Relating Siegel cusp forms to Siegel–Maaß forms

Jürg Kramer and Antareep Mandal*

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Abstract

In this paper we generalize a well-known isomorphism between the space of cusp forms of weight k for a Fuchsian subgroup of the first kind $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ and the space of certain Maaß forms of weight k for Γ to an isomorphism between the space of Siegel cusp forms of weight k for a subgroup $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$, which is commensurable with the Siegel modular group $\mathrm{Sp}_n(\mathbb{Z})$, and a suitable space of Siegel–Maaß forms of weight k for Γ.

1 Introduction

Let $\mathbb{H} := \{z = x + iy \in \mathbb{C} \,|\, y > 0\}$ denote the upper half-plane and $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ a Fuchsian subgroup of the first kind, which acts by fractional linear transformations on $\mathbb{H}$. Let $\mathcal{S}_k(\Gamma)$ denote the space of cusp forms of weight k for Γ and let $\mathcal{H}_k(\Gamma)$ denote the space of real-analytic automorphic forms of weight k for Γ, on which the Maaß Laplacian

$$\Delta_k := y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iky\frac{\partial}{\partial x}$$

of weight k acts. Then, it is well-known that there is an isomorphism

$$\mathcal{S}_k(\Gamma) \cong \ker\left(\Delta_k + \frac{k}{2}\left(1 - \frac{k}{2}\right)\mathrm{id}\right) \tag{1.1}$$

of $\mathbb{C}$-vector spaces, induced by the assignment $f \mapsto y^{k/2}f$, where the right-hand side consists of the Maaß forms in $\mathcal{H}_k(\Gamma)$ with eigenvalue $k/2(1-k/2)$ of Δ_k. This identification of two types of automorphic forms for Γ has various useful applications. For example, in the article [2], the isomorphism (1.1) was crucial in relating the sup-norm bound problem for cusp forms of weight k for Γ to bounds for the heat kernel for Δ_k on the quotient space $\Gamma\backslash\mathbb{H}$.

In this paper, we attempt a generalization of the isomorphism (1.1) to the Siegel modular setting, which, to our surprise, we could not find in the literature. Letting $\mathrm{Sym}_n(\mathbb{C})$ be the set of complex symmetric $(n \times n)$-matrices, we let $\mathbb{H}_n := \{Z = X + iY \in \mathrm{Sym}_n(\mathbb{C}) \,|\, Y > 0\}$ denote the Siegel upper half-space of degree $n \geq 1$ and we let $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ denote a subgroup acting by generalized fractional linear transformations on $\mathbb{H}_n$, which is commensurable with the Siegel modular group $\mathrm{Sp}_n(\mathbb{Z})$. Then, let $\mathcal{S}_k^n(\Gamma)$ denote the space of Siegel cusp forms of weight k and degree n for Γ and let $\mathcal{H}_k^n(\Gamma)$ be the space of real-analytic automorphic forms of weight k and degree n for Γ, on which the Siegel–Maaß Laplacian

$$\Delta_k := \mathrm{tr}\left(Y\left(\left(Y\frac{\partial}{\partial X}\right)^t\frac{\partial}{\partial X} + \left(Y\frac{\partial}{\partial Y}\right)^t\frac{\partial}{\partial Y}\right) - ikY\frac{\partial}{\partial X}\right) \tag{1.2}$$

of weight k acts. As the main result of this paper, we show in Corollary 5.4 the isomorphism

$$\mathcal{S}_k^n(\Gamma) \cong \ker\left(\Delta_k + \frac{nk}{4}(n - k + 1)\mathrm{id}\right), \tag{1.3}$$

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of $\mathbb{C}$-vector spaces, induced by the assignment $f \mapsto \det(Y)^{k/2} f$, thereby generalizing the isomorphism (1.1) to the Siegel modular setting. The right-hand side of (1.3) now consists of the Siegel–Maaß forms in $\mathcal{H}_k^n(\Gamma)$ with eigenvalue $nk(n-k+1)/4)$ of Δ_k.

In case $n = 1$, the isomorphism (1.1) is obtained as a by-product of the proof of the symmetry of the Maaß Laplacian Δ_k (see [4]). The most straightforward proof of the symmetry of Δ_k is obtained by constructing a suitable $\mathrm{SL}_2(\mathbb{R})$-invariant differential form using the raising or the lowering operators on $\mathbb{H}$, and then integrating it over the quotient space $\Gamma\backslash\mathbb{H}$ (see [1], p. 135). Generalizations of all these operators as well as their transformation behaviour under the action of the symplectic group $\mathrm{Sp}_n(\mathbb{R})$ to the Siegel modular setting have been provided by Maaß in [3]. However, in spite of all these crucial ingredients being around for a long time, we could not find in the literature a precise proof of the symmetry of the Siegel–Maaß Laplacian Δ_k of weight k. We provide a complete proof of the symmetry of Δ_k in Theorem 5.1, where we construct the appropriate $\mathrm{Sp}_n(\mathbb{R})$-invariant differential form on $\mathbb{H}_n$, which, while calculationally a bit cumbersome, is conceptually a rather straightforward piecing-together of Maaß's calculations. Our main result in Corollary 5.4 is then a consequence of Theorem 5.1.

As indicated above, the generalization of the isomorphism (1.1) will perspectively allow us, among others, to relate the sup-norm bound problem for Siegel cusp forms of weight k and degree n for Γ to bounds for the heat kernel for the Siegel–Maaß Laplacian Δ_k on the quotient space $\Gamma\backslash\mathbb{H}_n$.

This paper is organized as follows: In section 2, we provide a quick summary of the basics of the Siegel upper half-space and Siegel modular forms. In the subsequent two sections 3 and 4, we introduce and discuss the transformation behaviour of the relevant operators in the Siegel modular setting. This material is already present in [3], but due to sub-optimal typesetting, at places, it is hard to decipher. So we take this opportunity to redo these calculations along Maaß's lines and present them here for the reader's convenience. However, no claim of originality is made here on this material. Finally in section 5, piecing together Maaß's results, we construct the appropriate $\mathrm{Sp}_n(\mathbb{R})$-invariant differential form on $\mathbb{H}_n$ to show the symmetry of the Siegel–Maaß Laplacian Δ_k, and then use it to show the generalization (1.3) of the isomorphism (1.1).

2 Basic notations and definitions

For $n \in \mathbb{N}_{>0}$ and a commutative ring R, let $\mathrm{M}_n(R)$ denote the set of $(n \times n)$-matrices with entries in R and $\mathrm{Sym}_n(R)$ the set of symmetric matrices in $\mathrm{M}_n(R)$. The Siegel upper half-space $\mathbb{H}_n$ of degree n is then defined by

$$\mathbb{H}_n := \{Z = X + iY \in \mathrm{M}_n(\mathbb{C}) \,|\, X, Y \in \mathrm{Sym}_n(\mathbb{R}) :\, Y > 0\}.$$

The symplectic group $\mathrm{Sp}_n(\mathbb{R})$ of degree n is defined by

$$\mathrm{Sp}_n(\mathbb{R}) := \{\gamma \in \mathrm{M}_{2n}(\mathbb{R}) \,|\, \gamma^t J_n \gamma = J_n\},$$

where $J_n \in \mathrm{M}_{2n}(\mathbb{R})$ is the skew-symmetric matrix

$$J_n := \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$$

with $\mathbb{1}_n$ denoting the identity matrix of $\mathrm{M}_n(\mathbb{R})$. Writing an element $\gamma \in \mathrm{Sp}_n(\mathbb{R})$ in block form as

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $A, B, C, D \in \mathrm{M}_n(\mathbb{R})$, we can recast the relation $\gamma^t J_n \gamma = J_n$ as the set of relations

$$A^t C = C^t A, \quad B^t D = D^t B, \quad A^t D - C^t B = \mathbb{1}_n. \tag{2.1}$$

Observing that with $\gamma \in \mathrm{Sp}_n(\mathbb{R})$, we also have $\gamma^t \in \mathrm{Sp}_n(\mathbb{R})$, the set of relations (2.1) turns out to be equivalent to the relations

$$AB^t = BA^t, \quad CD^t = DC^t, \quad AD^t - BC^t = \mathbb{1}_n.$$

The group $\mathrm{Sp}_n(\mathbb{R})$ acts by the symplectic action

$$\mathbb{H}_n \ni Z \mapsto \gamma Z = (AZ+B)(CZ+D)^{-1} \qquad \left(\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})\right) \tag{2.2}$$

on $\mathbb{H}_n$. Using the equality

$$(AZ+B)(CZ+D)^{-1} = (CZ+D)^{-t}(AZ+B)^t, \tag{2.3}$$

which follows from equation (2.1), we compute

$$(CZ+D)^t \operatorname{Im}(\gamma Z)(C\overline{Z}+D) = \operatorname{Im}(Z),$$

from which we derive the important relation

$$\operatorname{Im}(\gamma Z) = (CZ+D)^{-t} \operatorname{Im}(Z)(C\overline{Z}+D)^{-1} \tag{2.4}$$

giving rise to the determinant relation

$$\det(\operatorname{Im}(\gamma Z)) = \frac{\det(\operatorname{Im}(Z))}{|\det(CZ+D)|^2}.$$

For the differential of the symplectic action (2.2), we compute using again equation (2.3)

$$\begin{aligned} \mathrm{d}(\gamma Z) &= A\,\mathrm{d}Z(CZ+D)^{-1} - (AZ+B)(CZ+D)^{-1}C\,\mathrm{d}Z(CZ+D)^{-1} \\ &= (CZ+D)^{-t}((CZ+D)^t A\,\mathrm{d}Z - (AZ+B)^t C\,\mathrm{d}Z)(CZ+D)^{-1}. \end{aligned}$$

Using once again equation (2.1), this gives

$$\mathrm{d}(\gamma Z) = (CZ+D)^{-t}\,\mathrm{d}Z\,(CZ+D)^{-1}. \tag{2.5}$$

The arclength $\mathrm{d}s_n^2$ and the volume form $\mathrm{d}\mu_n$ on $\mathbb{H}_n$ in terms of $Z = (z_{j,k})_{1\le j\le k\le n} \in \mathbb{H}_n$ are given by

$$\mathrm{d}s_n^2(Z) = \operatorname{tr}(Y^{-1}\,\mathrm{d}Z\,Y^{-1}\,\mathrm{d}\overline{Z}) \qquad (Z = X+iY),$$

$$\mathrm{d}\mu_n(Z) = \frac{\bigwedge\limits_{1\le j\le k\le n} \mathrm{d}x_{j,k}\wedge \mathrm{d}y_{j,k}}{\det(Y)^{n+1}} \qquad (z_{j,k} = x_{j,k}+iy_{j,k}).$$

From equations (2.4) and (2.5) it is obvious that the arclength $\mathrm{d}s_n^2$ and the volume form $\mathrm{d}\mu_n$ on $\mathbb{H}_n$ given by the above equations are invariant under the symplectic action. Corresponding to this metric, we have the Laplace–Beltrami operator Δ on $\mathbb{H}_n$, called the Siegel Laplacian, which is also invariant under the symplectic action.

