SUP-NORM BOUNDS FOR JACOBI CUSP FORMS

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Abstract. In this article, we give bounds for the natural invariant norm of cusp forms of real weight k and character χ for any cofinite Fuchsian subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$. Using the representation of Jacobi cusp forms of integral weight k and index m for the modular group $\Gamma_0 = \mathrm{SL}_2(\mathbb{Z})$ as linear combinations of modular forms of weight $k - \frac{1}{2}$ for some congruence subgroup of Γ_0 (depending on m) and suitable Jacobi theta functions, we derive bounds for the natural invariant norm of these Jacobi cusp forms. More specifically, letting $J_{k,m}^{\mathrm{cusp}}(\Gamma_0)$ denote the complex vector space of Jacobi cusp forms under consideration and $\|\cdot\|_{\mathrm{Pet}}$ the pointwise Petersson norm on $J_{k,m}^{\mathrm{cusp}}(\Gamma_0)$, we prove that for given $\epsilon > 0$, the bound

$$\sup_{(\tau,z)\in\mathbb{H}\times\mathbb{C}} \|f(\tau,z)\|_{\mathrm{Pet}} = O_\epsilon\big(k^{\frac{3}{4}} m^{1+\epsilon}\big)$$

holds for any $f \in J_{k,m}^{\mathrm{cusp}}(\Gamma_0)$, which is normalized with respect to the Petersson inner product, where the implied constant depends only on the choice of $\epsilon > 0$.

1. Introduction

1.1. **Background.** In general, bounds for automorphic forms and for their Fourier coefficients represent an area of great interest in number theory. More specifically, we mention in this respect the results of [FJK16], where J. Friedman, J. Jorgenson, and J. Kramer obtained optimal sup-norm bounds on average for cusp forms of even weight k for any cofinite Fuchsian subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$. These bounds turn out to be uniform with respect to the subgroup Γ. Moreover, in [FJK19], effective versions for these sup-norm bounds are given. With regard to sup-norm bounds for individual Hecke eigenforms of large level, we mention, for example, the results by V. Blomer and R. Holowinsky in [BH11].

So far, less attention has been devoted to the study of sup-norm bounds for Jacobi cusp forms. The first comprehensive study of Jacobi forms was undertaken by M. Eichler and D. Zagier in [EZ85]. Subsequently, various authors have built on their work. In contrast to their analytical approach, a geometrical approach to the theory of Jacobi forms was given by J. Kramer in [Kr91].

Let k, m be positive integers. A Jacobi form of weight k and index m for the modular group $\Gamma_0 := \mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function on the product $\mathbb{H} \times \mathbb{C}$ of the upper half-plane $\mathbb{H}$ with the complex plane $\mathbb{C}$ having a suitable transformation behaviour with respect to Γ_0 and vanishing "at infinity". We denote the complex vector space of Jacobi cusp forms of weight k and index m for Γ_0 by $J_{k,m}^{\mathrm{cusp}}(\Gamma_0)$. The pointwise Petersson norm of a Jacobi form $f \in J_{k,m}^{\mathrm{cusp}}(\Gamma_0)$ is then defined by

$$\|f(\tau,z)\|_{\mathrm{Pet}}^2 := |f(\tau,z)|^2 \,\mathrm{Im}(\tau)^k \, e^{-\frac{4\pi m \mathrm{Im}(z)^2}{\mathrm{Im}(\tau)}} \qquad (\tau \in \mathbb{H},\, z \in \mathbb{C}).$$

Let F be a Siegel cusp form of weight k for the Siegel modular group $\mathrm{Sp}_4(\mathbb{Z})$, and let $\{f_m\}_{m\geq 1}$ be the set of Jacobi forms appearing in the Fourier–Jacobi expansion of F, i. e.,

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$f_m \in J_{k,m}^{\mathrm{cusp}}(\Gamma_0)$. Then, for any $\epsilon > 0$, W. Kohnen proved the following sup-norm bound for the pointwise Petersson norm of f_m in [Ko93]

$$\sup_{(\tau,z)\in\mathbb{H}\times\mathbb{C}} \|f_m(\tau,z)\|_{\mathrm{Pet}} = O_{F,\epsilon}\big(m^{\frac{k}{2}-\frac{2}{9}+\epsilon}\big), \tag{1}$$

where the implied constant depends on the Siegel cusp form F and on the choice of $\epsilon > 0$. Motivated by the Ramanujan–Petersson conjecture, W. Kohnen then conjectured the bound

$$\sup_{(\tau,z)\in\mathbb{H}\times\mathbb{C}} \|f_m(\tau,z)\|_{\mathrm{Pet}} = O_{F,\epsilon}\big(m^{\frac{k-1}{2}+\epsilon}\big),$$

where the implied constant depends on the Siegel cusp form F and on the choice of $\epsilon > 0$.

More recently, P. Anamby and S. Das established in [AD23] a general sup-norm bound for the pointwise Petersson norm of $f \in J_{k,m}^{\mathrm{cusp}}(\Gamma_0)$, which is normalized with respect to the Petersson inner product, i. e., for which we have

$$\int\limits_{\Gamma_0\ltimes\mathbb{Z}^2\backslash\mathbb{H}\times\mathbb{C}} \|f(\tau,z)\|^2_{\mathrm{Pet}}\,\frac{\mathrm{d}\xi\wedge\mathrm{d}\eta\wedge\mathrm{d}x\wedge\mathrm{d}y}{\eta^3} = 1 \qquad (\tau=\xi+i\eta,\ z=x+iy).$$

Their bound is (see Theorem 1.4 in[AD23])

$$\sup_{(\tau,z)\in\mathbb{H}\times\mathbb{C}} \|f(\tau,z)\|_{\mathrm{Pet}} = O_{\epsilon}\big(k\,m\big), \tag{2}$$

where the implied constant depends on the choice of $\epsilon > 0$.

