \min(\text{rk}_i, \text{rk}_j)} y_k} \leq \frac{C'}{1 + y_i^{1/2} y_j^{1/2}}$$

hold.

Proof. As we have

$$W_i Y^{-1} W_j^t = \sum_{k,\ell=1}^g (W_i)_k (Y^{-1})_{k,\ell} (W_j)_\ell,$$

the first claimed inequality is an immediate consequence of Lemma 5.20 and Lemma 5.21. The second claimed inequality can then be easily derived. \square

For the sequel, we will need finer estimates. For this, we let $I \subseteq \{1, \dots, d\}$ denote a subset and we write $W_I = \sum_{j \in I} y_j W_j$. To uniformize our notation, we set $y_0 = 1$ and, given $I \subseteq \{1, \dots, d\}$, we put $I_0^c = \{0, \dots, d\} \setminus I$. For later purposes, we put $I^c = \{1, \dots, d\} \setminus I$ and note that $I_0^c = I^c \cup \{0\}$.

5.23. Lemma. *With the above notations, there is a constant $C > 0$ such that for all subsets $I \subseteq \{1, \dots, d\}$ the upper bound*

$$W_I Y^{-1} W_I^t \leq C \frac{\sum_{i \in I} y_i \sum_{j \in I_0^c} y_j}{\sum_{k=0}^d y_k}$$

holds.

Proof. By equation (5.7), we have that $W_I = -W_{I_0^c}$, from which we deduce

$$W_I Y^{-1} W_I^t = |W_I Y^{-1} W_{I_0^c}^t|.$$

By Lemma 5.22, there is a constant $C_1 > 0$ such that

$$(5.8) \quad \begin{aligned} W_I Y^{-1} W_I^t &\leq C_1 \sum_{i \in I} \sum_{j \in I_0^c} \frac{y_i y_j}{1 + \sum_{k: \text{rk}_k \geq \min(\text{rk}_i, \text{rk}_j)} y_k} \\ &\leq C_1 \sum_{\ell=1}^g \frac{\sum_{\substack{i \in I \\ \text{rk}_i = \ell}} y_i \sum_{\substack{j \in I_0^c \\ \text{rk}_j \geq \ell}} y_j + \sum_{\substack{i \in I \\ \text{rk}_i > \ell}} y_i \sum_{\substack{j \in I_0^c \\ \text{rk}_j = \ell}} y_j}{1 + \sum_{k: \text{rk}_k \geq \ell} y_k}. \end{aligned}$$

Before continuing, we observe that given $a, b, c, d \geq 0$ with $a + b > 0$, then the inequality

$$\frac{ab}{a+b} \leq \frac{(a+c)(b+d)}{a+b+c+d}$$

holds, since we easily verify that

$$(a+b)(a+c)(b+d) - ab(a+b+c+d) = a^2d + b^2c + acd + bcd \geq 0.$$

From this observation we can add the missing terms in the denominators of the quantity (5.8), as long as we add the same term to each of the factors of the corresponding numerator. Since we can further add positive terms to the numerators, we deduce

$$W_I Y^{-1} W_I^t \leq C_1 \sum_{\ell=1}^g \frac{2 \sum_{i \in I} y_i \sum_{j \in I_0^c} y_j}{\sum_{k=0}^d y_k},$$

which implies the result. \square

The key estimate of this subsection is the next result.

5.24. Proposition. *Let $J, K \subseteq \{1, \dots, d\}$ of cardinality r , $I = J \cap K$, and fix $\varepsilon > 0$. Then, with the above notations, there is a constant $C > 0$ such that for $(y_1, \dots, y_d) \in \mathbb{R}_{\geq \varepsilon}^d$, the upper bound*

$$\left| \det \left(\frac{\partial^2 \varphi}{\partial y_j \partial y_k} \right)_{\substack{j \in J \\ k \in K}} \right| \leq C \frac{\sum_{i \in I_0^c} y_i}{\sum_{i=0}^d y_i} \frac{1}{\prod_{j \in J} y_j^{1/2} \prod_{k \in K} y_k^{1/2}}$$

holds.

Proof. By Lemma 5.18, we have

$$(5.9) \quad \det \left(\frac{\partial^2 \varphi}{\partial y_j \partial y_k} \right)_{\substack{j \in J \\ k \in K}} = 2^r \det (W_j Y^{-1} W_k^t)_{\substack{j \in J \\ k \in K}}.$$

We now fix $i_0 \in I$ and note that the determinant of the Gram matrix on the right-hand side of (5.9) remains unchanged, if we replace both occurrences of W_{i_0} by $(1/y_{i_0})W_I$. Recalling that the determinant of a real symmetric positive definite matrix can be bounded by the product of its diagonal elements, we are led by means of Lemma 5.22 and Lemma 5.23 to the upper bound

$$\begin{aligned} \det \left(\frac{\partial^2 \varphi}{\partial y_j \partial y_k} \right)_{\substack{j \in J \\ k \in K}} &\leq \frac{C_{i_0}}{y_{i_0}^2} \frac{\sum_{i \in I} y_i \sum_{j \in I_0^c} y_j}{\sum_{k=0}^d y_k} \frac{1}{\prod_{\substack{j \in J \\ j \neq i_0}} y_j^{1/2} \prod_{\substack{k \in K \\ k \neq i_0}} y_k^{1/2}} \\ &= \frac{C_{i_0}}{y_{i_0}} \frac{\sum_{i \in I} y_i \sum_{j \in I_0^c} y_j}{\sum_{k=0}^d y_k} \frac{1}{\prod_{j \in J} y_j^{1/2} \prod_{k \in K} y_k^{1/2}}, \end{aligned}$$

where $C_{i_0} > 0$ is a constant; observe that in the application of Lemma 5.22 we are allowed to remove the summand 1 in the denominator because we have $(y_1, \dots, y_d) \in \mathbb{R}_{\geq \varepsilon}^d$. Since the inequality just obtained is true for each $i_0 \in I$, we find a constant $C > 0$ such that

$$\sum_{i_0 \in I} y_{i_0} \det \left(\frac{\partial^2 \varphi}{\partial y_j \partial y_k} \right)_{\substack{j \in J \\ k \in K}} \leq C \frac{\sum_{i \in I} y_i \sum_{j \in I_0^c} y_j}{\sum_{k=0}^d y_k} \frac{1}{\prod_{j \in J} y_j^{1/2} \prod_{k \in K} y_k^{1/2}},$$

from which the claim follows immediately. \square

5.25. Lemma. *Let $\varepsilon > 0$ be a real number and $m, n > 0$ integers. Then, there is a constant $C > 0$ such that for all $s_1, \dots, s_m, t_1, \dots, t_n \geq \varepsilon$, the bound*

$$\sum_{j=1}^m s_j \left(\prod_{k=1}^n t_k \right)^{1/n} \leq C \sum_{k=1}^n t_k \prod_{j=1}^m s_j$$

holds.