Definition 2.1. Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ be a subgroup commensurable with $\mathrm{Sp}_n(\mathbb{Z})$, i.e., the intersection $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ is a finite index subgroup of Γ as well as of $\mathrm{Sp}_n(\mathbb{Z})$. We let $\gamma_j \in \mathrm{Sp}_n(\mathbb{Z})$ $(j = 1, \ldots, h)$ denote a set of representatives for the left cosets of $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ in $\mathrm{Sp}_n(\mathbb{Z})$. Then, a *Siegel modular form of weight k and degree n for* Γ is a function $f\colon \mathbb{H}_n \longrightarrow \mathbb{C}$ satisfying the following conditions:

(i) f is holomorphic;

(ii) $f(\gamma Z) = \det(CZ+D)^k f(Z)$ for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$;

(iii) given $Y_0 \in \mathrm{Sym}_n(\mathbb{R})$ with $Y_0 > 0$, the quantities $\det(C_jZ+D_j)^{-k} f(\gamma_j Z)$ are bounded in the region $\{Z = X+iY \in \mathbb{H}_n \,|\, Y \ge Y_0\}$ for the set of representatives $\gamma_j = \left(\begin{smallmatrix} A_j & B_j \\ C_j & D_j \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$ $(j = 1, \ldots, h)$.

Moreover, a Siegel modular form f as above is called a *Siegel cusp form of weight k and degree n for* Γ if condition (iii) above is strengthened to the condition

(iii') given $Y_0 \in \mathrm{Sym}_n(\mathbb{R})$ with $Y_0 \gg 0$, the quantities $\det(C_jZ+D_j)^{-k} f(\gamma_j Z)$ become arbitrarily small in the region $\{Z = X+iY \in \mathbb{H}_n \,|\, Y \ge Y_0\}$ for the set of representatives $\gamma_j = \left(\begin{smallmatrix} A_j & B_j \\ C_j & D_j \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$ $(j = 1, \ldots, h)$.

Remark 2.2. The sets of Siegel modular forms and Siegel cusp forms of weight k and degree n for Γ have the structure of $\mathbb{C}$-vector spaces, which we denote by $\mathcal{M}_k^n(\Gamma)$ and $\mathcal{S}_k^n(\Gamma)$, respectively, and which turn out to be finite dimensional. Moreover, the space $\mathcal{S}_k^n(\Gamma)$ is equipped with the so-called Petersson inner product given by

$$\langle f, g\rangle := \int_{\Gamma\backslash\mathbb{H}_n} \det(Y)^k f(Z)\overline{g}(Z)\,\mathrm{d}\mu_n(Z) \qquad (f, g \in \mathcal{S}_k^n(\Gamma)),$$

making $\mathcal{S}_k^n(\Gamma)$ into a hermitian inner product space.

3 Siegel–Maaß Laplacian of weight (α, β)

In this section, we will recall from [3] various differential operators acting on smooth complex valued functions defined on $\mathbb{H}_n$. In particular, we will define the Siegel–Maaß Laplacian of weight (α, β), where $\alpha, \beta \in \mathbb{R}$. Letting $\alpha = k/2$ and $\beta = -k/2$ will then lead us to the Siegel–Maaß Laplacian Δ_k mentioned in formula (1.2) in the introduction. We point out that the Siegel Laplacian Δ mentioned in the previous section and the Siegel–Maaß Laplacian Δ_k are related by the formula

$$\Delta_k = \Delta - ik \operatorname{tr}\left(Y \frac{\partial}{\partial X} \right)$$

with the symmetric $(n \times n)$-matrix $\partial/\partial X$ of partial derivatives being defined below.

Given $Z = X + iY \in \mathbb{H}_n$, we start by introducing the following symmetric $(n \times n)$-matrices of partial derivatives:

$$\begin{aligned}
&\text{(i)} && \left(\frac{\partial}{\partial X} \right)_{j,k} := \frac{1 + \delta_{j,k}}{2} \frac{\partial}{\partial x_{j,k}}, \\
&\text{(ii)} && \left(\frac{\partial}{\partial Y} \right)_{j,k} := \frac{1 + \delta_{j,k}}{2} \frac{\partial}{\partial y_{j,k}}, \\
&\text{(iii)} && \frac{\partial}{\partial Z} := \frac{1}{2} \left(\frac{\partial}{\partial X} - i \frac{\partial}{\partial Y} \right), \\
&\text{(iv)} && \frac{\partial}{\partial \overline{Z}} := \frac{1}{2} \left(\frac{\partial}{\partial X} + i \frac{\partial}{\partial Y} \right),
\end{aligned}$$

where $\delta_{j,k}$ is the Kornecker delta symbol.

Definition 3.1. Following Maaß [3], we define, using the above notations, for arbitrary real numbers $\alpha, \beta \in \mathbb{R}$, the following $(n \times n)$-matrices of differential operators acting on smooth complex valued functions on $\mathbb{H}_n$:

$$\begin{aligned}
&\text{(i)} && K_\alpha := (Z - \overline{Z}) \frac{\partial}{\partial Z} + \alpha \mathbb{1}_n, \\
&\text{(ii)} && \Lambda_\beta := (Z - \overline{Z}) \frac{\partial}{\partial \overline{Z}} - \beta \mathbb{1}_n, \\
&\text{(iii)} && \Omega_{\alpha,\beta} := \Lambda_{\beta-(n+1)/2} K_\alpha + \alpha(\beta - (n+1)/2) \mathbb{1}_n, \\
&\text{(iv)} && \widetilde{\Omega}_{\alpha,\beta} := K_{\alpha-(n+1)/2} \Lambda_\beta + \beta(\alpha - (n+1)/2) \mathbb{1}_n.
\end{aligned}$$

Next, we want to expand $\Omega_{\alpha,\beta}$ and $\widetilde{\Omega}_{\alpha,\beta}$ in terms of $Z, \overline{Z}, \partial/\partial Z$, and $\partial/\partial \overline{Z}$. For that we need the following lemma.

Lemma 3.2. *Let $C, D \colon \mathbb{H}_n \longrightarrow \mathrm{M}_n(\mathbb{C})$ be smooth matrix valued functions depending on Z and $\overline{Z}$. Then, the following matrix operator identities hold:*
(i) *Assuming that $\partial C/\partial Z = 0$ and $\partial D/\partial Z = 0$, we have*

$$\frac{\partial}{\partial Z} (CZ + D)^t = \left((CZ + D) \frac{\partial}{\partial Z} \right)^t + \frac{1}{2}(n+1) C^t.$$

(ii) *Assuming that $\partial C/\partial \overline{Z} = 0$ and $\partial D/\partial \overline{Z} = 0$, we have*

$$\frac{\partial}{\partial \overline{Z}} (C\overline{Z} + D)^t = \left((C\overline{Z} + D) \frac{\partial}{\partial \overline{Z}} \right)^t + \frac{1}{2}(n+1) C^t.$$

Proof. Since part (ii) follows from part (i) by conjugation, we prove only (i). Let Φ be a matrix depending on Z and $\overline{Z}$ such that the product $(CZ + D)^t \Phi$ makes sense. Then, writing the (j, k)-th entry of the matrix $\partial/\partial Z (CZ + D)^t \Phi$ as the sum

$$\left(\frac{\partial}{\partial Z} (CZ + D)^t \Phi \right)_{j,k} = \sum_{l,m=1}^{n} \left(\frac{\partial}{\partial Z} \right)_{j,l} \left((CZ + D)^t_{l,m} \Phi_{m,k} \right)$$

and noting that $\partial Z/\partial z_{j,l} = (1-\delta_{j,l})E_{j,l} + E_{l,j}$, where $E_{j,k} \in \mathrm{M}_n(\mathbb{C})$ is the matrix with its (j,k)-th entry being 1 and the remaining entries being 0, elementary calculations lead us to the operator identity

$$\frac{\partial}{\partial Z}\,(CZ+D)^t = \left((CZ+D)\,\frac{\partial}{\partial Z}\right)^t + \frac{1}{2}(n+1)C^t,$$

which is what we needed to prove. ☐

Corollary 3.3. *For $Z \in \mathbb{H}_n$, the following operator identities hold:*

$$\text{(i)}\quad \frac{\partial}{\partial Z}\,(Z-\overline{Z}) = \left((Z-\overline{Z})\,\frac{\partial}{\partial Z}\right)^t + \frac{1}{2}(n+1)\mathbb{1}_n,$$

$$\text{(ii)}\quad \frac{\partial}{\partial \overline{Z}}\,(Z-\overline{Z}) = \left((Z-\overline{Z})\,\frac{\partial}{\partial \overline{Z}}\right)^t - \frac{1}{2}(n+1)\mathbb{1}_n.$$