1.2. **Main results.** The goal of this article is to provide new sup-norm bounds for the pointwise Petersson norm for Jacobi forms of integral weight k and integral index m for Γ_0, which are normalized with respect to the Petersson inner product. The main result in this respect is given in Theorem 4.4 and states for $f \in J_{k,m}^{\mathrm{cusp}}(\Gamma_0)$, which is normalized with respect to the Petersson inner product, that

$$\sup_{(\tau,z)\in\mathbb{H}\times\mathbb{C}} \|f(\tau,z)\|_{\mathrm{Pet}} = O_{\epsilon}\big(k^{\frac{3}{4}}\,m^{1+\epsilon}\big), \tag{3}$$

where the implied constant depends only on the choice of $\epsilon > 0$. For the proof, we essentially use the representation of the Jacobi cusp forms under consideration as linear combinations of modular forms of weight $k-\frac{1}{2}$ for some congruence subgroup of Γ_0 (depending on m) and suitable Jacobi theta functions; we then need to derive bounds for the pointwise Petersson norms of these functions to arrive at our result. Comparing our bound (3) with the bound (2) by P. Anamby and S. Das, we realize an improvement with regard to the polynomial growth in k by a factor of $k^{\frac{1}{4}}$, while the polynomial growth of the bound (2) in m is better by a factor of m^{ϵ} (for any $\epsilon > 0$).

In order to be able to derive our bound (3), we need sup-norm bounds for the pointwise Petersson norm of cusp forms of positive real weight k and character χ for any cofinite Fuchsian subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$. Such bounds have been derived for finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ by Steiner in [St16]. Although these bounds are uniform with respect to the index of the finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$, there are mild restrictions for a direct application of the bounds derived in [St16] to our setting. On the other hand, such bounds could also be derived from [FJK16] with some extra work. However, we provide here new, alternative proofs for the results of [FJK16] applying to any positive real weight k and any character χ by using the Bergman kernel for the modular curve associated to Γ.

More specifically, given $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ a Fuchsian subgroup, $k \in \mathbb{R}_{>0}$, and $\chi\colon \Gamma \to \mathbb{C}^{\times}$ a character, we let $S_{k,\chi}(\Gamma)$ denote the space of cusp forms of weight k and character χ for Γ.

Denoting by d_k the dimension of $S_{k,\chi}(\Gamma)$ and letting $\{f_1,\ldots,f_{d_k}\}$ be an orthonormal basis of $S_{k,\chi}(\Gamma)$ with respect to the Petersson inner product, the Bergman kernel associated to $S_{k,\chi}(\Gamma)$ is then defined by

$$B_{k,\chi}(\tau,\tau') := \sum_{j=1}^{d_k} f_j(\tau)\overline{f_j(\tau')};$$

it is straightforward that this definition does not depend on the choice of an orthonormal basis of $S_{k,\chi}(\Gamma)$. The pointwise Petersson norm of the Bergman kernel is defined by

$$\|B_{k,\chi}(\tau,\tau')\|_{\mathrm{Pet}} = |B_{k,\chi}(\tau,\tau')|\,(\mathrm{Im}(\tau)\mathrm{Im}(\tau'))^{\frac{k}{2}},$$

which gives on the diagonal

$$\|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}} = \sum_{j=1}^{d_k} \|f_j(\tau)\|^2_{\mathrm{Pet}}. \tag{4}$$

As a second main result of this article, we establish in Theorem 3.3, assuming that $k\in\mathbb{R}_{\geq 5}$, for Γ being cocompact without elliptic elements the bound

$$\sup_{z\in\mathbb{H}} \|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}} = O_\Gamma(k);$$

moreover, for Γ being cofinite, we give the bound

$$\sup_{z\in\mathbb{H}} \|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}} = O_\Gamma\big(k^{\frac{3}{2}}\big),$$

where the implied constants depend only on the Fuchsian subgroup Γ. Due to the relation (4), these results reprove the sup-norm bounds on average obtained in [FJK16], but now for any real weight $k\in\mathbb{R}_{\geq 5}$ and any character χ. Based on these results, we are then also able to prove the uniformity of the above bounds with respect to the subgroup Γ in Theorem 3.5.

1.3. **Outline.** Let us briefly outline the contents of this article. In the subsequent, second section we collect all the necessary prerequisites for the sequel of the paper. In particular, we introduce the definitions of cusp forms and Jacobi cusp forms together with their respective (pointwise) Petersson inner products. Furthermore, we define the Bergman kernel for modular curves and state its basic properties. We close the section by recalling asymptotics of Bergman kernels associated to tensor powers of holomorphic line bundles on compact complex Kähler manifolds due to [Bo96], which are crucial in the derivation of the bound (3).

The third section is devoted to the revisiting of the sup-norm bounds on average obtained in [FJK16], but now for any real weight $k\in\mathbb{R}_{\geq 5}$ and any character χ. Here, the proofs of Theorem 3.3 and Theorem 3.5 are provided.

In the fourth section, the bound (3) is proven in Theorem 4.4. In addition to some straightforward bounds established for the relative L^2-norm of classical Jacobi theta functions, the above mentioned asymptotics of Bergman kernels associated to tensor powers of holomorphic line bundles corresponding to these Jacobi theta functions are crucial in the derivation of the proof of Theorem 4.4.

2. Preliminaries

2.1. **Hyperbolic metric.** Let $\mathbb{H} := \{\tau \in \mathbb{C} \,|\, \tau = \xi + i\eta,\, \eta > 0\}$ be the upper half-plane. We denote by $\mathrm{d}s^2_{\mathrm{hyp}}(\tau)$ the line element and by $\mu_{\mathrm{hyp}}(\tau)$ the volume form corresponding to the hyperbolic metric on $\mathbb{H}$, which is compatible with the complex structure of $\mathbb{H}$ and has constant curvature equal to -1. Locally on $\mathbb{H}$, we have

$$\mathrm{d}s^2_{\mathrm{hyp}}(\tau) = \frac{\mathrm{d}\xi^2 + \mathrm{d}\eta^2}{\eta^2} \quad \text{and} \quad \mu_{\mathrm{hyp}}(\tau) = \frac{\mathrm{d}\xi \wedge \mathrm{d}\eta}{\eta^2}\,.$$

For $\tau, \tau' \in \mathbb{H}$, we let $\mathrm{dist}_{\mathrm{hyp}}(\tau,\tau')$ denote the hyperbolic distance between these two points. For later purposes, it is useful to introduce the displacement function

$$\sigma(\tau,\tau') := \cosh^2\left(\frac{\mathrm{dist}_{\mathrm{hyp}}(\tau,\tau')}{2}\right) = \frac{|\tau - \overline{\tau}'|^2}{4\,\mathrm{Im}(\tau)\mathrm{Im}(\tau')}\,. \tag{5}$$

2.2. **Quotient space.** Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian subgroup acting by fractional linear transformations on $\mathbb{H}$. Let X_Γ be the quotient space $\Gamma\backslash\mathbb{H}$ and g_Γ the genus of X_Γ. In the sequel, we identify X_Γ with a fundamental domain $\mathcal{F}_\Gamma \subset \mathbb{H}$ for the group Γ, which we assume to be closed and connected.