Proof. Without loss of generality, we can assume that $s_1 \geq s_j$ for $j = 1, \dots, m$. Since $s_j \geq \varepsilon$ for $j = 1, \dots, m$, we obtain

$$\sum_{j=1}^m s_j \leq m s_1 \leq m s_1 \frac{s_2}{\varepsilon} \cdots \frac{s_m}{\varepsilon} = \frac{m}{\varepsilon^{m-1}} \prod_{j=1}^m s_j.$$

On the other hand, using the inequality between the geometric and the arithmetic mean, leads to

$$\left(\prod_{k=1}^n t_k \right)^{1/n} \leq \frac{1}{n} \sum_{k=1}^n t_k.$$

Putting the above two inequalities together yields

$$\sum_{j=1}^m s_j \left(\prod_{k=1}^n t_k \right)^{1/n} \leq \frac{m}{\varepsilon^{m-1}} \sum_{k=1}^n t_k \prod_{j=1}^m s_j,$$

which proves the claim. \square

5.6. The arithmetic self-intersection of the adelic arithmetic line bundle $\overline{\mathcal{F}}_{k,m,N}$. We return to the setting introduced in Subsection 3.3. For this, we realize $\mathcal{B}_{g,N}$, as in Subsection 5.4 preceding Lemma 5.14, as an open dense subset \mathcal{U} of a projective arithmetic variety \mathcal{X} of relative dimension $d = g(g+1)/2 + g$ over $\text{Spec}(\mathbb{Z})$. The difference $\mathcal{X} \setminus \mathcal{B}_{g,N}$ is then the support of an effective divisor \mathcal{B} , namely the union of the fibers over the primes dividing N together with the union of the closure of the components of the boundary divisor of the toroidal compactification $\overline{\mathcal{B}}_{g,N,\Pi}$ under consideration. As usual, we write $X = \mathcal{X}(\mathbb{C})$, i. e., $X = \overline{\mathcal{B}}_{g,N,\Pi}(\mathbb{C})$, for the associated complex manifold, $U = \mathcal{B}_{g,N}(\mathbb{C})$, and B for the divisor induced by \mathcal{B} on X . We upgrade

\mathcal{B} to an arithmetic boundary divisor $\overline{\mathcal{B}} = (\mathcal{B}, g_B)$ by means of a suitable Green function g_B of continuous type for B .

As discussed in Subsection 5.4, the adelic arithmetic line bundle $\overline{\mathcal{L}}_N$ is integrable on $\mathcal{B}_{g,N}$, while the example presented in Subsection 4.1 suggests that the adelic arithmetic line bundle $\overline{\omega}_N$ is not integrable on $\mathcal{A}_{g,N}$. However, we can change the metric $\|\cdot\|_{\text{Fal}}$ on $\overline{\omega}_N$ to obtain an integrable adelic arithmetic line bundle $\overline{\omega}'_N$ on $\mathcal{A}_{g,N}$ as follows. By the definition of \mathcal{A}_g^* , we can choose $r \in \mathbb{N}$ large enough so that $\omega^{\otimes r}$ becomes a very ample line bundle on \mathcal{A}_g^* giving rise to the closed immersion $\iota: \mathcal{A}_g^* \hookrightarrow \mathbb{P}_{\mathbb{Z}}^n$ for suitable $n \in \mathbb{N}$. Choosing next a smooth projective admissible cone decomposition Σ of $\overline{\mathcal{C}}_g$ and extending the two morphisms ρ'_N and ρ' in diagram (5.6) to the respective compactifications, we obtain the diagram

$$\overline{\mathcal{A}}_{g,N,\Sigma} \xrightarrow{\rho'_{N,\Sigma}} \overline{\mathcal{A}}_{g,\Sigma} \xrightarrow{\rho'_\Sigma} \mathcal{A}_g^*.$$

By construction, we find that

$$\omega_N = (\rho'_\Sigma \circ \rho'_{N,\Sigma})^* \omega = (\rho'_\Sigma \circ \rho'_{N,\Sigma})^* \iota^* \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n}(1)^{\otimes 1/r} = (\iota \circ \rho'_\Sigma \circ \rho'_{N,\Sigma})^* \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n}(1)^{\otimes 1/r}.$$

By now equipping $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n}(1)$ with the Fubini–Study metric $\|\cdot\|_{\text{FS}}$, we obtain after tensoring $1/r$ -times and pull-back by $\iota \circ \rho'_\Sigma \circ \rho'_{N,\Sigma}$ the desired integrable adelic arithmetic line bundle $\overline{\omega}'_N$ on $\mathcal{A}_{g,N}$, since the adelic arithmetic line bundle $\overline{\mathcal{O}}_{\mathbb{P}_{\mathbb{Z}}^n}(1) = (\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n}(1), \|\cdot\|_{\text{FS}})$ is integrable on $\mathbb{P}_{\mathbb{Z}}^n$. In this way, next to the adelic arithmetic line bundle $\overline{\mathcal{J}}_{k,m,N}$, we obtain the integrable adelic arithmetic line bundle

$$\overline{\mathcal{J}}'_{k,m,N} := \pi_N^* \overline{\omega}'_N^{\otimes k} \otimes \overline{\mathcal{L}}_N^{\otimes m}$$

on $\mathcal{B}_{g,N}$.