Proof. As $\partial\overline{Z}/\partial Z = 0$, we can choose $C = \mathbb{1}_n$ and $D = -\overline{Z}$ in Lemma 3.2 (i), from which the first claimed formula follows. The second formula follows analogously. ☐

Using the above corollary, one can expand $\Omega_{\alpha,\beta}$ and $\widetilde{\Omega}_{\alpha,\beta}$ as

$$\Omega_{\alpha,\beta} = (Z-\overline{Z})\left((Z-\overline{Z})\,\frac{\partial}{\partial \overline{Z}}\right)^t \frac{\partial}{\partial Z} + \alpha(Z-\overline{Z})\,\frac{\partial}{\partial \overline{Z}} - \beta(Z-\overline{Z})\,\frac{\partial}{\partial Z}\,,$$

$$\widetilde{\Omega}_{\alpha,\beta} = (Z-\overline{Z})\left((Z-\overline{Z})\,\frac{\partial}{\partial Z}\right)^t \frac{\partial}{\partial \overline{Z}} + \alpha(Z-\overline{Z})\,\frac{\partial}{\partial \overline{Z}} - \beta(Z-\overline{Z})\,\frac{\partial}{\partial Z}\,.$$

Then, $\Omega_{\alpha,\beta}$ and $\widetilde{\Omega}_{\alpha,\beta}$ are related by the identity

$$\widetilde{\Omega}_{\alpha,\beta} = (Z-\overline{Z})\big((Z-\overline{Z})^{-1}\Omega_{\alpha,\beta}\big)^t.$$

Definition 3.4. The operator $\Delta_{\alpha,\beta} := -\operatorname{tr}(\Omega_{\alpha,\beta}) = -\operatorname{tr}(\widetilde{\Omega}_{\alpha,\beta})$ is called the *Siegel–Maaß Laplacian of weight* (α,β).

4 Transformation behaviour of Maaß operators

Recall that the symplectic action of $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$ on the point $Z \in \mathbb{H}_n$ is given by

$$\gamma Z = (AZ+B)(CZ+D)^{-1} = (CZ+D)^{-t}(AZ+B)^t;$$

to avoid cumbersome notation, we will use in this paper sometimes the shorthand $Z^\gamma := \gamma Z$. In this section, we will then study the transformation behaviour of the Maaß operators introduced in Definition 3.1 by expressing the operators K_α^γ, Λ_β^γ, $\Omega_{\alpha,\beta}^\gamma$ obtained by replacing $Z, \overline{Z}$ by $Z^\gamma, \overline{Z}^\gamma$ in K_α, Λ_β, $\Omega_{\alpha,\beta}$, respectively, as they operate on smooth complex valued functions defined on $\mathbb{H}_n$.

We begin by investigating how the matrix differential operators $\partial/\partial Z$ and $\partial/\partial\overline{Z}$ transform under the symplectic action of γ on Z. From equation (2.5), we know that the differential $\mathrm{d}Z$ transforms like

$$\mathrm{d}Z^\gamma = (CZ+D)^{-t}\,\mathrm{d}Z\,(CZ+D)^{-1}.$$

Therefore, as the differential of a smooth function $\varphi\colon \mathbb{H}_n \longrightarrow \mathbb{C}$ depending only on Z is given by $\mathrm{d}\varphi = \operatorname{tr}(\partial\varphi/\partial Z\,\mathrm{d}Z)$, we have

$$\frac{\partial\varphi}{\partial Z} = (CZ+D)^{-1}\,\frac{\partial\varphi}{\partial Z^\gamma}\,(CZ+D)^{-t},$$

i.e., the operator $\partial/\partial Z$ transforms as

$$\frac{\partial}{\partial Z^\gamma} = (CZ+D)\left((CZ+D)\,\frac{\partial}{\partial Z}\right)^t. \tag{4.1}$$

By conjugation, the operator $\partial/\partial\overline{Z}$ transforms as

$$\frac{\partial}{\partial\overline{Z}^{\gamma}}=(C\overline{Z}+D)\bigg((C\overline{Z}+D)\,\frac{\partial}{\partial\overline{Z}}\bigg)^{t}. \tag{4.2}$$

Next we need to know how to differentiate $\det(Z-\overline{Z})$ and $\det(CZ+D)$ with respect to Z, which we carry out in the next two lemmas.

Lemma 4.1. *The matrix identity*

$$\frac{\partial\det(Z-\overline{Z})}{\partial Z}=\det(Z-\overline{Z})(Z-\overline{Z})^{-1}$$

holds.

Proof. Since $Y=\frac{1}{2i}(Z-\overline{Z})\in\mathrm{Sym}_n(\mathbb{R})$, it can be diagonalized with orthogonal matrices. Thus, let $Y=U^t\Lambda U$, where $\Lambda\in\mathrm{M}_n(\mathbb{R})$ is a diagonal matrix with diagonal entries equal to the eigenvalues $\lambda_1,\ldots,\lambda_n$ of Y and $U\in\mathrm{O}_n(\mathbb{R})$. Therefore, differentiating $\det(Y)$ with respect to the entries of $Y=(y_{j,k})_{1\leq j,k\leq n}$, we have

$$\frac{\partial\det(Y)}{\partial y_{j,k}}=\frac{\partial\prod_{l=1}^{n}\lambda_l}{\partial y_{j,k}}=\det(Y)\,\mathrm{tr}\left(\Lambda^{-1}\,\frac{\partial\Lambda}{\partial y_{j,k}}\right).$$

Now writing $\mathrm{tr}(\Lambda^{-1}\,\partial\Lambda/\partial y_{j,k})$ as $\mathrm{tr}(Y^{-1}\,\partial Y/\partial y_{j,k})$ and using the fact that

$$\frac{\partial Y}{\partial y_{j,k}}=(1-\delta_{j,k})E_{j,k}+E_{k,j},$$

we obtain

$$\frac{\partial\det(Y)}{\partial y_{j,k}}=\det(Y)(2-\delta_{j,k})(Y^{-1})_{j,k}.$$

Now since $((1+\delta_{j,k})/2)(2-\delta_{j,k})=1$, we have $\partial\det(Y)/\partial Y=\det(Y)Y^{-1}$. This is equivalent to the identity

$$\frac{\partial\det(Z-\overline{Z})}{\partial Z}=\det(Z-\overline{Z})(Z-\overline{Z})^{-1},$$

which is what we needed to prove. □

Lemma 4.2. *Let* $\gamma=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\mathrm{Sp}_n(\mathbb{R})$. *Then, the matrix identity*

$$\frac{\partial\det(CZ+D)}{\partial Z}=\det(CZ+D)(CZ+D)^{-1}C=\det(CZ+D)C^t(CZ+D)^{-t}$$

holds.

Proof. For $1\leq j,k\leq n$, let $u_{j,k}\colon\mathbb{H}_n\longrightarrow\mathbb{C}$ be smooth scalar valued functions; then, $U:=(u_{j,k})_{1\leq j,k\leq n}$ becomes a smooth matrix valued function on $\mathbb{H}_n$. Moreover, let $\varphi\colon\mathrm{M}_n(\mathbb{C})\longrightarrow\mathbb{C}$ be a smooth scalar valued function. Then, differentiating the function $\varphi\circ U\colon\mathbb{H}_n\longrightarrow\mathbb{C}$ with respect to the entries $z_{j,k}$ of Z, we have

$$\frac{\partial\varphi(U(Z))}{\partial z_{j,k}}=\sum_{l,m=1}^{n}\frac{\partial\varphi}{\partial u_{l,m}}\,\frac{\partial u_{l,m}}{\partial z_{j,k}}=\sum_{l,m=1}^{n}\left(\frac{\partial\varphi}{\partial U}\right)^{t}_{m,l}\left(\frac{\partial U}{\partial z_{j,k}}\right)_{l,m}=\sum_{m=1}^{n}\left(\left(\frac{\partial\varphi}{\partial U}\right)^{t}\frac{\partial U}{\partial z_{j,k}}\right)_{m,m}.$$

Thus, the chain rule of differentiation in this case takes the form

$$\frac{\partial\varphi(U(Z))}{\partial z_{j,k}}=\mathrm{tr}\left(\left(\frac{\partial\varphi}{\partial U}\right)^{t}\frac{\partial U}{\partial z_{j,k}}\right).$$

Note that since we do not assume U to be symmetric beforehand, in this case we have $(\partial/\partial U)_{j,k} = \partial/\partial u_{j,k}$ instead of $(\partial/\partial U)_{j,k} = ((1+\delta_{j,k})/2)\partial/\partial u_{j,k}$.

Now putting $\varphi(Z) = \det(Z)$ and $U(Z) = CZ + D$, by the above formula, we have

$$\frac{\partial \det(CZ+D)}{\partial z_{j,k}} = \operatorname{tr}\left(\left(\frac{\partial \det(CZ+D)}{\partial(CZ+D)}\right)^t \frac{\partial(CZ+D)}{\partial z_{j,k}}\right).$$

We already derived a formula for differentiating the determinant of a symmetric matrix by its entries. The structure of symmetry actually complicates the calculation as its entries are no longer independent. For a square matrix U, not assumed to be symmetric beforehand, the calculation can be simplified by considering the cofactor expansion of a determinant. Let $\widetilde{U} = (\widetilde{u}_{j,l})_{1\le j,l\le n}$ be the cofactor matrix of U. Then, we have

$$\frac{\partial \det(U)}{\partial u_{j,k}} = \frac{\partial}{\partial u_{j,k}} \sum_{l=1}^{n} u_{j,l}\widetilde{u}_{j,l} = \widetilde{u}_{j,k}.$$

Here we exploit the property that since we delete the j-th row (and the l-th column) to build the cofactor $\widetilde{u}_{j,l}$, it must be independent of $u_{j,k}$. This does not hold for a symmetric matrix. Now, since $U^{-1} = 1/\det(U)\,\widetilde{U}^t$, we have

$$\frac{\partial \det(U)}{\partial U} = \widetilde{U} = \det(U)\,U^{-t}.$$

Thus, going back to our initial calculation, we arrive at

$$\begin{aligned}\frac{\partial \det(CZ+D)}{\partial z_{j,k}} &= \det(CZ+D)\operatorname{tr}\left((CZ+D)^{-1}\frac{\partial(CZ+D)}{\partial z_{j,k}}\right)\\ &= \det(CZ+D)\sum_{l,m=1}^{n}(CZ+D)^{-1}_{l,m}\frac{\partial(CZ+D)_{m,l}}{\partial z_{j,k}}.\end{aligned}$$

Now, entrywise partial differentiation with respect to entries $z_{j,k}$ of Z followed by an elementary calculation with taking care of the ensuing Kronecker delta symbols leads us to the matrix identity

$$\frac{\partial \det(CZ+D)}{\partial Z} = \det(CZ+D)(CZ+D)^{-1}C = \det(CZ+D)C^t(CZ+D)^{-t},$$

which is what we needed to prove. □

Lemmas 4.1 and 4.2 prepare the groundwork for calculating the transformation behaviour of the Maaß operators, which we undertake one by one in the subsequent three propositions.