Denote by

$$\mathcal{P}_\Gamma = \{p_1, \ldots, p_s\}$$

the set of cusps of $\mathcal{F}_\Gamma$. Let $\sigma_{\mathcal{P},j} \in \mathrm{SL}_2(\mathbb{R})$ be a scaling matrix of the cusp p_j, that is, $p_j = \sigma_{\mathcal{P},j} i\infty$ with stabilizer subgroup Γ_{p_j} described as

$$\sigma^{-1}_{\mathcal{P},j}\Gamma_{p_j}\sigma_{\mathcal{P},j} = \begin{cases} \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle, & \text{if } -\mathrm{id} \notin \Gamma, \\ \left\langle \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle, & \text{if } -\mathrm{id} \in \Gamma, \end{cases} \qquad (j = 1, \ldots, s). \tag{6}$$

For $Y > 0$, we let $\mathcal{F}^Y_j \subset \mathcal{F}_\Gamma$ denote the neighborhood of the cusp p_j characterized by

$$\sigma^{-1}_{\mathcal{P},j}\mathcal{F}^Y_j = \{\tau = \xi + i\eta \in \mathbb{H} \,|\, -1/2 \leq \xi \leq 1/2,\, \eta \geq Y\} \qquad (j = 1, \ldots, s).$$

With these notations, we define $\mathcal{F}_Y$ to be the closure of the complement of the union $\mathcal{F}^Y_1 \cup \ldots \cup \mathcal{F}^Y_s$ in $\mathcal{F}_\Gamma$, i.e.,

$$\mathcal{F}_Y := \mathrm{cl}\big(\mathcal{F}_\Gamma \setminus \big(\mathcal{F}^Y_1 \cup \ldots \cup \mathcal{F}^Y_s\big)\big), \tag{7}$$

which is compact; we note that $\mathcal{F}_Y = \mathcal{F}_\Gamma$, if Γ is cocompact. We choose $0 < m_Y < M_Y$ such that for all $\tau \in \mathcal{F}_Y$ the inequalities

$$m_Y \leq \mathrm{Im}(\sigma^{-1}_{\mathcal{P},j}\tau) \leq M_Y$$

hold for all $j = 1, \ldots, s$; we note that m_Y and M_Y depend on the choice of Y.

Denote by

$$\mathcal{E}_\Gamma = \{e_1, \ldots, e_t\}$$

the set of elliptic fixed points of $\mathcal{F}_\Gamma$. Let Γ_{e_j} and m_j denote the stabilizer subgroup and order of the elliptic fixed point e_j, respectively.

We denote the hyperbolic length of the shortest closed geodesic on X_Γ by ℓ_Γ. For a domain $D \subset \mathbb{H}$, we denote its hyperbolic diameter by $\mathrm{diam}_{\mathrm{hyp}}(D)$ and its hyperbolic volume by

$\mathrm{vol}_{\mathrm{hyp}}(D)$. Finally, the injectivity radius r_Γ is defined by

$$r_\Gamma := \inf\left\{\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau)\,\middle|\,\tau\in\mathcal{F}_\Gamma,\,\gamma\in\Gamma\setminus\left(\bigcup_{j=1}^{s}\Gamma_{p_j}\cup\bigcup_{j=1}^{t}\Gamma_{e_j}\right)\right\}. \tag{8}$$

We note that if X_Γ is compact without elliptic fixed points, i. e., $\mathcal{P}_\Gamma=\mathcal{E}_\Gamma=\emptyset$, then the injectivity radius r_Γ equals the length of the shortest closed geodesic ℓ_Γ of X_Γ.

2.3. **Cusp forms and Bergman kernel.** For $k\in\mathbb{R}_{>0}$ and a character $\chi\colon\Gamma\to\mathbb{C}^\times$, we let $S_{k,\chi}(\Gamma)$ denote the space of cusp forms of weight k and character χ for Γ, i. e., the space of holomorphic functions $f\colon\mathbb{H}\to\mathbb{C}$, which have the transformation behavior

$$f(\gamma z)(cz+d)^{-k}=\chi(\gamma)f(z)$$

for all $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$, and which vanish at all the cusps of $\mathcal{F}_\Gamma$. Given $f\in S_{k,\chi}(\Gamma)$, we define

$$\|f(\tau)\|^2_{\mathrm{Pet}}:=|f(\tau)|^2\,\eta^k\qquad(\tau=\xi+i\eta),$$

which defines a Γ-invariant function on $\mathbb{H}$ called the pointwise Petersson norm of f.

The space $S_{k,\chi}(\Gamma)$ is equipped with the Petersson inner product

$$\langle f_1,f_2\rangle_{\mathrm{Pet}}:=\int\limits_{\mathcal{F}_\Gamma}f_1(\tau)\overline{f_2(\tau)}\,\eta^k\mu_{\mathrm{hyp}}(\tau)\qquad(f_1,f_2\in S_{k,\chi}(\Gamma)). \tag{9}$$

Let d_k denote the dimension of $S_{k,\chi}(\Gamma)$ and let $\{f_1,\ldots,f_{d_k}\}$ be an orthonormal basis of $S_{k,\chi}(\Gamma)$ with respect to the Petersson inner product. Then, the Bergman kernel associated to $S_{k,\chi}(\Gamma)$ is defined by

$$B_{k,\chi}(\tau,\tau'):=\sum_{j=1}^{d_k}f_j(\tau)\overline{f_j(\tau')}.$$

It is obvious that this definition does not depend on the choice of an orthonormal basis of $S_{k,\chi}(\Gamma)$.