In the next step, we shift from the language of adelic arithmetic line bundles to adelic arithmetic divisors on $\mathcal{B}_{g,N}$. For this, we choose a rational \mathbb{Q} -section s_ω of ω_N and a rational \mathbb{Q} -section $s_{\mathcal{L}}$ of \mathcal{L}_N , and define the adelic arithmetic divisors

$$\begin{aligned} \overline{\mathcal{D}}_{k,m,N} &= (\mathcal{D}_{k,m,N}, g_{k,m,N}) := \widehat{\text{div}}_{\overline{\mathcal{J}}_{k,m,N}}(s_\omega^{\otimes k} \otimes s_{\mathcal{L}}^{\otimes m}), \\ \overline{\mathcal{D}}'_{k,m,N} &= (\mathcal{D}_{k,m,N}, g'_{k,m,N}) := \widehat{\text{div}}_{\overline{\mathcal{J}}'_{k,m,N}}(s_\omega^{\otimes k} \otimes s_{\mathcal{L}}^{\otimes m}). \end{aligned}$$

By construction, the adelic arithmetic divisors $\overline{\mathcal{D}}'_{k,m,N}$ are integrable on $\mathcal{B}_{g,N}$, so their arithmetic self-intersection product can be defined using the techniques developed in [YZ21]. However, we are interested in the arithmetic self-intersection product of the adelic arithmetic divisors $\overline{\mathcal{D}}_{k,m,N}$, which cannot be defined by these techniques, although they have the same divisorial part as the adelic arithmetic divisors $\overline{\mathcal{D}}'_{k,m,N}$. In order to solve this problem, we will use the generalized arithmetic self-intersection product of the adelic arithmetic divisors $\overline{\mathcal{D}}_{k,m,N}$ involving finite energies as developed in Subsection 4.2. For this, we need some more notations. First, we find by the linearity that

$$\overline{\mathcal{D}}_{k,m,N} = k\overline{\mathcal{D}}_{1,0,N} + m\overline{\mathcal{D}}_{0,1,N}, \quad \overline{\mathcal{D}}'_{k,m,N} = k\overline{\mathcal{D}}'_{1,0,N} + m\overline{\mathcal{D}}_{0,1,N}$$

and

$$\mathcal{D}_{k,m,N} = k\mathcal{D}_{1,0,N} + m\mathcal{D}_{0,1,N}, \quad g_{k,m,N} = kg_{1,0,N} + mg_{0,1,N}, \quad g'_{k,m,N} = kg'_{1,0,N} + mg_{0,1,N}.$$

Next, we choose a sufficiently large arithmetic reference divisor $\overline{\mathcal{E}} = (\mathcal{E}, g_E)$ on \mathcal{X} with g_E a Green function of smooth type for E such that

$$(5.10) \quad \overline{\mathcal{E}} \geq \overline{\mathcal{B}}, \quad \overline{\mathcal{E}} \geq \overline{\mathcal{D}}'_{1,0,N} + 2\overline{\mathcal{B}}, \quad \overline{\mathcal{E}} \geq \overline{\mathcal{D}}_{0,1,N} + 2\overline{\mathcal{B}}.$$

On the associated complex manifold $X = \mathcal{X}(\mathbb{C})$, we then define the functions

$$\varphi_{1,0,N} = g_{1,0,N} - g_E \quad \text{and} \quad \varphi'_{1,0,N} = g'_{1,0,N} - g_E,$$

and let $\omega_0 = \omega_E(g_E)$. By construction, the function $\varphi'_{1,0,N}$ is ω_0 -plurisubharmonic.

We claim that $\varphi_{1,0,N} \in \mathcal{E}(X, \omega_0, \varphi'_{1,0,N})$: To prove the claim, we start by noting that $g_{1,0,N} = o(g_B)$ by arguing as in Lemma 5.12 and $g_E = O(g_B)$ since $E \geq \mathcal{D}_{1,0,N} + 2B$, from which we derive

$$\varphi_{1,0,N} = g_{1,0,N} - g_E < \infty.$$

Next, observing that the rational section s_ω of ω_N induces a meromorphic section $s = s_{\omega, \mathbb{C}}$ of $\omega_N \otimes_{\mathbb{Z}[1/N, \zeta_N]} \mathbb{C}$ over $\mathcal{A}_{g, N}(\mathbb{C})$, we recall that

$$\log \|s\|_{\text{inv}}^2 = \log |s|^2 + \log \det(Y),$$

from which we compute

$$\begin{aligned} \text{dd}^c \varphi_{1,0,N} + \omega_0 &= \text{dd}^c (g_{1,0,N} - g_E) + \omega_0 = \text{dd}^c g_{1,0,N} - \text{dd}^c g_E + \omega_0 \\ &= -\text{dd}^c \log \|s\|_{\text{inv}}^2 + \delta_E = -\text{dd}^c \log |s|^2 + \delta_{D_{1,0,N}} - \text{dd}^c \log \det(Y) + \delta_C, \end{aligned}$$

where $C = E - D_{1,0,N} \geq 0$. By the meromorphicity of the section s , we therefore arrive at the inequality

$$\text{dd}^c \varphi_{0,1,N} + \omega_0 \geq -\text{dd}^c \log \det(Y)$$

on $\mathcal{A}_{g, N}(\mathbb{C})$. Now, we recall from [BV04, pp. 71–73] that the function $-\log \det(Y)$ is convex on \mathbb{H}_g , whence

$$\text{dd}^c \varphi_{0,1,N} + \omega_0 \geq 0,$$

from which we conclude that $\varphi_{0,1,N}$ is ω_0 -plurisubharmonic on X . Finally, since the natural invariant metric $\|\cdot\|_{\text{inv}}$ is a good metric in the sense of Mumford [Mum77], we find that $\varphi_{1,0,N}$ has relative full mass with respect to $\varphi'_{1,0,N}$. Therefore, we end up with $\varphi_{1,0,N} \in \mathcal{E}(X, \omega_0, \varphi'_{1,0,N})$.

More generally, multiplying (5.10) by $(k+m)$, we obtain

$$(k+m)\overline{\mathcal{E}} \geq k\overline{\mathcal{D}}'_{1,0,N} + 2k\overline{\mathcal{B}} + m\overline{\mathcal{D}}_{0,1,N} + 2m\overline{\mathcal{B}} = \overline{\mathcal{D}}'_{k,m,N} + 2(k+m)\overline{\mathcal{B}}.$$

By then defining the functions

$$\varphi_{k,m,N} = g_{k,m,N} - (k+m)g_E \quad \text{and} \quad \varphi'_{k,m,N} = g'_{k,m,N} - (k+m)g_E,$$

one shows in the same way as above that $\varphi'_{k,m,N}$ is $(k+m)\omega_0$ -plurisubharmonic and that $\varphi_{k,m,N} \in \mathcal{E}(X, (k+m)\omega_0, \varphi'_{k,m,N})$.