Proposition 4.3. *Let $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$ and $\varphi\colon \mathbb{H}_n \longrightarrow \mathbb{C}$ be a smooth function. Then, the operator K_α^γ obtained by replacing $Z \in \mathbb{H}_n$ in K_α by $Z^\gamma = \gamma Z$ is related to the operator K_α by the identity*

$$\begin{aligned}&K_\alpha^\gamma\big(\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta \varphi(Z)\big)\\ &\qquad = \det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta (C\overline{Z}+D)^{-t} K_\alpha \varphi(Z)(CZ+D)^t.\end{aligned}$$

Proof. From the definition of K_α^γ, we have

$$\begin{aligned}&K_\alpha^\gamma\big(\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta \varphi(Z)\big)\\ &\qquad = \left((Z^\gamma - \overline{Z}^\gamma)\frac{\partial}{\partial Z^\gamma} + \alpha \mathbb{1}_n\right)\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta \varphi(Z).\end{aligned}$$

Then, expanding $\partial/\partial Z^\gamma$ by means of equation (4.1) gives

$$\begin{aligned}K_\alpha^\gamma\big(\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta \varphi(Z)\big) &= \alpha \det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta \varphi(Z)\mathbb{1}_n\\ &+ (C\overline{Z}+D)^{-t}(Z-\overline{Z})\frac{\partial}{\partial Z}\big(\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta\varphi(Z)\big)(CZ+D)^t. \qquad (4.3)\end{aligned}$$

Now, focusing on the second line of the above equality and using Lemma 4.2, we get

$$\begin{aligned}&\frac{\partial}{\partial Z}\big(\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\varphi(Z)\big)\\&\qquad=\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\bigg(\alpha\varphi(Z)C^{t}(CZ+D)^{-t}+\frac{\partial\varphi}{\partial Z}\bigg).\end{aligned}$$

Multiplying the above equation from the left by $(Z-\overline{Z})$ gives

$$\begin{aligned}&(Z-\overline{Z})\,\frac{\partial}{\partial Z}\big(\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\varphi(Z)\big)\\&\qquad=\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\bigg(\alpha\varphi(Z)(ZC^{t}-\overline{Z}C^{t})(CZ+D)^{-t}+(Z-\overline{Z})\,\frac{\partial\varphi}{\partial Z}\bigg).\end{aligned}$$

Now writing $(ZC^{t}-\overline{Z}C^{t})=(CZ+D)^{t}-(C\overline{Z}+D)^{t}$ and using the definition of K_{α} on the right-hand side of the above equation, we have

$$\begin{aligned}&(Z-\overline{Z})\,\frac{\partial}{\partial Z}\big(\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\varphi(Z)\big)\\&\qquad=\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\big(K_{\alpha}\varphi(Z)-\alpha\varphi(Z)(C\overline{Z}+D)^{t}(CZ+D)^{-t}\big).\end{aligned}$$

Therefore, multiplying on the left by $(C\overline{Z}+D)^{-t}$ and on the right by $(CZ+D)^{t}$, we obtain

$$\begin{aligned}&(C\overline{Z}+D)^{-t}(Z-\overline{Z})\,\frac{\partial}{\partial Z}\big(\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\varphi(Z)\big)(CZ+D)^{t}\\&\qquad=\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\big((C\overline{Z}+D)^{-t}K_{\alpha}\varphi(Z)(CZ+D)^{t}-\alpha\varphi(Z)\mathbb{1}_{n}\big).\end{aligned}$$

Combining the last equality with equation (4.3), leads to the identity

$$\begin{aligned}&K_{\alpha}^{\gamma}\big(\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\varphi(Z)\big)\\&\qquad=\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}(C\overline{Z}+D)^{-t}K_{\alpha}\varphi(Z)(CZ+D)^{t},\end{aligned}$$

which is what we had set out to prove. □

Proposition 4.4. *Let* $\gamma=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\mathrm{Sp}_{n}(\mathbb{R})$ *and* $\varphi\colon\mathbb{H}_{n}\longrightarrow\mathbb{C}$ *be a smooth function. Then, the operator* Λ_{β}^{γ} *obtained by replacing* $Z\in\mathbb{H}_{n}$ *in* Λ_{β} *by* $Z^{\gamma}=\gamma Z$ *is related to the operator* Λ_{β} *by the identity*

$$\begin{aligned}&\Lambda_{\beta}^{\gamma}\big(\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\varphi(Z)\big)\\&\qquad=\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}(CZ+D)^{-t}\Lambda_{\beta}\varphi(Z)(C\overline{Z}+D)^{t}.\end{aligned}$$

Proof. Since $\overline{K}_{\beta}=-\Lambda_{\beta}$, the required identity follows from Proposition 4.3 by complex conjugation. □

Proposition 4.5. *Let* $\gamma=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\mathrm{Sp}_{n}(\mathbb{R})$ *and* $\varphi\colon\mathbb{H}_{n}\longrightarrow\mathbb{C}$ *be a smooth function. Then, the operator* $\Omega_{\alpha,\beta}^{\gamma}$ *obtained by replacing* $Z\in\mathbb{H}_{n}$ *in* $\Omega_{\alpha,\beta}$ *by* $Z^{\gamma}=\gamma Z$ *is related to the operator* $\Omega_{\alpha,\beta}$ *by the identity*

$$\begin{aligned}&\Omega_{\alpha,\beta}^{\gamma}\big(\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\varphi(Z)\big)\\&\qquad=\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}(CZ+D)^{-t}\Omega_{\alpha,\beta}\varphi(Z)(CZ+D)^{t}.\end{aligned}\tag{4.4}$$

Proof. To prove the proposition, we first need to calculate

$$\frac{\partial}{\partial\overline{Z}^{\gamma}}\big(K_{\alpha}^{\gamma}\big(\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\varphi(Z)\big)\big),$$

which, upon expanding $\partial/\partial\overline{Z}^{\gamma}$ by means of equation (4.2) and using Proposition 4.3, becomes

$$(C\overline{Z}+D)\bigg((C\overline{Z}+D)\,\frac{\partial}{\partial\overline{Z}}\bigg)^{t}\big(\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}(C\overline{Z}+D)^{-t}K_{\alpha}\varphi(Z)(CZ+D)^{t}\big).$$

Using Lemma 4.2, the above expression becomes

$$\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta \Big(\beta(C\overline{Z}+D)C^t(C\overline{Z}+D)^{-t}K_\alpha\varphi(Z)(CZ+D)^t + (C\overline{Z}+D)\Big((C\overline{Z}+D)\frac{\partial}{\partial\overline{Z}}\Big)^t\big((C\overline{Z}+D)^{-t}K_\alpha\varphi(Z)(CZ+D)^t\big)\Big).$$

Now, using Lemma 3.2 (ii), we have

$$\begin{aligned}&\Big((C\overline{Z}+D)\frac{\partial}{\partial\overline{Z}}\Big)^t\big((C\overline{Z}+D)^{-t}K_\alpha\varphi(Z)(CZ+D)^t\big)\\ &\qquad = \frac{\partial}{\partial\overline{Z}}\big(K_\alpha\varphi(Z)\big)(CZ+D)^t - \frac{1}{2}(n+1)C^t(C\overline{Z}+D)^{-t}K_\alpha\varphi(Z)(CZ+D)^t,\end{aligned}$$

and thus arrive from the above calculation at

$$\begin{aligned}&\frac{\partial}{\partial\overline{Z}^\gamma}\big(K_\alpha^\gamma\big(\det(CZ+D)^\alpha\det(C\overline{Z}+D)^\beta\varphi(Z)\big)\big)\\ &\qquad = \det(CZ+D)^\alpha\det(C\overline{Z}+D)^\beta\Big((\beta-(n+1)/2)(C\overline{Z}+D)C^t(C\overline{Z}+D)^{-t}\times\\ &\qquad\quad \times K_\alpha\varphi(Z)(CZ+D)^t + (C\overline{Z}+D)\frac{\partial}{\partial\overline{Z}}\big(K_\alpha\varphi(Z)\big)(CZ+D)^t\Big).\end{aligned}$$

Then, multiplying both sides from the left by $(Z^\gamma - \overline{Z}^\gamma)$ and using

$$(Z^\gamma - \overline{Z}^\gamma) = (CZ+D)^{-t}(Z-\overline{Z})(C\overline{Z}+D)^{-1}$$

on the right-hand side, we get

$$\begin{aligned}&(Z^\gamma-\overline{Z}^\gamma)\frac{\partial}{\partial\overline{Z}^\gamma}\big(K_\alpha^\gamma\big(\det(CZ+D)^\alpha\det(C\overline{Z}+D)^\beta\varphi(Z)\big)\big)\\ &\quad = \det(CZ+D)^\alpha\det(C\overline{Z}+D)^\beta\Big((\beta-(n+1)/2)(CZ+D)^{-t}(Z-\overline{Z})C^t(C\overline{Z}+D)^{-t}\times\\ &\qquad \times K_\alpha\varphi(Z)(CZ+D)^t + (CZ+D)^{-t}(Z-\overline{Z})\frac{\partial}{\partial\overline{Z}}\big(K_\alpha\varphi(Z)\big)(CZ+D)^t\Big).\end{aligned}$$