The Bergman kernel $B_{k,\chi}(\tau,\tau')$ is a holomorphic cusp form of weight k and character χ for Γ in the τ-variable, and an anti-holomorphic cusp form of weight k and character $\overline{\chi}$ for Γ in the τ'-variable. Hence, the pointwise Petersson norm of the Bergman kernel is given by

$$\|B_{k,\chi}(\tau,\tau')\|_{\mathrm{Pet}}=|B_{k,\chi}(\tau,\tau')|\,(\eta\eta')^{\frac{k}{2}},$$

which is a Γ-invariant function on $\mathbb{H}\times\mathbb{H}$ with respect to both variables.

Moreover, $B_{k,\chi}(\tau,\tau')$ is the reproducing kernel for $S_{k,\chi}(\Gamma)$, i. e., we have

$$\int\limits_{\mathcal{F}_\Gamma}B_{k,\chi}(\tau,\tau')f(\tau')\eta'^k\mu_{\mathrm{hyp}}(\tau')=f(\tau)\qquad(\tau'=\xi'+i\eta')$$

for any $f\in S_{k,\chi}(\Gamma)$. Therefore, for $k\in\mathbb{R}_{>3}$, the Bergman kernel $B_{k,\chi}(\tau,\tau')$ can also be represented in the following form (see Proposition 1.3 on p. 77 in [Fr90])

$$B_{k,\chi}(\tau,\tau')=\frac{(2i)^k(k-1)}{4\pi}\sum_{\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma}\frac{1}{(\tau-\gamma\overline{\tau}')^k}\frac{1}{\chi(\gamma)(c\overline{\tau}'+d)^k}\,. \tag{10}$$

Note that the formula for the Bergman kernel given in [Fr90] is missing a factor of $(2i)^k$.

2.4. **Counting function.** Given $\tau \in \mathbb{H}$ and $\rho \in \mathbb{R}_{\geq 0}$, we recall from [JL95] the counting function

$$N_\Gamma(\tau;\rho) := |\mathcal{N}_\Gamma(\tau;\rho)|,$$

where

$$\mathcal{N}_\Gamma(\tau;\rho) := \bigg\{\gamma \in \Gamma \setminus \bigg(\bigcup_{j=1}^{s} \Gamma_{p_j} \cup \bigcup_{j=1}^{t} \Gamma_{e_j}\bigg) \,\bigg|\, \mathrm{dist}_{\mathrm{hyp}}(\tau, \gamma\tau) \leq \rho\bigg\}.$$

Let now f be a positive, smooth, and decreasing function on $\mathbb{R}_{\geq 0}$. Then, adapting the arguments from [JL95] to Fuchsian subgroups of $\mathrm{SL}_2(\mathbb{R})$, we have for any $\tau \in \mathbb{H}$ and any $\delta \geq r_\Gamma/2$ the inequality

$$\int_0^\infty f(\rho)\,\mathrm{d}N_\Gamma(\tau;\rho) \leq \int_0^\delta f(\rho)\,\mathrm{d}N_\Gamma(\tau;\rho) + \frac{2|\mathrm{Cent}(\Gamma)|\cosh(r_\Gamma/4)}{\sinh(r_\Gamma/4)}\sinh(\delta)f(\delta) + \frac{|\mathrm{Cent}(\Gamma)|}{2\sinh^2(r_\Gamma/4)}\int_\delta^\infty f(\rho)\sinh\big(\rho + r_\Gamma/2\big)\,\mathrm{d}\rho; \tag{11}$$

here $\mathrm{Cent}(\Gamma)$ denotes the center of Γ. Note that our definition (8) of injectivity radius differs from the one used in [JL95] by a factor of 2, and the inequality (11) takes this fact into account.

2.5. **Jacobi forms.** For $k, m \in \mathbb{N}$, we let $J_{k,m}^{\mathrm{cusp}}(\Gamma_0)$ denote the space of Jacobi cusp forms of weight k and index m for $\Gamma_0 = \mathrm{SL}_2(\mathbb{Z})$, i. e., the space of holomorphic functions $f\colon \mathbb{H}\times\mathbb{C} \to \mathbb{C}$, which have the transformation behaviour

$$f\Big(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\Big)(c\tau+d)^{-k}\exp\Big(2\pi i m\Big(\lambda^2\tau + 2\lambda z - \frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}\Big)\Big) = f(\tau,z)$$

for all $\big[\big(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\big), (\lambda,\mu)\big] \in \Gamma_0 \ltimes \mathbb{Z}^2$, and which have a Fourier expansion of the form

$$f(\tau,z) = \sum_{\substack{n\in\mathbb{N},\, r\in\mathbb{Z}\\ 4mn-r^2>0}} c(n,r)\,q^n\zeta^r \qquad (q = e^{2\pi i\tau},\ \zeta = e^{2\pi i z}).$$

Given $f \in J_{k,m}^{\mathrm{cusp}}(\Gamma_0)$, we define

$$\|f(\tau,z)\|_{\mathrm{Pet}}^2 := |f(\tau,z)|^2\,\eta^k\,e^{-\frac{4\pi m y^2}{\eta}} \qquad (\tau = \xi + i\eta,\ z = x + iy),$$

which defines a $\Gamma_0 \ltimes \mathbb{Z}^2$-invariant function on $\mathbb{H}\times\mathbb{C}$ called the pointwise Petersson norm of f.

Let $\mathcal{D}_{\Gamma_0}$ denote a fundamental domain of the quotient space $Y_{\Gamma_0} := \Gamma_0 \ltimes \mathbb{Z}^2\backslash\mathbb{H}\times\mathbb{C}$, which is a 2-dimensional complex manifold. The space $J_{k,m}^{\mathrm{cusp}}(\Gamma_0)$ is equipped with the Petersson inner product

$$\langle f_1, f_2\rangle_{\mathrm{Pet}} := \int_{\mathcal{D}_{\Gamma_0}} f_1(\tau,z)\overline{f_2(\tau,z)}\,\eta^k\,e^{-\frac{4\pi m y^2}{\eta}}\,\frac{\mathrm{d}\xi\wedge\mathrm{d}\eta\wedge\mathrm{d}x\wedge\mathrm{d}y}{\eta^3} \qquad (f_1, f_2 \in J_{k,m}^{\mathrm{cusp}}(\Gamma_0)). \tag{12}$$