We are now able to state the first main theorem of this section.

5.26. Theorem. *For non-negative integers k, m , the adelic arithmetic divisor $\overline{\mathcal{D}}_{k,m,N}$ on $\mathcal{B}_{g,N}$ has finite energy with respect to $\overline{\mathcal{D}}'_{k,m,N}$. Therefore, the generalized arithmetic intersection product $\overline{\mathcal{D}}_{k,m,N}^{d+1}$ is a well-defined real number.*

Proof. In order to prove Theorem 5.26, it is enough to show that (using the preceding notation)

$$\varphi_{k,m,N} \in \mathcal{E}^1(X, (k+m)\omega_0, \varphi'_{k,m,N}).$$

Recalling that $X = \overline{\mathcal{B}}_{g,N,\Pi}(\mathbb{C})$ and $U = \mathcal{B}_{g,N}(\mathbb{C})$, and taking into account Lemma 2.43, we are left to show that

$$\int_X (\varphi_{k,m,N} - \varphi'_{k,m,N}) \langle (\text{dd}^c \varphi_{k,m,N} + (k+m)\omega_0)^{\wedge d} \rangle > -\infty.$$

By the multilinearity of the non-pluripolar product recalled in Theorem 2.32 (ii), it is enough to show that for any $0 \leq r \leq d$, we have

$$\int_X (\varphi_{1,0,N} - \varphi'_{1,0,N}) \langle (\text{dd}^c \varphi_{1,0,N} + \omega_0)^{\wedge d-r} \wedge (\text{dd}^c \varphi_{0,1,N} + \omega_0)^{\wedge r} \rangle > -\infty.$$

Since the restrictions of the currents $\text{dd}^c \varphi_{1,0,N} + \omega_0$ and $\text{dd}^c \varphi_{0,1,N} + \omega_0$ to U are smooth, we have

$$\begin{aligned} &\int_X (\varphi_{1,0,N} - \varphi'_{1,0,N}) \langle (\text{dd}^c \varphi_{1,0,N} + \omega_0)^{\wedge d-r} \wedge (\text{dd}^c \varphi_{0,1,N} + \omega_0)^{\wedge r} \rangle \\ &= \int_U (\varphi_{1,0,N} - \varphi'_{1,0,N}) (\text{dd}^c \varphi_{1,0,N} + \omega_0)^{\wedge d-r} \wedge (\text{dd}^c \varphi_{0,1,N} + \omega_0)^{\wedge r}. \end{aligned}$$

We will now prove the claimed boundedness result locally on the coordinate neighbourhoods $V \subseteq X$ introduced in Remark 5.10. Recall that in order to define such a coordinate neighbourhood V around a point $p_{\overline{\tau}}$ of a stratum $\overline{\mathcal{B}}_{g,N,\overline{\tau}}(\mathbb{C})$ of $\overline{\mathcal{B}}_{g,N,\Pi}(\mathbb{C})$, we started from an integral basis

$$(5.11) \quad \{(Y_1, \beta_1), \dots, (Y_d, \beta_d)\}$$

of a maximal cone $\tau' \in \Pi$ that has τ as a face, together with a pair (Y_0, β_0) with Y_0 a real symmetric positive definite $(g \times g)$ -matrix and a row vector $\beta_0 \in \mathbb{R}^{(1,g)}$. After possibly replacing the smooth projective admissible cone decomposition Π of \tilde{C}_g by its barycentric subdivision, we may assume by [BGHdJ18, Lemma 3.5] without loss of generality that the matrices Y_0, \dots, Y_d satisfy the flag condition of Definition 5.19 for each maximal cone $\tau' \in \Pi$. The coordinate neighbourhood V with coordinates (q_1, \dots, q_d) around the point $p_{\overline{\tau}}$ is then characterized as follows:

- (i) There is a subset $L \subseteq \{1, \dots, d\}$ such that $V \setminus U$ is described by the equation $\prod_{j \in L} q_j = 0$.
- (ii) There are real numbers $0 < r_1, \dots, r_d < 1/e$ such that the set of pairs

$$V' := \left\{ (Z, W) = i(Y_0, \beta_0) + \sum_{j=1}^d z_j(Y_j, \beta_j) \left| \begin{array}{l} z_j \in \mathbb{C} : \quad 2\pi \operatorname{Im}(z_j) > -\log(r_j), \quad j \in L \\ |z_j| < r_j, \quad j \notin L \end{array} \right. \right\}$$

constitutes an open subset of $\mathbb{H}_g \times \mathbb{C}^{(1,g)}$.

- (iii) By choosing V' small enough, the uniformization map $\mathbb{H}_g \times \mathbb{C}^{(1,g)} \rightarrow \mathcal{B}_{g,N}(\mathbb{C})$ maps the open subset $V' \subseteq \mathbb{H}_g \times \mathbb{C}^{(1,g)}$ to the coordinate neighbourhood $V \subseteq X$ centered at $p_{\overline{\tau}}$ by means of the assignment

$$(z_1, \dots, z_d) \mapsto (q_1, \dots, q_d), \text{ where } q_j = \begin{cases} e^{2\pi i z_j}, & j \in L, \\ z_j, & j \notin L, \end{cases}$$

so that V' maps surjectively onto $V \cap U$.

We note that we need to keep the order prescribed by the flag condition, so we cannot assume that the normal crossing divisor $V \setminus U$ is given by the vanishing of the first coordinates.