Next, writing the expression $(Z-\overline{Z})C^t$ on the right-hand side of the above equation as $(CZ+D)^t - (C\overline{Z}+D)^t$, we can rewrite the above equation as

$$\begin{aligned}&(Z^\gamma-\overline{Z}^\gamma)\frac{\partial}{\partial\overline{Z}^\gamma}\big(K_\alpha^\gamma\big(\det(CZ+D)^\alpha\det(C\overline{Z}+D)^\beta\varphi(Z)\big)\big) = \det(CZ+D)^\alpha\det(C\overline{Z}+D)^\beta\times\\ &\quad\times\Big((\beta-(n+1)/2)\big((C\overline{Z}+D)^{-t}K_\alpha\varphi(Z)(CZ+D)^t - (CZ+D)^{-t}K_\alpha\varphi(Z)(CZ+D)^t\big)\\ &\qquad + (CZ+D)^{-t}(Z-\overline{Z})\frac{\partial}{\partial\overline{Z}}\big(K_\alpha\varphi(Z)\big)(CZ+D)^t\Big).\end{aligned}$$

Now shifting the first term on the right-hand side to the left and using the transformation behaviour of K_α derived in Proposition 4.3, we arrive at

$$\begin{aligned}&\Big((Z^\gamma-\overline{Z}^\gamma)\frac{\partial}{\partial\overline{Z}^\gamma} - (\beta-(n+1)/2)\mathbb{1}_n\Big)\big(K_\alpha^\gamma\big(\det(CZ+D)^\alpha\det(C\overline{Z}+D)^\beta\varphi(Z)\big)\big)\\ &\quad = \det(CZ+D)^\alpha\det(C\overline{Z}+D)^\beta(CZ+D)^{-t}\Big((Z-\overline{Z})\frac{\partial}{\partial\overline{Z}} - (\beta-(n+1)/2)\mathbb{1}_n\Big)\times\\ &\qquad \times K_\alpha\varphi(Z)(CZ+D)^t,\end{aligned}$$

which, by Definition 3.1 and the transformation behaviour of Λ_β given in Proposition 4.4, gives

$$\begin{aligned}&\Lambda^\gamma_{\beta-(n+1)/2}K^\gamma_\alpha\big(\det(CZ+D)^\alpha\det(C\overline{Z}+D)^\beta\varphi(Z)\big)\\ &\qquad = \det(CZ+D)^\alpha\det(C\overline{Z}+D)^\beta(CZ+D)^{-t}\Lambda_{\beta-(n+1)/2}K_\alpha\varphi(Z)(CZ+D)^t,\end{aligned}$$

which, by definition of $\Omega^\gamma_{\alpha,\beta}$, yields the desired identity. □

Remark 4.6. Let $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$ and $\varphi\colon \mathbb{H}_n \longrightarrow \mathbb{C}$ be a smooth function. Taking traces on both sides of equation (4.4) leads to the following transformation behaviour of the Siegel–Maaß Laplacian $\Delta_{\alpha,\beta}$

$$\Delta^{\gamma}_{\alpha,\beta}\big(\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\varphi(Z)\big) = \det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\Delta_{\alpha,\beta}\varphi(Z).$$

Now, if the smooth function φ satisfies the functional equation

$$\varphi(Z^{\gamma}) = \det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\varphi(Z),$$

the transformation behaviour of $\Delta_{\alpha,\beta}$ leads to the identity

$$\Delta^{\gamma}_{\alpha,\beta}\varphi(Z^{\gamma}) = \det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\Delta_{\alpha,\beta}\varphi(Z).$$

Definition 4.7. Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ be a subgroup commensurable with $\mathrm{Sp}_n(\mathbb{Z})$, i.e., the intersection $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ is a finite index subgroup of Γ as well as of $\mathrm{Sp}_n(\mathbb{Z})$. We let $\gamma_j \in \mathrm{Sp}_n(\mathbb{Z})$ $(j = 1,\ldots,h)$ denote a set of representatives for the left cosets of $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ in $\mathrm{Sp}_n(\mathbb{Z})$. We then let $\mathcal{V}^n_{\alpha,\beta}(\Gamma)$ denote the space of all functions $\varphi\colon \mathbb{H}_n \longrightarrow \mathbb{C}$ satisfying the following conditions:

(i) φ is real-analytic;

(ii) $\varphi(\gamma Z) = \det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\varphi(Z)$ for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$;

(iii) given $Y_0 \in \mathrm{Sym}_n(\mathbb{R})$ with $Y_0 > 0$, there exist $M \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ such that the inequalities

$$|\det(C_jZ+D_j)^{-\alpha}\det(C_j\overline{Z}+D_j)^{-\beta}\varphi(\gamma_jZ)| \leq M\,\mathrm{tr}(Y)^N$$

hold in the region $\{Z = X+iY \in \mathbb{H}_n \mid Y \geq Y_0\}$ for the set of representatives $\gamma_j = \left(\begin{smallmatrix} A_j & B_j \\ C_j & D_j \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$ $(j = 1,\ldots,h)$.

Remark 4.8. For $\varphi \in \mathcal{V}^n_{\alpha,\beta}(\Gamma)$, we set

$$\|\varphi\|^2 := \int\limits_{\Gamma\backslash\mathbb{H}_n} \det(Y)^{\alpha+\beta}|\varphi(Z)|^2\,\mathrm{d}\mu_n(Z),$$

whenever it is defined. In this way we obtain the Hilbert space

$$\mathcal{H}^n_{\alpha,\beta}(\Gamma) := \big\{\varphi \in \mathcal{V}^n_{\alpha,\beta}(\Gamma) \,\big|\, \|\varphi\| < \infty\big\}$$

equipped with the inner product

$$\langle\varphi,\psi\rangle = \int\limits_{\Gamma\backslash\mathbb{H}_n} \det(Y)^{\alpha+\beta}\varphi(Z)\overline{\psi}(Z)\,\mathrm{d}\mu_n(Z) \qquad (\varphi,\psi \in \mathcal{H}^n_{\alpha,\beta}(\Gamma)).$$

We note that in order to enable $\|\varphi\| < \infty$, the exponent $N \in \mathbb{N}$ in part (iii) of Definition 4.7 has to be 0. Moreover, we note that Remark 4.6 shows that the Siegel–Maaß Laplacian $\Delta_{\alpha,\beta}$ acts on the Hilbert space $\mathcal{H}^n_{\alpha,\beta}(\Gamma)$.

Definition 4.9. Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ be a subgroup commensurable with $\mathrm{Sp}_n(\mathbb{Z})$. The elements of the Hilbert space $\mathcal{H}^n_{\alpha,\beta}(\Gamma)$ are called *automorphic forms of weight* (α,β) *and degree* n *for* Γ. Moreover, if $\varphi \in \mathcal{H}^n_{\alpha,\beta}(\Gamma)$ is an eigenform of $\Delta_{\alpha,\beta}$, it is called a *Siegel–Maaß form of weight* (α,β) *and degree* n *for* Γ.

Corollary 4.10. *Let* $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ *be a subgroup commensurable with* $\mathrm{Sp}_n(\mathbb{Z})$ *and* $\varphi \in \mathcal{H}^n_{\alpha,\beta}(\Gamma)$. *Then, we have for all* $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$

(i) $K^{\gamma}_{\alpha}\varphi(Z^{\gamma}) = \det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}(C\overline{Z}+D)^{-t}K_{\alpha}\varphi(Z)(CZ+D)^{t}$,

(ii) $\Lambda^{\gamma}_{\beta}\varphi(Z^{\gamma}) = \det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}(CZ+D)^{-t}\Lambda_{\beta}\varphi(Z)(C\overline{Z}+D)^{t}$,

(iii) $\Omega^{\gamma}_{\alpha,\beta}\varphi(Z^{\gamma}) = \det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}(CZ+D)^{-t}\Omega_{\alpha,\beta}\varphi(Z)(CZ+D)^{t}$.

Proof. The proof is an immediate consequence of Propositions 4.3–4.5 and the definition of the Hilbert space $\mathcal{H}^n_{\alpha,\beta}(\Gamma)$. ☐

5 Symmetry of the Siegel–Maaß Laplacian of weight (α, β)

Let $\mathrm{d}Z := (\mathrm{d}z_{j,k})_{1\le j,k\le n}$ denote the $(n\times n)$-matrix consisting of differential forms of degree 1 and let $[\mathrm{d}Z] := \bigwedge_{1\le j\le k\le n} \mathrm{d}z_{j,k}$ denote the differential form of degree $n(n+1)/2$ at $Z \in \mathbb{H}_n$. We introduce an $(n\times n)$-matrix $\{\mathrm{d}Z\}$ consisting of differential forms of degree $(n(n+1)/2-1)$, namely

$$\{\mathrm{d}Z\}_{j,k} := \frac{1+\delta_{j,k}}{2}\,\varpi_{j,k},$$

where $\varpi_{j,k}$ is defined by

$$\varpi_{j,k} := \varepsilon_{j,k} \bigwedge_{\substack{1\le l\le m\le n\\ (l,m)\neq(j,k)}} \mathrm{d}z_{l,m} \qquad (1\le j\le k\le n)$$

in case $j \le k$ and $\varpi_{j,k} = \varpi_{k,j}$ in case $j > k$ with the sign $\varepsilon_{j,k} = \pm 1$ determined by $\mathrm{d}z_{j,k}\wedge\varpi_{j,k} = [\mathrm{d}Z]$. It is easy to see that

$$\mathrm{d}Z \wedge \{\mathrm{d}Z\} = \frac{1}{2}(n+1)[\mathrm{d}Z]\mathbb{1}_n.$$

Let now $\gamma = \left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$. Since we have $\mathrm{d}Z^\gamma = (CZ+D)^{-t}\,\mathrm{d}Z\,(CZ+D)^{-1}$ and $[\mathrm{d}Z^\gamma] = \det(CZ+D)^{-(n+1)}[\mathrm{d}Z]$, we derive from the relation

$$\mathrm{d}Z^\gamma \wedge \{\mathrm{d}Z^\gamma\} = \frac{1}{2}(n+1)[\mathrm{d}Z^\gamma]\mathbb{1}_n$$

that the matrix $\{\mathrm{d}Z\}$ has the transformation behaviour

$$\{\mathrm{d}Z^\gamma\} = \det(CZ+D)^{-(n+1)}(CZ+D)\{\mathrm{d}Z\}(CZ+D)^t.$$

Next we shall use these differential forms to show that the Siegel–Maaß Laplacian $\Delta_{\alpha,\beta}$ acts as a symmetric operator on the Hilbert space $\mathcal{H}^n_{\alpha,\beta}(\Gamma)$.