For $f \in J_{k,m}^{\mathrm{cusp}}(\Gamma_0)$, one has the decomposition

$$f(\tau,z) = \sum_{\mu=0}^{2m-1}\varphi_\mu(\tau)\vartheta_{\mu,m}(\tau,z), \tag{13}$$

where the function φ_μ is a cusp form of weight $(k-\frac{1}{2})$ for the finite index subgroup $\Gamma_1 := \Gamma_0(4m)$ of Γ_0, and $\vartheta_{\mu,m}$ is the Jacobi theta function

$$\vartheta_{\mu,m}(\tau,z) := \sum_{n\in\mathbb{Z}} e^{2\pi i m\tau\left(n-\frac{\mu}{2m}\right)^2+2\pi i z(2mn-\mu)}. \tag{14}$$

As we will see below, the theta functions $\vartheta_{\mu,m}$ ($\mu=0,\ldots,2m-1$) arise for fixed $\tau\in\mathbb{H}$ as global sections of a suitable line bundle on the elliptic curve associated to τ. In fact, it is shown in Theorem 5.1 of [EZ85] that the decomposition (14) gives rise to the isomorphism

$$J^{\mathrm{cusp}}_{k,m}(\Gamma_0) \cong \mathcal{V}_{k-\frac{1}{2}}(\Gamma_0),$$

where $\mathcal{V}_{k-\frac{1}{2}}(\Gamma_0)$ denotes the complex vector space of vector-valued cusp forms of weight $(k-\frac{1}{2})$ with suitable transformation behaviour with respect to Γ_0.

Let now

$$f_1(\tau,z) = \sum_{\mu=0}^{2m-1}\varphi_{\mu,1}(\tau)\vartheta_{\mu,m}(\tau,z) \quad\text{and}\quad f_2(\tau,z) = \sum_{\mu=0}^{2m-1}\varphi_{\mu,2}(\tau)\vartheta_{\mu,m}(\tau,z)$$

be two Jacobi cusp forms of weight k and index m for Γ_0. Then, the decomposition (13) gives rise to the equality (see Theorem 5.3 in [EZ85])

$$\langle f_1,f_2\rangle_{\mathrm{Pet}} = \frac{1}{\sqrt{4m}}\int_{\mathcal{F}_{\Gamma_0}}\sum_{\mu=0}^{2m-1}\varphi_{\mu,1}(\tau)\overline{\varphi_{\mu,2}(\tau)}\,\eta^{k-\frac{1}{2}}\,\frac{\mathrm{d}\xi\wedge\mathrm{d}\eta}{\eta^2}, \tag{15}$$

where we recall that $\mathcal{F}_{\Gamma_0}$ is a fundamental domain for the quotient space $X_{\Gamma_0}=\Gamma_0\backslash\mathbb{H}$.

2.6. **Asymptotics for Bergman kernels.** In this subsection, we recall asymptotics of Bergman kernels associated to tensor powers of holomorphic line bundles on compact complex Kähler manifolds, which are used in section 4 to derive bounds for theta functions.

Let (M,ω) be a compact complex Kähler manifold of dimension n with positive closed $(1,1)$-form ω. Let $\mathcal{L}$ be a positive hermitian holomorphic line bundle on M and let $H^0(M,\mathcal{L}^{\otimes m})$ denote the vector space of global holomorphic sections its m-th tensor power $\mathcal{L}^{\otimes m}$ for $m\in\mathbb{Z}_{\geq 1}$. Let $|\cdot|_{\mathcal{L}^{\otimes m}}$ and $\langle\cdot,\cdot\rangle_{\mathrm{L}^2,\mathcal{L}^{\otimes m}}$ denote the pointwise hermitian metric and the L^2-inner product on $H^0(M,\mathcal{L}^{\otimes m})$, respectively.

Let $\{s_j\}$ denote an orthonormal basis of $H^0(M,\mathcal{L}^{\otimes m})$ with respect to the L^2-inner product. For any $z\in M$, the function

$$B_{\mathcal{L}^{\otimes m}}(z) := \sum_j |s_j(z)|^2_{\mathcal{L}^{\otimes m}}$$

is called the Bergman kernel associated to the line bundle $\mathcal{L}^{\otimes m}$. We note that the above definition is independent on the choice of an orthonormal basis of $H^0(M,\mathcal{L}^{\otimes m})$.

Let

$$\mathrm{c}_1(\mathcal{L},|\cdot|_{\mathcal{L}})(z) := -\frac{i}{2\pi}\partial_z\partial_{\overline{z}}\log|s(z)|^2_{\mathcal{L}} \tag{16}$$

denote the curvature form of the line bundle $\mathcal{L}$ at the point $z\in M$, where s is any meromorphic section of $\mathcal{L}$. At any $z\in M$, there exists a coordinate chart around the point z such that

$$\omega(z) = \sum_{j=1}^n\frac{i}{2}\cdot\mathrm{d}z_j\wedge\mathrm{d}\overline{z}_j \quad\text{and}\quad \mathrm{c}_1(\mathcal{L},|\cdot|_{\mathcal{L}})(z) = \sum_{j=1}^n\frac{i}{2}\cdot\alpha_j\cdot\mathrm{d}z_j\wedge\mathrm{d}\overline{z}_j.$$

The complex numbers $\alpha_1,\dots,\alpha_n$ are called the eigenvalues of the curvature form $\mathrm{c}_1(\mathcal{L},|\cdot|_{\mathcal{L}})(z)$ at the point $z\in M$. We set

$$\det_{\omega}\big(\mathrm{c}_1(\mathcal{L},|\cdot|_{\mathcal{L}})(z)\big):=\prod_{j=1}^{n}\alpha_j.$$

Since the line bundle $\mathcal{L}$ is positive, we have $\alpha_j>0$ for $j=1,\dots,n$. Finally, we recall from Theorem 2.1 in [Bo96] the bound

$$B_{\mathcal{L}^{\otimes m}}(z)=\det_{\omega}\big(\mathrm{c}_1(\mathcal{L},|\cdot|_{\mathcal{L}})(z)\big)m^n+O(m^{n-1}),\tag{17}$$

provided that $\mathcal{L}$ is a positive line bundle for any $z\in M$.

3. Sup-norm bounds for cusp forms revisited

Refining arguments of [AM17] and [AM18], we first derive bounds for the Bergman kernel along the diagonal.