Our task now is to prove that the integral

$$(5.12) \quad \int_{V \cap U} (\varphi_{1,0,N} - \varphi'_{1,0,N})(\operatorname{dd}^c \varphi_{1,0,N} + \omega_0)^{\wedge d-r} \wedge (\operatorname{dd}^c \varphi_{0,1,N} + \omega_0)^{\wedge r}$$

is finite. In fact, we will show that the integrand of (5.12) is absolutely integrable by bounding its absolute value by an integrable function. We need some more notation. We write

$$\eta_1 = (\varphi_{1,0,N} - \varphi'_{1,0,N})(\operatorname{dd}^c \varphi_{1,0,N} + \omega_0)^{\wedge d-r} \quad \text{and} \quad \eta_2 = (\operatorname{dd}^c \varphi_{0,1,N} + \omega_0)^{\wedge r};$$

then the integrand of (5.12) becomes $\eta = \eta_1 \wedge \eta_2$. We further write

$$\eta_2 = \sum_{J,K} \eta_{2,J,K} \quad \text{with} \quad \eta_{2,J,K} = f_{2,J,K} dq_J \wedge d\overline{q}_K,$$

where the sum runs over all subsets $J, K \subseteq \{1, \dots, d\}$ of cardinality r so that if $J = \{j_1, \dots, j_r\}$, then $dq_J = dq_{j_1} \wedge \dots \wedge dq_{j_r}$, and similarly for $d\overline{q}_K$. This gives

$$\eta = \sum_{J,K} \eta_{1,J^c,K^c} \wedge \eta_{2,J,K},$$

where $J^c = \{1, \dots, d\} \setminus J$ and $K^c = \{1, \dots, d\} \setminus K$; we also write

$$\eta_{1,J^c,K^c} = f_{1,J^c,K^c} dq_{J^c} \wedge d\overline{q}_{K^c}.$$

Since the natural invariant metric $\|\cdot\|_{\text{inv}}$ on the Hodge bundle $\overline{\omega}_N$ is a good metric in the sense of Mumford [Mum77] (see also [BGKK05]), we know that the differential forms η_{1,J^c,K^c} have log-log growth when approaching the boundary $V \setminus U$. This means that there is a constant $C > 0$ and an integer $M \geq 0$ such that the upper bound

$$(5.13) \quad |f_{1,J^c,K^c}| \leq C \frac{(\sum_{i=1}^d \log(\log |q_i|^2))^M}{\prod_{j \in J^c} |q_j \log |q_j|| \prod_{k \in K^c} |q_k \log |q_k||}$$

holds on $V \cap U$.

Next, we need to find a bound for the functions $f_{2,J,K}$ on $V \cap U$. For this, we recall that the rational section $s_{\mathcal{L}}$ of \mathcal{L}_N induces a meromorphic section $s = s_{\mathcal{L},\mathbb{C}}$ of $\mathcal{L}_N \otimes_{\mathbb{Z}[1/N, \zeta_N]} \mathbb{C}$ over U . As

(Z, W) ranges through the open subset $V' \subseteq \mathbb{H}_g \times \mathbb{C}^{(1,g)}$ with coordinates (z_1, \dots, z_d) as described above, we compute

$$\log \|s\|_{\text{inv}}^2 = \log |s|^2 - 4\pi \operatorname{Im}(W) \operatorname{Im}(Z)^{-1} \operatorname{Im}(W)^t = \log |s|^2 - 4\pi \beta Y^{-1} \beta^t,$$

where

$$Y = Y(y_1, \dots, y_d) = Y_0 + \sum_{j=1}^d y_j Y_j \quad \text{and} \quad \beta = \beta(y_1, \dots, y_d) = \beta_0 + \sum_{j=1}^d y_j \beta_j,$$

recalling that $z_j = x_j + iy_j$ ($j = 1, \dots, d$). From this, we deduce the following equalities on the open subset V' (which maps surjectively onto $V \cap U$)

$$\begin{aligned} \operatorname{dd}^c \varphi_{0,1,N} + \omega_0 &= \operatorname{dd}^c (g_{0,1,N} - g_E) + \omega_0 = \operatorname{dd}^c g_{0,1,N} - \operatorname{dd}^c g_E + \omega_0 \\ &= -\operatorname{dd}^c \log \|s\|_{\text{inv}}^2 + \delta_E = -\operatorname{dd}^c \log |s|^2 + \delta_{D_{0,1,N}} + 4\pi \operatorname{dd}^c \beta Y^{-1} \beta^t + \delta_C, \end{aligned}$$

where $C = E - D_{0,1,N} \geq 0$ such that $|C_{\mathbb{C}}| \subseteq V \setminus U$. By the meromorphicity of the section s , we therefore have the equality

$$(5.14) \quad \operatorname{dd}^c \varphi_{0,1,N} + \omega_0 = 4\pi \operatorname{dd}^c \beta Y^{-1} \beta^t$$

on the open subset V' .

Now, we replace the summand Y_0 of Y by $Y_0 - \sum_{j \notin L} r'_j Y_j$ with r'_j being strictly smaller than r_j , which is still a real symmetric positive definite matrix by the definition of the open subset V' . Then, we replace z_j by $z_j + ir'_j$ for $j \notin L$. After possibly shrinking V' once again, we can thus assume that $y_j \geq \varepsilon > 0$ for all $j \in \{1, \dots, d\}$ and that the uniformization map is given by $q_j = z_j - ir'_j$ for all $j \notin L$. By recalling the function $\varphi: \mathbb{R}_{\geq \varepsilon}^d \rightarrow \mathbb{R}$ introduced in Subsection 5.5, we deduce from equation (5.14) that

$$(5.15) \quad f_{2,J,K} = (2i)^r \det \left(\frac{\partial^2 \varphi}{\partial y_j \partial y_k} \right)_{\substack{j \in J \\ k \in K}} \prod_{\substack{j \in J \\ k \in K}} \frac{\partial y_j}{\partial q_j} \frac{\partial y_k}{\partial \bar{q}_k}.$$

Using Proposition 5.24, we find that there are constants $C, C' > 0$ such that the bound

$$(5.16) \quad \begin{aligned} |f_{2,J,K}| &\leq C \frac{\sum_{i \in I_0^c} y_i}{\sum_{i=0}^d y_i} \frac{1}{\prod_{j \in J} y_j^{1/2} \prod_{k \in K} y_k^{1/2}} \left| \prod_{\substack{j \in J \\ k \in K}} \frac{\partial y_j}{\partial q_j} \frac{\partial y_k}{\partial \bar{q}_k} \right| \\ &\leq C' \frac{\sum_{i \in I_0^c} |\log |q_i||}{\sum_{i=0}^d |\log |q_i||} \frac{1}{\prod_{j \in J} |\log |q_j||^{1/2} \prod_{k \in K} |\log |q_k||^{1/2}} \prod_{\substack{j \in J \\ k \in K}} \frac{1}{|q_j q_k|}, \end{aligned}$$

holds on $V \cap U$; recall that $I = J \cap K$ and $I_0^c = \{0, \dots, d\} \setminus I$, and that we use the convention $y_0 = \log |q_0| = 1$.