Theorem 5.1. *Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ be a subgroup commensurable with $\mathrm{Sp}_n(\mathbb{Z})$ and let $\varphi, \psi \in \mathcal{H}^n_{\alpha,\beta}(\Gamma)$ be compactly supported. Then, we have the formula*

$$\langle -\Delta_{\alpha,\beta}\varphi, \psi\rangle = \int\limits_{\Gamma\backslash\mathbb{H}_n} \det(Y)^{\alpha+\beta}\,\mathrm{tr}\left(\Lambda_\beta\varphi(Z)\overline{\Lambda}_\beta\overline{\psi}(Z)\right)\mathrm{d}\mu_n(Z) + n\beta(\alpha-(n+1)/2)\langle\varphi,\psi\rangle.$$

In particular, this formula establishes the relation

$$\langle \Delta_{\alpha,\beta}\varphi, \psi\rangle = \langle \varphi, \Delta_{\alpha,\beta}\psi\rangle,$$

which shows that the Siegel–Maaß Laplacian $\Delta_{\alpha,\beta}$ acts as a symmetric operator on the Hilbert space $\mathcal{H}^n_{\alpha,\beta}(\Gamma)$.

Proof. We start by considering the differential form

$$\omega(Z) := \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\overline{\psi}(Z)\,\mathrm{tr}\left(\Lambda_\beta\varphi(Z)(Z-\overline{Z})\{\mathrm{d}Z\}\right)\wedge[\mathrm{d}\overline{Z}].$$

Let $\gamma = \left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right) \in \Gamma$. Then, the transformation formulas

(a) $\det(Z^\gamma - \overline{Z}^\gamma)^{\alpha+\beta-(n+1)}$
$= \det(CZ+D)^{-(\alpha+\beta-(n+1))}\det(C\overline{Z}+D)^{-(\alpha+\beta-(n+1))}\det(Z-\overline{Z})^{\alpha+\beta-(n+1)}$,

(b) $\overline{\psi}(Z^\gamma) = \det(CZ+D)^\beta\det(C\overline{Z}+D)^\alpha\overline{\psi}(Z)$,

(c) $\mathrm{tr}\left(\Lambda_\beta^\gamma\varphi(Z^\gamma)(Z^\gamma-\overline{Z}^\gamma)\{\mathrm{d}Z^\gamma\}\right)$
$= \det(CZ+D)^{\alpha-(n+1)}\det(C\overline{Z}+D)^\beta\,\mathrm{tr}\left(\Lambda_\beta\varphi(Z)(Z-\overline{Z})\{\mathrm{d}Z\}\right)$,

(d) $[\mathrm{d}\overline{Z}^\gamma] = \det(C\overline{Z}+D)^{-(n+1)}[\mathrm{d}\overline{Z}]$

show that $\omega(Z^\gamma) = \omega(Z)$ for all $\gamma \in \Gamma$, i.e., $\omega(Z)$ is a Γ-invariant differential form on $\mathbb{H}_n$, and hence can be considered as a differential form on the quotient space $\Gamma\backslash\mathbb{H}_n$. Since the automorphic forms φ, ψ are real-analytic, the differential form ω is a smooth differential form. Therefore, by Stokes' theorem, we have

$$\int\limits_{\Gamma\backslash\mathbb{H}_n} \mathrm{d}\omega(Z) = \int\limits_{\partial\Gamma\backslash\mathbb{H}_n} \omega(Z).$$

As φ, ψ are compactly supported, the integral on the right-hand side of the above equation vanishes, which gives

$$\int\limits_{\Gamma\backslash\mathbb{H}_n} \mathrm{d}\omega(Z) = 0. \tag{5.1}$$

As we shall see, by explicitly computing $\mathrm{d}\omega(Z)$, the vanishing of the above integral will lead to the formula claimed in the theorem.

For the computation of $\mathrm{d}\omega(Z)$, we set $\rho := \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\overline{\psi}(Z)$, $P := \Lambda_\beta\varphi(Z)$, and $Q := (Z-\overline{Z})$. Then, we obtain

$$\omega(Z) = \rho\,\mathrm{tr}(P\,Q\,\{\mathrm{d}Z\}) \wedge [\mathrm{d}\overline{Z}] = \sum_{j,k,l=1}^{n} \rho\, p_{j,k}\, q_{k,l}\, \{\mathrm{d}Z\}_{l,j} \wedge [\mathrm{d}\overline{Z}].$$

Taking exterior derivatives on both sides leads to

$$\begin{aligned}
\mathrm{d}\omega(Z) &= \sum_{j,k,l=1}^{n} \frac{\partial}{\partial z_{l,j}} (\rho\, p_{j,k}\, q_{k,l})\, \mathrm{d}z_{l,j} \wedge \frac{1+\delta_{l,j}}{2} \varpi_{l,j} \wedge [\mathrm{d}\overline{Z}] \\
&= \sum_{j,k,l=1}^{n} \frac{1+\delta_{l,j}}{2} \frac{\partial}{\partial z_{l,j}} (\rho\, p_{j,k}\, q_{k,l})\, [\mathrm{d}Z] \wedge [\mathrm{d}\overline{Z}] \\
&= \sum_{j,k,l=1}^{n} \left(\frac{\partial}{\partial Z}\right)_{l,j} (\rho\, p_{j,k}\, q_{k,l})\, [\mathrm{d}Z] \wedge [\mathrm{d}\overline{Z}].
\end{aligned} \tag{5.2}$$

Now a term by term differentiation in the last expression on the right-hand side of the above equation allows us to write it as the sum of the three traces

$$\sum_{j,k,l=1}^{n} \left(\frac{\partial}{\partial Z}\right)_{l,j} (\rho\, p_{j,k}\, q_{k,l}) = \mathrm{tr}\left(\frac{\partial \rho}{\partial Z}\, P\, Q\right) + \rho\,\mathrm{tr}\left(\frac{\partial}{\partial Z}\, P\, Q\right) + \rho\,\mathrm{tr}\left(P^t\, \frac{\partial}{\partial Z}\, Q\right), \tag{5.3}$$

which we calculate one by one next.

(i) We begin by considering

$$\frac{\partial \rho}{\partial Z} = \frac{\partial}{\partial Z}\left(\det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\overline{\psi}(Z)\right),$$

which, by Lemma 4.1, calculates to

$$\frac{\partial \rho}{\partial Z} = (\alpha+\beta-(n+1))\det(Z-\overline{Z})^{\alpha+\beta-(n+1)}(Z-\overline{Z})^{-1}\,\overline{\psi}(Z) + \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\, \frac{\partial\overline{\psi}(Z)}{\partial Z}.$$

Now multiplying both sides of the above equation on the right by $P\,Q = \Lambda_\beta\varphi(Z)(Z-\overline{Z})$ and taking the trace gives

$$\begin{aligned}
\mathrm{tr}\left(\frac{\partial \rho}{\partial Z}\, P\, Q\right) = \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\Bigg(&(\alpha+\beta-(n+1))\,\mathrm{tr}\left((Z-\overline{Z})^{-1}\overline{\psi}(Z)\Lambda_\beta\varphi(Z)(Z-\overline{Z})\right) \\
&+ \mathrm{tr}\left(\frac{\partial\overline{\psi}(Z)}{\partial Z}\,\Lambda_\beta\varphi(Z)(Z-\overline{Z})\right)\Bigg),
\end{aligned}$$

which, upon rearranging the terms inside the traces on the right-hand side by cyclically permuting them, becomes

$$\operatorname{tr}\left(\frac{\partial\rho}{\partial Z}\,P\,Q\right) = \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\Big((\alpha+\beta-(n+1))\operatorname{tr}\big(\Lambda_\beta\varphi(Z)\overline{\psi}(Z)\big) + \operatorname{tr}\Big(\Lambda_\beta\varphi(Z)(Z-\overline{Z})\,\frac{\partial\overline{\psi}(Z)}{\partial Z}\Big)\Big). \tag{5.4}$$

(ii) Next, we consider the second trace

$$\rho\operatorname{tr}\left(\frac{\partial}{\partial Z}\,P\,Q\right) = \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\overline{\psi}(Z)\operatorname{tr}\left(\frac{\partial}{\partial Z}\,\Lambda_\beta\varphi(Z)(Z-\overline{Z})\right)$$

in equation (5.3), which, again through rearrangement of the terms inside the trace by a cyclical permutation, takes the form

$$\rho\operatorname{tr}\left(\frac{\partial}{\partial Z}\,P\,Q\right) = \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\operatorname{tr}\left((Z-\overline{Z})\,\frac{\partial}{\partial Z}\,\Lambda_\beta\varphi(Z)\overline{\psi}(Z)\right). \tag{5.5}$$