Proposition 3.1. *With notations as above, let $\Gamma\subset\mathrm{SL}_2(\mathbb{R})$ be a cocompact Fuchsian subgroup without elliptic elements. Then, for $k\in\mathbb{R}_{\geq 5}$ and $\tau\in\mathbb{H}$, we have the bound*

$$\|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}}\leq\frac{k-1}{2\pi}+\frac{3(k-1)}{\pi\cosh^{k-4}(r_\Gamma/4)}\bigg(1+\frac{1}{\sinh^2(r_\Gamma/4)}\bigg).$$

Proof. Letting $k\in\mathbb{R}_{\geq 5}$ and considering the Bergman kernel (10) on the diagonal, we derive by means of relation (5) the bound

$$\begin{aligned}\|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}}&=\frac{2^k(k-1)}{4\pi}\Bigg|\sum_{\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma}\frac{1}{(\tau-\gamma\overline{\tau})^k}\frac{1}{\chi(\gamma)(c\overline{\tau}+d)^k}\Bigg|\,\mathrm{Im}(\tau)^k\\&\leq\frac{k-1}{4\pi}\sum_{\gamma\in\Gamma}\bigg(\frac{4\,\mathrm{Im}(\tau)\mathrm{Im}(\gamma\tau)}{|\tau-\gamma\overline{\tau}|^2}\bigg)^{k/2}=\frac{k-1}{4\pi}\sum_{\gamma\in\Gamma}\frac{1}{\cosh^k(\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau)/2)}\\&=\frac{k-1}{4\pi}\bigg(|\mathrm{Cent}(\Gamma)|+\sum_{\gamma\in\Gamma\setminus\mathrm{Cent}(\Gamma)}\frac{1}{\cosh^k(\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau)/2)}\bigg).\end{aligned}\tag{18}$$

Substituting $\delta=r_\Gamma/2$ in inequality (11) and using the fact that $|\mathrm{Cent}(\Gamma)|\leq 2$, we derive

$$\begin{aligned}&\sum_{\gamma\in\Gamma\setminus\mathrm{Cent}(\Gamma)}\frac{1}{\cosh^k(\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau)/2)}\\&\leq\int_0^{r_\Gamma/2}\frac{\mathrm{d}N_\Gamma(\tau;\rho)}{\cosh^k(\rho/2)}+\frac{8}{\cosh^{k-2}(r_\Gamma/4)}+\frac{1}{\sinh^2(r_\Gamma/4)}\int_{r_\Gamma/2}^{\infty}\frac{\sinh(\rho+r_\Gamma/2)}{\cosh^k(\rho/2)}\,\mathrm{d}\rho.\end{aligned}\tag{19}$$

From the defining equation (8) of the injectivity radius r_Γ, we find for the first term of (19) that

$$\int_0^{r_\Gamma/2}\frac{\mathrm{d}N_\Gamma(\tau;\rho)}{\cosh^k(\rho/2)}=0.\tag{20}$$

With regard to the third term of (19), we recall the bound (12) in [AM17], which states for any $k \in \mathbb{R}_{\geq 5}$ and any $\delta \geq 0$ (note that we have replaced $2k$ by k) that

$$\frac{1}{\sinh^2(r_\Gamma/4)} \int_\delta^\infty \frac{\sinh(\rho + r_\Gamma/2)}{\cosh^k(\rho/2)}\,\mathrm{d}\rho$$
$$\leq \frac{4}{(k-2)\cosh^{k-2}(\delta/2)}\left(2 + \frac{1}{\sinh^2(r_\Gamma/4)}\right) + \frac{8}{(k-4)\cosh^{k-4}(\delta/2)} \cdot \frac{1}{\sinh^2(r_\Gamma/4)}. \tag{21}$$

From the elementary inequality $\cosh^{k-4}(r_\Gamma/4) \leq \cosh^{k-2}(r_\Gamma/4)$ and recalling that $k \in \mathbb{R}_{\geq 5}$, we now derive from (21) with $\delta = r_\Gamma/2$ the bound

$$\frac{1}{\sinh^2(r_\Gamma/4)} \int_{r_\Gamma/2}^\infty \frac{\sinh(\rho + r_\Gamma/2)}{\cosh^k(\rho/2)}\,\mathrm{d}\rho$$
$$\leq \frac{4}{(k-2)\cosh^{k-2}(r_\Gamma/4)}\left(2 + \frac{1}{\sinh^2(r_\Gamma/4)}\right) + \frac{8}{(k-4)\cosh^{k-4}(r_\Gamma/4)} \cdot \frac{1}{\sinh^2(r_\Gamma/4)}$$
$$\leq \frac{4}{\cosh^{k-4}(r_\Gamma/4)}\left(1 + \frac{1}{\sinh^2(r_\Gamma/4)}\right) + \frac{8}{(k-4)\cosh^{k-4}(r_\Gamma/4)} \cdot \frac{1}{\sinh^2(r_\Gamma/4)}. \tag{22}$$

Combining the bounds (18), (19) with (20), (22), and using the fact that $k \in \mathbb{R}_{\geq 5}$, completes the proof of the proposition. □

Proposition 3.2. *With notations as above, let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a cofinite Fuchsian subgroup. Then, for $k \in \mathbb{R}_{\geq 5}$ and $\tau \in \mathbb{H}$, we have the bound*

$$\|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}} \leq \frac{k-1}{2\pi} + \frac{3(k-1)}{\pi\cosh^{k-4}(r_\Gamma/4)}\left(1 + \frac{1}{\sinh^2(r_\Gamma/4)}\right)$$
$$+ \frac{k-1}{4\pi}\sum_{e_j \in \mathcal{E}_\Gamma}(m_j - 1) + \frac{2(k-1)}{\sqrt{\pi}} \cdot \frac{\Gamma((k-1)/2)}{\Gamma(k/2)} \sum_{p_j \in \mathcal{P}_\Gamma} \mathrm{Im}(\sigma_{\mathcal{P},j}^{-1}\tau),$$

where $\sigma_{\mathcal{P},j}$ is the scaling matrix associated to the cusp $p_j \in \mathcal{P}_\Gamma$ defined in (6).