The idea to prove the boundedness of the integral (5.12) is now simple. We observe that an integral of the form

$$\int_{|q| \leq 1} \frac{dq \wedge d\bar{q}}{|q|^2 |\log |q||^{2a}}$$

is convergent as long as $a > 1/2$. This shows that the respective integrals over the components of η_2 would be divergent because for each term of the form dq_i/q_i , we only have a term of the form $\log |q_i|$ of degree $-1/2$, while the respective integrals over the components of η_1 have an excess of convergence because for each term of the form dq_i/q_i , we have a term of the form $\log |q_i|$ of degree -1 , which is more than what is needed. Luckily, the factor $\sum_{i \in I_0^c} |\log |q_i|| / \sum_{i=0}^d |\log |q_i||$ in (5.16) will allow us to transfer the excess of convergence from η_1 to η_2 .

Before doing so, we observe that the integrand of (5.12) vanishes unless $r = g$. Indeed, the differential form η_1 is the pull-back of a differential form on the base $\mathcal{A}_{g,N}(\mathbb{C})$ that has complex dimension $d' = d - g$. Therefore, η_1 vanishes unless $d - r \leq d' = d - g$. In other words, η_1 vanishes unless $r \geq g$. On the other hand, equation (5.15) shows that η_2 vanishes unless $r \leq g$, since the

rank of the Hessian matrix of φ is at most g by Lemma 5.18. We thus conclude that the integrand of the integral (5.12) vanishes unless $r = g$.

Finally, we have all the ingredients in place to prove Theorem 5.26. Let $J, K \subseteq \{1, \dots, d\}$ be subsets of cardinality g . As before, we write $I = J \cap K$, and we further set $I' = J^c \cap K^c$ and $I'' = \{1, \dots, d\} \setminus (I \cup I')$. Combining the upper bounds (5.13) and (5.16) on $V \cap U$, we obtain

$$|f_{1,J^c,K^c} f_{2,J,K}| \leq C \frac{\sum_{i \in I_0^c} |\log |q_i||}{\sum_{i=0}^d |\log |q_i||} \frac{1}{\prod_{j \in I} |\log |q_j|| \prod_{k \in I''} |\log |q_k||^{3/2} \prod_{\ell \in I'} |\log |q_\ell||^2} \prod_{i=1}^d \frac{1}{|q_i|^2},$$

where $C > 0$ is again a constant. Here we see that we have a potential problem of convergence with respect to the integration along the variables q_j for $j \in I$, which can be resolved using Lemma 5.25. Namely, using this lemma we deduce that there is a further constant $C' > 0$ such that

$$\begin{aligned} \frac{\sum_{i \in I_0^c} |\log |q_i||}{\sum_{i=0}^d |\log |q_i||} \left(\prod_{j \in I} |\log |q_j|| \right)^{1/n} &\leq C' \frac{\sum_{j \in I} |\log |q_j||}{\sum_{i=0}^d |\log |q_i||} \prod_{i \in I_0^c} |\log |q_i|| \leq \\ C' \prod_{i \in I_0^c} |\log |q_i|| &= C' \prod_{i \in I^c} |\log |q_i|| = C' \prod_{k \in I''} |\log |q_k|| \prod_{\ell \in I'} |\log |q_\ell||, \end{aligned}$$

where n denotes the cardinality of I . Rearranging terms and taking the fourth root immediately leads to the inequality

$$\left(\frac{\sum_{i \in I_0^c} |\log |q_i||}{\sum_{i=0}^d |\log |q_i||} \right)^{1/4} \frac{1}{\prod_{k \in I''} |\log |q_k||^{1/4} \prod_{\ell \in I'} |\log |q_\ell||^{1/4}} \leq C' \frac{1}{\prod_{j \in I} |\log |q_j||^{1/4n}}.$$

With this inequality at hand and observing that $\sum_{i \in I_0^c} |\log |q_i|| / \sum_{i=0}^d |\log |q_i|| \leq 1$, we arrive with a further constant $C'' > 0$ at the bound

$$|f_{1,J^c,K^c} f_{2,J,K}| \leq C'' \frac{1}{\prod_{j \in I} |\log |q_j||^{1+1/4n} \prod_{k \in I''} |\log |q_k||^{1+1/4} \prod_{\ell \in I'} |\log |q_\ell||^{1+3/4}} \prod_{i=1}^d \frac{1}{|q_i|^2},$$

which is integrable on $V \cap U$ and thus proves Theorem 5.26. \square

Once we know that the arithmetic self-intersection product of the adelic arithmetic line bundle $\overline{\mathcal{F}}_{k,m,N}$ of Siegel–Jacobi forms of weight k , index m , and level N is well-defined, it can easily be computed using the functorial properties discussed in Subsection 4.3. As in the beginning of Subsection 5.6, let s_ω be a rational \mathbb{Q} -section of ω_N and $s_{\mathcal{L}}$ a rational \mathbb{Q} -section of \mathcal{L}_N , which allow us to define the adelic arithmetic divisor

$$\overline{\mathcal{D}}_{k,m,N} = \widehat{\text{div}}_{\overline{\mathcal{F}}_{k,m,N}}(s_\omega^{\otimes k} \otimes s_{\mathcal{L}}^{\otimes m})$$

on $\mathcal{B}_{g,N}$ corresponding to the adelic arithmetic line bundle $\overline{\mathcal{F}}_{k,m,N}$. In the same way, we define the adelic arithmetic divisor

$$\overline{\mathcal{E}}_{k,N} = \widehat{\text{div}}_{\overline{\omega}_N^{\otimes k}}(s_\omega^{\otimes k})$$

on $\mathcal{A}_{g,N}$ corresponding to the adelic arithmetic line bundle $\overline{\omega}_N^{\otimes k}$ of Siegel modular forms of weight k and level N .

5.27. Theorem. *With the above notations, we have*

$$\overline{\mathcal{D}}_{k,m,N}^{d+1} = \frac{(d+1)!}{(d'+1)!} k^{d'+1} m^{g2g} \overline{\mathcal{E}}_{1,N}^{d'+1},$$

where we recall that $d' = g(g+1)/2$ and $d = d' + g$.