(iii) Finally, we consider the third trace

$$\rho\operatorname{tr}\left(P^t\,\frac{\partial}{\partial Z}\,Q\right) = \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\overline{\psi}(Z)\operatorname{tr}\left(\big(\Lambda_\beta\varphi(Z)\big)^t\left(\frac{\partial}{\partial Z}\,(Z-\overline{Z})\right)\mathbb{1}_n\right)$$

in equation (5.3). By the first operator identity in Corollary 3.3, we have the matrix identity

$$\left(\frac{\partial}{\partial Z}\,(Z-\overline{Z})\right)\mathbb{1}_n = \left((Z-\overline{Z})\,\frac{\partial}{\partial Z}\right)^t\mathbb{1}_n + \frac{1}{2}(n+1)\mathbb{1}_n = \frac{1}{2}(n+1)\mathbb{1}_n,$$

which gives, upon rearrangement of the scalar quantities, the identity

$$\rho\operatorname{tr}\left(P^t\,\frac{\partial}{\partial Z}\,Q\right) = \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\frac{1}{2}(n+1)\operatorname{tr}\big(\Lambda_\beta\varphi(Z)\overline{\psi}(Z)\big). \tag{5.6}$$

Now, adding up equations (5.4)–(5.6), it follows from equation (5.3) that

$$\sum_{j,k,l=1}^{n}\left(\frac{\partial}{\partial Z}\right)_{l,j}(\rho\,p_{j,k}\,q_{k,l}) = \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\Big((\alpha+\beta-(n+1)/2)\operatorname{tr}\big(\Lambda_\beta\varphi(Z)\overline{\psi}(Z)\big) + \operatorname{tr}\Big(\Lambda_\beta\varphi(Z)(Z-\overline{Z})\,\frac{\partial\overline{\psi}(Z)}{\partial Z}\Big) + \operatorname{tr}\Big((Z-\overline{Z})\,\frac{\partial}{\partial Z}\,\Lambda_\beta\varphi(Z)\overline{\psi}(Z)\Big)\Big).$$

Rearranging terms on the right-hand side of the last expression, leads to

$$\sum_{j,k,l=1}^{n}\left(\frac{\partial}{\partial Z}\right)_{l,j}(\rho\,p_{j,k}\,q_{k,l}) = \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\Big(\operatorname{tr}\Big(\Lambda_\beta\varphi(Z)\Big((Z-\overline{Z})\,\frac{\partial}{\partial Z}+\beta\mathbb{1}_n\Big)\overline{\psi}(Z)\Big) + \operatorname{tr}\Big((Z-\overline{Z})\,\frac{\partial}{\partial Z}+(\alpha-(n+1)/2)\mathbb{1}_n\Big)\Lambda_\beta\varphi(Z)\overline{\psi}(Z)\Big).$$

Identifying the operator $(Z-\overline{Z})\partial/\partial Z+\beta\mathbb{1}_n$ on the right-hand side of the above equation as $-\overline{\Lambda}_\beta$ and the operator $(Z-\overline{Z})\partial/\partial Z+(\alpha-(n+1)/2)\mathbb{1}_n$ as $K_{\alpha-(n+1)/2}$, we can rewrite the right-hand side of the above equation as

$$\det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\Big(-\operatorname{tr}\big(\Lambda_\beta\varphi(Z)\,\overline{\Lambda}_\beta\overline{\psi}(Z)\big) + \operatorname{tr}\big(K_{\alpha-(n+1)/2}\,\Lambda_\beta\varphi(Z)\overline{\psi}(Z)\big)\Big),$$

which, by definition of $\widetilde{\Omega}_{\alpha,\beta}$, is equal to

$$\det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\Big(\operatorname{tr}\big(\widetilde{\Omega}_{\alpha,\beta}-\beta(\alpha-(n+1)/2)\mathbb{1}_n\big)\varphi(Z)\overline{\psi}(Z) - \operatorname{tr}\big(\Lambda_\beta\varphi(Z)\overline{\Lambda}_\beta\overline{\psi}(Z)\big)\Big).$$

In total, we get

$$\sum_{j,k,l=1}^{n}\left(\frac{\partial}{\partial Z}\right)_{l,j}(\rho\, p_{j,k}\, q_{k,l}) = \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\Big(-\Delta_{\alpha,\beta}\,\varphi(Z)\overline{\psi}(Z) - \operatorname{tr}\big(\Lambda_\beta\varphi(Z)\overline{\Lambda}_\beta\overline{\psi}(Z)\big)$$
$$- n\beta(\alpha-(n+1)/2)\,\varphi(Z)\overline{\psi}(Z)\Big).$$

Thus, substituting $\sum_{j,k,l=1}^{n}(\partial/\partial Z)_{l,j}(\rho\, p_{j,k}\, q_{k,l})$ back into equation (5.2), we arrive at

$$\mathrm{d}\omega(Z) = \det(Z-\overline{Z})^{\alpha+\beta}\Big(-\Delta_{\alpha,\beta}\,\varphi(Z)\overline{\psi}(Z) - \operatorname{tr}\big(\Lambda_\beta\varphi(Z)\overline{\Lambda}_\beta\overline{\psi}(Z)\big)$$
$$- n\beta(\alpha-(n+1)/2)\,\varphi(Z)\overline{\psi}(Z)\Big)\frac{[\mathrm{d}Z]\wedge[\mathrm{d}\overline{Z}]}{\det(Z-\overline{Z})^{n+1}}.$$

Now, noting that the volume form

$$\det(Z-\overline{Z})^{\alpha+\beta}\frac{[\mathrm{d}Z]\wedge[\mathrm{d}\overline{Z}]}{\det(Z-\overline{Z})^{n+1}}$$

is just a constant multiple of $\det(Y)^{\alpha+\beta}\mathrm{d}\mu_n(Z)$, it follows readily from the vanishing result (5.1) that

$$\langle -\Delta_{\alpha,\beta}\varphi,\psi\rangle = \int_{\Gamma\backslash\mathbb{H}_n}\det(Y)^{\alpha+\beta}\operatorname{tr}\big(\Lambda_\beta\varphi(Z)\overline{\Lambda}_\beta\overline{\psi}(Z)\big)\,\mathrm{d}\mu_n(Z) + n\beta(\alpha-(n+1)/2)\langle\varphi,\psi\rangle,$$

which is the claimed formula.

Using the latter formula, we compute

$$\begin{aligned}\langle\varphi,-\Delta_{\alpha,\beta}\psi\rangle &= \overline{\langle -\Delta_{\alpha,\beta}\psi,\varphi\rangle}\\ &= \int_{\Gamma\backslash\mathbb{H}_n}\det(Y)^{\alpha+\beta}\,\overline{\operatorname{tr}\big(\Lambda_\beta\psi(Z)\overline{\Lambda}_\beta\overline{\varphi}(Z)\big)}\,\mathrm{d}\mu_n(Z) + n\beta(\alpha-(n+1)/2)\overline{\langle\psi,\varphi\rangle}\\ &= \int_{\Gamma\backslash\mathbb{H}_n}\det(Y)^{\alpha+\beta}\operatorname{tr}\big(\Lambda_\beta\varphi(Z)\overline{\Lambda}_\beta\overline{\psi}(Z)\big)\,\mathrm{d}\mu_n(Z) + n\beta(\alpha-(n+1)/2)\langle\varphi,\psi\rangle\\ &= \langle -\Delta_{\alpha,\beta}\varphi,\psi\rangle,\end{aligned}$$

which proves the claimed symmetry of the Siegel–Maaß Laplacian $\Delta_{\alpha,\beta}$. ☐

Corollary 5.2. *Let $\Gamma\subset\mathrm{Sp}_n(\mathbb{R})$ be a subgroup commensurable with $\mathrm{Sp}_n(\mathbb{Z})$ and let $\varphi\in\mathcal{H}^n_{\alpha,\beta}(\Gamma)$ be a Siegel–Maaß form of weight (α,β) and degree n for Γ. Then, if φ is an eigenform of $\Delta_{\alpha,\beta}$ with eigenvalue λ, we have $\lambda\in\mathbb{R}$ and $\lambda\geq n\beta(\alpha-(n+1)/2)$.*

Furthermore, φ has eigenvalue $\lambda=\beta(\alpha-(n+1)/2)$ if and only if $\varphi(Z)=\det(Y)^{-\beta}f(Z)$, where $f\colon\mathbb{H}_n\longrightarrow\mathbb{C}$ is a holomorphic function satisfying

$$f(\gamma Z) = \det(CZ+D)^{\alpha-\beta}f(Z)$$

for all $\gamma=\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\in\Gamma$. Moreover, if $\beta<0$, then f is a Siegel cusp form of weight $\alpha-\beta$ and degree n for Γ.