Proof. For $k \in \mathbb{R}_{\geq 5}$ and $\tau \in \mathbb{H}$, using the bound (18) and the fact that $|\mathrm{Cent}(\Gamma)| \leq 2$, we derive

$$\|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}} \leq \frac{k-1}{2\pi} + \frac{k-1}{4\pi} \sum_{\gamma \in \Gamma \setminus \left(\bigcup_{e_j \in \mathcal{E}_\Gamma} \Gamma_{e_j} \cup \bigcup_{p_j \in \mathcal{P}_\Gamma} \Gamma_{p_j}\right)} \frac{1}{\cosh^k(\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau)/2)}$$
$$+ \frac{k-1}{4\pi} \sum_{e_j \in \mathcal{E}_\Gamma} \sum_{\gamma \in \Gamma_{e_j} \setminus \mathrm{Cent}(\Gamma)} \frac{1}{\cosh^k(\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau)/2)}$$
$$+ \frac{k-1}{4\pi} \sum_{p_j \in \mathcal{P}_\Gamma} \sum_{\gamma \in \Gamma_{p_j} \setminus \mathrm{Cent}(\Gamma)} \frac{1}{\cosh^k(\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau)/2)}. \tag{23}$$

Adapting our arguments from Proposition 3.1 to the second summand on the right-hand side of (23), we arrive at the bound

$$\frac{k-1}{4\pi}\sum_{\gamma\in\Gamma\setminus\left(\bigcup_{e_j\in\mathcal{E}_\Gamma}\Gamma_{e_j}\cup\bigcup_{p_j\in\mathcal{P}_\Gamma}\Gamma_{p_j}\right)}\frac{1}{\cosh^k(\mathrm{dist_{hyp}}(\tau,\gamma\tau)/2)}$$

$$\leq\frac{3(k-1)}{\pi\cosh^{k-4}(r_\Gamma/4)}\left(1+\frac{1}{\sinh^2(r_\Gamma/4)}\right). \tag{24}$$

For the third term on the right-hand side of (23), we trivially have the bound

$$\frac{k-1}{4\pi}\sum_{e_j\in\mathcal{E}_\Gamma}\sum_{\gamma\in\Gamma_{e_j}\setminus\mathrm{Cent}(\Gamma)}\frac{1}{\cosh^k(\mathrm{dist_{hyp}}(\tau,\gamma\tau)/2)}\leq\frac{k-1}{4\pi}\sum_{e_j\in\mathcal{E}_\Gamma}(m_j-1). \tag{25}$$

From the definition of the scaling matrix (6) and using the fact that $|\mathrm{Cent}(\Gamma)|\leq 2$, we find

$$\frac{k-1}{4\pi}\sum_{p_j\in\mathcal{P}_\Gamma}\sum_{\gamma\in\Gamma_{p_j}\setminus\mathrm{Cent}(\Gamma)}\frac{1}{\cosh^k(\mathrm{dist_{hyp}}(\tau,\gamma\tau)/2)}$$

$$\leq\frac{k-1}{2\pi}\sum_{p_j\in\mathcal{P}_\Gamma}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{\cosh^k(\mathrm{dist_{hyp}}(\sigma_{\mathcal{P},j}^{-1}\tau,\sigma_{\mathcal{P},j}^{-1}\tau+n)/2)}. \tag{26}$$

We now recall the bound (18) in [AM17], which gives for $k\in\mathbb{R}_{\geq5}$, $p_j\in\mathcal{P}_\Gamma$, and $\tau,\tau'\in\mathbb{H}$ (note that we have replaced $2k$ by k) the bound

$$\frac{k-1}{2\pi}\sum_{p_j\in\mathcal{P}_\Gamma}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{\cosh^k(\mathrm{dist_{hyp}}(\sigma_{\mathcal{P},j}^{-1}\tau,\sigma_{\mathcal{P},j}^{-1}\tau'+n)/2)}$$

$$\leq\frac{k-1}{\sqrt{\pi}}\cdot\frac{\Gamma\big((k-1)/2\big)}{\Gamma(k/2)}\sum_{p_j\in\mathcal{P}_\Gamma}\frac{\big(4\,\mathrm{Im}(\sigma_{\mathcal{P},j}^{-1}\tau)\mathrm{Im}(\sigma_{\mathcal{P},j}^{-1}\tau')\big)^{k/2}}{\big(\mathrm{Im}(\sigma_{\mathcal{P},j}^{-1}\tau)+\mathrm{Im}(\sigma_{\mathcal{P},j}^{-1}\tau')\big)^{k-1}}. \tag{27}$$

Substituting $\tau=\tau'$ in (27) and combining it with (26), we arrive for the fourth term on the right-hand side of (23) at the bound

$$\frac{k-1}{4\pi}\sum_{p_j\in\mathcal{P}_\Gamma}\sum_{\gamma\in\Gamma_{p_j}\setminus\mathrm{Cent}(\Gamma)}\frac{1}{\cosh^k(\mathrm{dist_{hyp}}(\tau,\gamma\tau)/2)}$$

$$\leq\frac{2(k-1)}{\sqrt{\pi}}\cdot\frac{\Gamma\big((k-1)/2\big)}{\Gamma(k/2)}\sum_{p_j\in\mathcal{P}_\Gamma}\mathrm{Im}(\sigma_{\mathcal{P},j}^{-1}\tau). \tag{28}$$

Combining the bounds (24), (25), and (28) with (23) completes the proof of the proposition. □

Theorem 3.3. *With notations as above, let* $\Gamma\subset\mathrm{SL}_2(\mathbb{R})$ *be a cofinite Fuchsian subgroup and* $k\in\mathbb{R}_{\geq5}$. *Then, if* Γ *is cocompact without elliptic elements, we have the bound*

$$\sup_{\tau\in\mathbb{H}}\|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}}=O_\Gamma(k). \tag{29}$$

Moreover, if Γ *is cofinite, we have the bound*

$$\sup_{\tau\in\mathbb{H}}\|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}}=O_\Gamma\big(k^{\frac{3}{2}}\big). \tag{30}$$

The implied constants in the bounds (29) *and* (30) *depend only on the Fuchsian subgroup* Γ.