Proof. By the multilinearity of the arithmetic intersection product, we compute

$$\overline{\mathcal{D}}_{k,m,N}^{d+1} = \sum_{j=0}^{d+1} \binom{d+1}{j} k^{d+1-j} m^j \overline{\mathcal{D}}_{1,0,N}^{d+1-j} \overline{\mathcal{D}}_{0,1,N}^j.$$

The next step is the global arithmetic analogue of the fact used in course of the proof of Theorem 5.26 that the arithmetic intersection products $\overline{\mathcal{D}}_{1,0,N}^{d+1-j} \overline{\mathcal{D}}_{0,1,N}^j$ vanish unless $j = g$. Indeed, let $[2]: \mathcal{B}_{g,N} \rightarrow \mathcal{B}_{g,N}$ be the morphism given by fiberwise multiplication by 2, which has degree 2^{2g} . We will then compute the arithmetic intersection product $([2]_{\text{adel}}^* \overline{\mathcal{D}}_{1,0,N})^{d+1-j} ([2]_{\text{adel}}^* \overline{\mathcal{D}}_{0,1,N})^j$ in two ways. In the first case, we use that $\overline{\mathcal{D}}_{1,0,N}$ is the pull-back of an adelic arithmetic divisor from the base $\mathcal{A}_{g,N}$, whence $[2]_{\text{adel}}^* \overline{\mathcal{D}}_{1,0,N} = \overline{\mathcal{D}}_{1,0,N}$, while Lemma 5.14 shows that $[2]_{\text{adel}}^* \overline{\mathcal{D}}_{0,1,N} \sim 2^2 \overline{\mathcal{D}}_{0,1,N}$ (linear equivalence of adelic arithmetic divisors). Therefore, we find in the first case

$$([2]_{\text{adel}}^* \overline{\mathcal{D}}_{1,0,N})^{d+1-j} ([2]_{\text{adel}}^* \overline{\mathcal{D}}_{0,1,N})^j = 2^{2j} \overline{\mathcal{D}}_{1,0,N}^{d+1-j} \overline{\mathcal{D}}_{0,1,N}^j.$$

In the second case, we use Proposition 4.9 to deduce

$$([2]_{\text{adel}}^* \overline{\mathcal{D}}_{1,0,N})^{d+1-j} ([2]_{\text{adel}}^* \overline{\mathcal{D}}_{0,1,N})^j = 2^{2g} \overline{\mathcal{D}}_{1,0,N}^{d+1-j} \overline{\mathcal{D}}_{0,1,N}^j,$$

which proves the claim. As a consequence, we derive

$$\overline{\mathcal{D}}_{k,m,N}^{d+1} = \binom{d+1}{g} k^{d'+1} m^g \overline{\mathcal{D}}_{1,0,N}^{d'+1} \overline{\mathcal{D}}_{0,1,N}^g.$$

Finally, we use that for any fiber F of the projection $\mathcal{B}_{g,N} \rightarrow \mathcal{A}_{g,N}$, the restriction of \mathcal{L}_N to F equals twice the principal polarization of the abelian variety F . We deduce that $\deg(\mathcal{D}_{0,1,N}|_F) = 2^g g!$. Therefore, Proposition 4.8 implies that

$$\overline{\mathcal{E}}_{k,m,N}^{d+1} = \binom{d+1}{g} k^{d'+1} m^g 2^g g! \overline{\mathcal{D}}_{1,0,N}^{d'+1} = \frac{(d+1)!}{(d'+1)!} k^{d'+1} m^g 2^g \overline{\mathcal{E}}_{1,N}^{d'+1},$$

which proves the theorem. \square

5.28. Remark. When $g = 1$, the arithmetic self-intersection product of the line bundle of modular forms has been computed independently by U. Kühn in [Küh01] and J.-B. Bost in [Bos99]. After normalizing the Faltings metric $\|\cdot\|_{\text{Fal}}$ on the Hodge bundle $\overline{\omega}_N$ by the factor $1/(4\pi)$, they obtained

$$\overline{\mathcal{E}}_{1,N}^2 = \frac{2}{[\text{SL}_2(\mathbb{Z}) : \Gamma(N)]} \left(\frac{1}{2} \zeta_{\mathbb{Q}}(-1) + \zeta'_{\mathbb{Q}}(-1) \right) = -\deg(\mathcal{E}_{1,N}) \left(\frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + \frac{1}{2} \right),$$

where $\zeta_{\mathbb{Q}}(s)$ denotes the classical Riemann zeta function. For general g and $N = 1$, upon suitable normalizations, it is conjectured (see also [MR02]) that a formula of the type

$$\overline{\mathcal{E}}_{1,1}^{d'+1} = -\deg(\mathcal{E}_{1,1}) \left(\frac{\zeta'_{\mathbb{Q}}(1-2g)}{\zeta_{\mathbb{Q}}(1-2g)} + \frac{\zeta'_{\mathbb{Q}}(3-2g)}{\zeta_{\mathbb{Q}}(3-2g)} + \dots + \frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + a \log(2) + b \right)$$

with $a, b \in \mathbb{Q}$ holds. This has been mostly verified in the paper [JvP22] by B. Jung and A. Pippich for $g = 2$. There is further confirmation of an analogue of this formula for Hilbert modular surfaces in the paper [BBGK07]. Recall that Siegel's computation of the volume of the fundamental domain of $\text{Sp}(2g, \mathbb{Z})$ gives

$$\deg(\mathcal{E}_{1,1}) = (-1)^{d'} \zeta_{\mathbb{Q}}(1-2g) \cdot \zeta_{\mathbb{Q}}(3-2g) \cdot \dots \cdot \zeta_{\mathbb{Q}}(-1).$$

We wonder whether Theorem 5.27 might be used to compute the arithmetic self-intersection product of the line bundle of Siegel modular forms by an inductive argument.