Proof. Since $\varphi\in\mathcal{H}^n_{\alpha,\beta}(\Gamma)$ is an eigenform of $\Delta_{\alpha,\beta}$ with eigenvalue λ, i.e., we have $(\Delta_{\alpha,\beta}+\lambda\,\mathrm{id})\varphi=0$, we compute using Theorem 5.1

$$\begin{aligned}\lambda\langle\varphi,\varphi\rangle &= \langle -\Delta_{\alpha,\beta}\varphi,\varphi\rangle\\ &= \int_{\Gamma\backslash\mathbb{H}_n}\det(Y)^{\alpha+\beta}\operatorname{tr}\big(|\Lambda_\beta\varphi(Z)|^2\big)\,\mathrm{d}\mu_n(Z) + n\beta(\alpha-(n+1)/2)\langle\varphi,\varphi\rangle.\end{aligned}$$

This immediately implies that $\lambda \in \mathbb{R}$. Furthermore, since $\operatorname{tr}(|\Lambda_\beta \varphi(Z)|^2) \geq 0$, we conclude that

$$\lambda \geq n\beta(\alpha - (n+1)/2).$$

To prove the second part of the corollary, we observe that the above equation shows that the equality $\lambda = n\beta(\alpha - (n+1)/2)$ is equivalent to

$$\int\limits_{\Gamma\backslash\mathbb{H}_n} \det(Y)^{\alpha+\beta} \operatorname{tr}\left(|\Lambda_\beta \varphi(Z)|^2\right) \mathrm{d}\mu_n(Z) = 0.$$

Since $\operatorname{tr}(|\Lambda_\beta \varphi(Z)|^2) \geq 0$, the above integral vanishes if and only if $\operatorname{tr}(|\Lambda_\beta \varphi(Z)|^2) = 0$. Now, as the matrix

$$\Lambda_\beta \varphi(Z) = (Z - \overline{Z}) \frac{\partial \varphi}{\partial \overline{Z}} - \beta \varphi(Z) \mathbb{1}_n$$

is similar to the complex symmetric matrix

$$S(Z) := 2i\, Y^{1/2} \frac{\partial \varphi}{\partial \overline{Z}} Y^{1/2} - \beta \varphi(Z) \mathbb{1}_n,$$

as we have the relation $\Lambda_\beta \varphi(Z) = Y^{1/2} S(Z) Y^{-1/2}$, the matrix $|\Lambda_\beta \varphi(Z)|^2$ becomes similar to the positive semidefinite hermitian matrix $S(Z)\overline{S}(Z)$, which is diagonalizable with non-negative real eigenvalues. Therefore, the condition $\operatorname{tr}(S(Z)\overline{S}(Z)) = \operatorname{tr}(|\Lambda_\beta \varphi(Z)|^2) = 0$ is equivalent to the vanishing of all the eigenvalues of $S(Z)\overline{S}(Z)$, which is equivalent to the vanishing of $S(Z)$ and hence of $\Lambda_\beta \varphi(Z)$. All in all, this proves that the equality $\lambda = n\beta(\alpha - (n+1)/2)$ is equivalent to the vanishing condition $\Lambda_\beta \varphi = 0$.

Continuing, we now set $f(Z) := \det(Y)^\beta \varphi(Z)$, and compute

$$\frac{\partial f}{\partial \overline{Z}} = \beta \det(Y)^{\beta-1} \frac{\partial \det(Y)}{\partial \overline{Z}} \varphi(Z) + \det(Y)^\beta \frac{\partial \varphi}{\partial \overline{Z}}\,.$$

Since we have

$$\frac{\partial \det(Y)}{\partial \overline{Z}} = \frac{1}{2}\left(\frac{\partial}{\partial X} + i\,\frac{\partial}{\partial Y}\right) \det(Y) = \frac{i}{2}\, \frac{\partial \det(Y)}{\partial Y} = \frac{i}{2} \det(Y) Y^{-1},$$

the above equality becomes

$$\begin{aligned} \frac{\partial f}{\partial \overline{Z}} &= \frac{i\beta}{2} \det(Y)^\beta Y^{-1} \varphi(Z) + \det(Y)^\beta \frac{\partial \varphi}{\partial \overline{Z}} \\ &= -\frac{i}{2} \det(Y)^\beta Y^{-1}\left(- \beta\varphi(Z)\mathbb{1}_n + 2iY\, \frac{\partial \varphi}{\partial \overline{Z}}\right) \\ &= -\frac{i}{2} \det(Y)^\beta Y^{-1}\left((Z-\overline{Z})\, \frac{\partial \varphi}{\partial \overline{Z}} - \beta\varphi(Z)\mathbb{1}_n\right) \\ &= -\frac{i}{2} \det(Y)^\beta Y^{-1} \Lambda_\beta \varphi(Z). \end{aligned}$$

In total, this shows that $\partial f/\partial \overline{Z} = 0$, i.e., the function f is holomorphic, if and only if $\Lambda_\beta \varphi(Z) = 0$, which, by the previous argument, is equivalent to $\varphi \in \mathcal{H}^n_{\alpha,\beta}(\Gamma)$ being a Siegel–Maaß form with eigenvalue $\lambda = \beta(\alpha - (n+1)/2)$.

Furthermore, as the function $\varphi \in \mathcal{H}^n_{\alpha,\beta}(\Gamma)$ has the transformation behaviour

$$\varphi(\gamma Z) = \det(CZ + D)^\alpha \det(C\overline{Z} + D)^\beta \varphi(Z)$$

for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$, the function $f(Z) = \det(Y)^\beta \varphi(Z) = \det(\operatorname{Im}(Z))^\beta \varphi(Z)$ has the transformation behaviour

$$\begin{aligned} f(\gamma Z) &= \det(\operatorname{Im}(\gamma Z))^\beta \varphi(\gamma Z) \\ &= \left(\frac{\det(\operatorname{Im}(Z))}{|\det(CZ+D)|^2}\right)^\beta \det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta \varphi(Z) \\ &= \det(CZ+D)^{\alpha-\beta} \det(\operatorname{Im}(Z))^\beta \varphi(Z) \\ &= \det(CZ+D)^{\alpha-\beta} f(Z), \end{aligned}$$

as claimed.

Finally, letting $\gamma_j = \left(\begin{smallmatrix} A_j & B_j \\ C_j & D_j \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$ $(j = 1, \ldots, h)$ be a set of representatives for the left cosets of $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ in $\mathrm{Sp}_n(\mathbb{Z})$, Remark 4.8 shows that given $Y_0 \in \mathrm{Sym}_n(\mathbb{R})$ with $Y_0 > 0$, the quantities

$$|\det(C_jZ + D_j)^{-\alpha} \det(C_j\overline{Z} + D_j)^{-\beta}\varphi(\gamma_jZ)|$$

have to be bounded in the region $\{Z = X + iY \in \mathbb{H}_n \,|\, Y \geq Y_0\}$. Therefore, if $\beta < 0$, this implies that given $Y_0 \in \mathrm{Sym}_n(\mathbb{R})$ with $Y_0 \gg 0$, the quantities

$$|\det(C_jZ + D_j)^{-(\alpha-\beta)} f(\gamma_jZ)| = |\det(C_jZ + D_j)^{-\alpha} \det(C_j\overline{Z} + D_j)^{-\beta} \det(\mathrm{Im}(\gamma_jZ))^{\beta}\varphi(\gamma_jZ)|$$

will become arbitrarily small in the region $\{Z = X + iY \in \mathbb{H}_n \,|\, Y \geq Y_0\}$. In other words, f is indeed a Siegel cusp form of weight $\alpha - \beta$ and degree n for Γ.

With all this, the proof of the corollary is complete. □

Remark 5.3. For $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ a subgroup commensurable with $\mathrm{Sp}_n(\mathbb{Z})$ and $\alpha = k/2$, $\beta = -k/2$ with $k \in \mathbb{N}_{>0}$, we denote the Hilbert space $\mathcal{H}^n_{\alpha,\beta}(\Gamma)$ simply by $\mathcal{H}^n_k(\Gamma)$. Similarly, we write for the operator $\Omega_{\alpha,\beta}$ simply Ω_k, which becomes

$$\begin{aligned}\Omega_k &= (Z - \overline{Z})\biggl((Z - \overline{Z})\,\frac{\partial}{\partial \overline{Z}}\biggr)^t \frac{\partial}{\partial Z} + \frac{k}{2}(Z - \overline{Z})\,\frac{\partial}{\partial \overline{Z}} + \frac{k}{2}(Z - \overline{Z})\,\frac{\partial}{\partial Z} \\ &= -Y\biggl(\biggl(Y\,\frac{\partial}{\partial X}\biggr)^t \frac{\partial}{\partial X} + \biggl(Y\,\frac{\partial}{\partial Y}\biggr)^t \frac{\partial}{\partial Y}\biggr) + ikY\,\frac{\partial}{\partial X}\,.\end{aligned}$$

Finally, we write for the operator $\Delta_{\alpha,\beta}$ simply Δ_k and call it the Siegel–Maaß Laplacian of weight k; it is given as

$$\Delta_k = \mathrm{tr}\biggl(Y\biggl(\biggl(Y\,\frac{\partial}{\partial X}\biggr)^t \frac{\partial}{\partial X} + \biggl(Y\,\frac{\partial}{\partial Y}\biggr)^t \frac{\partial}{\partial Y}\biggr) - ikY\,\frac{\partial}{\partial X}\biggr).$$

We note that the transformation behaviour of a Siegel–Maaß form φ of weight k and degree n for Γ takes the form

$$\varphi(\gamma Z) = \left(\frac{\det(CZ + D)}{\det(C\overline{Z} + D)}\right)^{k/2} \varphi(Z),$$

where $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$.

In the last corollary, we summarize the main results about Siegel–Maaß forms of weight k and degree n for Γ.

Corollary 5.4. *Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ be a subgroup commensurable with $\mathrm{Sp}_n(\mathbb{Z})$ and let $\varphi \in \mathcal{H}^n_k(\Gamma)$ be a Siegel–Maaß form of weight k and degree n for Γ. Then, if φ is an eigenform of Δ_k with eigenvalue λ, we have $\lambda \in \mathbb{R}$ and*

$$\lambda \geq \frac{nk}{4}(n - k + 1),$$

with equality attained if and only if the function φ is of the form $\varphi(Z) = \det(Y)^{k/2} f(Z)$ for some Siegel cusp form $f \in \mathcal{S}^n_k(\Gamma)$ of weight k and degree n for Γ. In other words, there is an isomorphism

$$\mathcal{S}^n_k(\Gamma) \cong \ker\left(\Delta_k + \frac{nk}{4}(n - k + 1)\mathrm{id}\right)$$

of $\mathbb{C}$-vector spaces, induced by the assignment $f \mapsto \det(Y)^{k/2} f$.

Proof. The proof is an immediate consequence of Corollray 5.2 by setting $\alpha = k/2$ and $\beta = -k/2$. □

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