REFERENCES

- [BEGHdJ21] A. M. Botero, J. I. Burgos Gil, D. Holmes, and R. de Jong. Chern-Weil and Hilbert-Samuel formulae for singular hermitian line bundles. arXiv:2112.09007 [math.AG], 2021.
- [BBGK07] J. Bruinier, J. I. Burgos Gil, and U. Kühn. Borchers products and arithmetic intersection theory on Hilbert modular surfaces. *Duke Math. J.*, 139(1):1–88, 2007.
- [BEGZ10] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi. Monge–Ampère equations in big cohomology classes. *Acta Math.*, 205(2):199–262, 2010.
- [BFJ16] S. Boucksom, C. Favre, and M. Jonsson. Singular semipositive metrics in non-Archimedean geometry. *J. Algebraic Geom.*, 25(1):77–139, 2016.
- [BGHdJ18] J. I. Burgos Gil, D. Holmes, and R. de Jong. Singularities of the biextension metric for families of abelian varieties. *Forum Math. Sigma*, 6:e12, 56, 2018.

- [BGKK05] J. I. Burgos Gil, J. Kramer, and U. Kühn. Arithmetic characteristic classes of automorphic vector bundles. *Documenta Math.*, 10:619–716, 2005.
- [BGKK07] J. I. Burgos Gil, J. Kramer, and U. Kühn. Cohomological arithmetic Chow rings. *J. Inst. Math. Jussieu*, 6(1):1–172, 2007.
- [BGKK16] J. I. Burgos Gil, J. Kramer, and U. Kühn. The singularities of the invariant metric on the Jacobi line bundle. In *Recent advances in Hodge theory*, volume 427 of *London Math. Soc. Lecture Note Ser.*, pages 45–77. Cambridge Univ. Press, Cambridge, 2016.
- [BGMP16] J. I. Burgos Gil, A. Moriwaki, P. Philippon, and M. Sombra. Arithmetic positivity on toric varieties. *J. Algebraic Geom.*, 25(2):201–272, 2016.
- [BGPS14] J. I. Burgos Gil, P. Philippon, and M. Sombra. Arithmetic geometry of toric varieties. Metrics, measures and heights. *Astérisque*, 360:vi+222, 2014.
- [Bos99] J.-B. Bost. Potential theory and Lefschetz theorems for arithmetic surfaces. *Ann. Sci. École Norm. Sup. (4)*, 32(2), 1999.
- [Bou04] S. Boucksom. Divisorial Zariski decompositions on compact complex manifolds. *Ann. Sci. École Norm. Sup. (4)*, 37(1):45–76, 2004.
- [BV04] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge University Press, Cambridge, 2004.
- [DDNL18a] T. Darvas, E. Di Nezza, and C. H. Lu. L^1 metric geometry of big cohomology classes. *Ann. Inst. Fourier (Grenoble)*, 68(7):3053–3086, 2018.
- [DDNL18b] T. Darvas, E. Di Nezza, and C. H. Lu. Monotonicity of nonpluripolar products and complex Monge–Ampère equations with prescribed singularity. *ANALYSIS AND PDE*, 11(8):2049–2087, 2018.
- [DDNL23] T. Darvas, E. Di Nezza, and C. H. Lu. Relative pluripotential theory on compact Kähler manifolds. arXiv:2303.11584, 2023.
- [Dem92a] J. P. Demailly. Regularization of closed positive currents and intersection theory. *J. Algebraic Geom.*, 1(3):361–409, 1992.
- [Dem92b] J. P. Demailly. Singular Hermitian metrics on positive line bundles. In *Complex algebraic varieties (Bayreuth, 1990)*, volume 1507 of *Lecture Notes in Math.*, pages 87–104. Springer, Berlin, 1992.
- [Dem12] J. P. Demailly. Complex Analytic and Differential Geometry, 2012. Unpublished book available at <https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf>.
- [DF20] N. B. Dang and C. Favre. Intersection theory of nef b-divisor classes. arXiv:2007.04549 [math.AG], 2020.
- [DN15] E. Di Nezza. Stability of Monge–Ampère energy classes. *J. Geom. Anal.*, 25:2565–2589, 2015.
- [FC90] G. Faltings and C. L. Chai. *Degeneration of abelian varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990. With an appendix by David Mumford.
- [GS90] H. Gillet and C. Soulé. Arithmetic intersection theory. *Inst. Hautes Études Sci. Publ. Math.*, 72:93–174 (1991), 1990.
- [GZ17] V. Guedj and A. Zeriahi. *Degenerate complex Monge–Ampère equations*, volume 26 of *EMS Tracts in Mathematics*. EMS, 2017.
- [Hör94] L. Hörmander. *Notions of convexity*, volume 127 of *Progress in Mathematics*. Birkhäuser, 1994.
- [JvP22] B. Jung and A.-M. von Pippich. The arithmetic volume of the moduli space of abelian surfaces. arXiv:2205.11864 [math.AG], 2022.
- [Kim10] D. Kim. L^2 extension of adjoint line bundle sections. *Ann. Inst. Fourier (Grenoble)*, 60(4):1435–1477, 2010.
- [Kra95] J. Kramer. An arithmetic theory of Jacobi forms in higher dimensions. *J. Reine Angew. Math.*, 458:157–182, 1995.
- [KRY06] S. Kudla, S. M. Rapoport, and T. Yang. *Modular forms and special cycles on Shimura curves*, volume 161 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2006.
- [Küh01] U. Kühn. Generalized arithmetic intersection numbers. *J. Reine Angew. Math.*, 534:209–236, 2001.
- [MR02] V. Maillot and D. Roessler. Conjectures sur les dérivées logarithmiques des fonctions L d’Artin aux entiers négatifs. *Math. Res. Lett.*, 9(5-6):715–724, 2002.
- [Mum77] D. Mumford. Hirzebruch’s proportionality theorem in the noncompact case. *Invent. Math.*, 42:239–272, 1977.
- [NO90] J. Noguchi and T. Ochiai. *Geometric function theory in several complex variables*, volume 80 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1990.
- [Sie36] C. L. Siegel. The volume of the fundamental domain for some infinite groups. *Trans. Amer. Math. Soc.*, 39(2):209–218, 1936.
- [Sou92] C. Soulé. *Lectures on Arakelov geometry*, volume 33 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1992. With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer.
- [YZ21] X. Yuan and S. W. Zhang. Adelic line bundles over quasi-projective varieties. arXiv:2105.13587 [math.NT], 2021.
- [Zha95] S. Zhang. Positive line bundles on arithmetic varieties. *Journal of AMS*, 8(1):187–221, 1995.
- [Zie89] C. Ziegler. Jacobi forms of higher degree. *Abh. Math. Sem. Univ. Hamburg*, 59:191–224, 1989.

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