Proof. When Γ is cocompact without elliptic elements, the claimed bound (29) follows directly from Proposition 3.1.
Let next Γ be a cofinite Fuchsian subgroup. From the proof of Theorem 6.1 in [FJK16], it follows that

$$\sup_{\tau\in\mathbb{H}} \|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}} = \sup_{\substack{\tau\in\partial\mathcal{F}_Y\\ Y=k/(2\pi)}} \|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}}, \tag{31}$$

where $\partial\mathcal{F}_Y$ denotes the boundary of the truncated fundamental domain $\mathcal{F}_Y$ defined in (7). Combining Proposition 3.2 with (31) and employing the fact that

$$\frac{\Gamma\big((k-1)/2\big)}{\Gamma(k/2)} = O\bigg(\frac{1}{\sqrt{k}}\bigg),$$

we derive

$$\sup_{\tau\in\mathbb{H}} \|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}} \leq \frac{k-1}{2\pi} + \frac{3(k-1)}{\pi\cosh^{k-4}(r_\Gamma/4)}\bigg(1+\frac{1}{\sinh^2(r_\Gamma/4)}\bigg) + (k-1)\,C_{\Gamma,\mathrm{ell}} + k^{\frac{3}{2}}\,C_{\Gamma,\mathrm{par}}$$

for some positive constants $C_{\Gamma,\mathrm{ell}}, C_{\Gamma,\mathrm{par}}$, which depend only on the Fuchsian subgroup Γ. This completes the proof of the theorem. □

Corollary 3.4. *With notations as above, let $\Gamma\subset\mathrm{SL}_2(\mathbb{R})$ be a cofinite Fuchsian subgroup. For $k\in\mathbb{R}_{\geq 5}$, let $f\in S_{k,\chi}(\Gamma)$ be a cusp form, which is normalized with respect to the Petersson inner product* (9). *If Γ is cocompact without elliptic elements, we have the bound*

$$\sup_{\tau\in\mathbb{H}} \|f(\tau)\|^2_{\mathrm{Pet}} = O_\Gamma(k). \tag{32}$$

Moreover, if Γ is cofinite, we have the bound

$$\sup_{\tau\in\mathbb{H}} \|f(\tau)\|^2_{\mathrm{Pet}} = O_\Gamma\big(k^{\frac{3}{2}}\big). \tag{33}$$

The implied constants in the bounds (32) *and* (33) *depend only on the Fuchsian subgroup Γ.*

Proof. Choose an orthonormal basis $\{f_1=f,\ldots,f_{d_k}\}$ of $S_{k,\chi}(\Gamma)$ with respect to the Petersson inner product (9). For $\tau\in\mathbb{H}$, we then have the bound

$$\|f(\tau)\|^2_{\mathrm{Pet}} \leq \|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}}.$$

The proof of the corollary now immediately follows from Theorem 3.3. □

Theorem 3.5. *With notations as above, let $\Gamma_0\subset\mathrm{SL}_2(\mathbb{R})$ be a fixed cofinite Fuchsian subgroup and let $\Gamma\subseteq\Gamma_0$ be a finite index subgroup of Γ_0. For $k\in\mathbb{R}_{\geq 5}$, let $f\in S_{k,\chi}(\Gamma)$ be a cusp form, which is normalized with respect to the Petersson inner product* (9). *If Γ_0 is cocompact without elliptic elements, we have the bound*

$$\sup_{\tau\in\mathbb{H}} \|f(\tau)\|^2_{\mathrm{Pet}} = O_{\Gamma_0}(k). \tag{34}$$

Moreover, if Γ_0 is cofinite, we have the bound

$$\sup_{\tau\in\mathbb{H}} \|f(\tau)\|^2_{\mathrm{Pet}} = O_{\Gamma_0}\big(k^{\frac{3}{2}}\big). \tag{35}$$

The implied constants in the bounds (34) *and* (35) *depend only on the Fuchsian subgroup Γ_0.*

Proof. Choose an orthonormal basis $\{f_1 = f, \ldots, f_{d_k}\}$ of $S_{k,\chi}(\Gamma)$ with respect to the Petersson inner product (9).

Let now Γ_0 be a cocompact Fuchsian subgroup without elliptic elements. From the proof of Proposition 3.1,we derive the bound

$$\begin{aligned}\sup_{\tau\in\mathbb{H}} \|f(\tau)\|^2_{\mathrm{Pet}} &\leq \sup_{\tau\in\mathbb{H}} \|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}} \\ &\leq \frac{k-1}{4\pi} \sup_{\tau\in\mathbb{H}} \sum_{\gamma\in\Gamma} \frac{1}{\cosh^k(\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau)/2)} \\ &\leq \frac{k-1}{4\pi} \sup_{\tau\in\mathbb{H}} \sum_{\gamma\in\Gamma_0} \frac{1}{\cosh^k(\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau)/2)} = O_{\Gamma_0}(k),\end{aligned}$$

which completes the proof of the theorem in the case that Γ_0 is cocompact without elliptic elements.

Let next Γ_0 be a cofinite Fuchsian subgroup. Given $Y > 0$, we recall from (7) the fundamental domain decomposition

$$\mathcal{F}_\Gamma = \mathcal{F}_Y \cup \left(\mathcal{F}_1^Y \cup \ldots \cup \mathcal{F}_s^Y\right),$$

where $\mathcal{F}_Y$ is a compact subset of $\mathcal{F}_\Gamma$ and the $\mathcal{F}_j^Y$'s are neighborhoods of the cusps $p_j \in \mathcal{P}_\Gamma$ $(j = 1, \ldots, s)$. Choosing Y large enough, we can assume without loss of generality in the sequel that the neighborhoods $\mathcal{F}_j^Y$ are pairwise disjoint. We now first provide a bound for the pointwise Petersson norm of f, when τ ranges across the compact set $\mathcal{F}_Y$, and subsequently we compute bounds for the pointwise Petersson norm of f, when τ ranges across the neighborhoods $\mathcal{F}_j^Y$ of the cusps for fixed, large enough Y.

Adapting arguments from the proof of Proposition 3.2, we obtain the bound

$$\begin{aligned}\sup_{\tau\in\mathcal{F}_Y} \|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}} &\leq \frac{k-1}{4\pi} \sup_{\tau\in\mathcal{F}_Y} \sum_{\gamma\in\Gamma} \frac{1}{\cosh^k(\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau)/2)} \\ &\leq \frac{k-1}{4\pi} \sup_{\tau\in\mathcal{F}_Y} \sum_{\gamma\in\Gamma_0} \frac{1}{\cosh^k(\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau)/2)} \\ &\leq \frac{k-1}{2\pi} + \frac{3(k-1)}{\pi\cosh^{k-4}(r_{\Gamma_0,Y}/4)}\left(1 + \frac{1}{\sinh^2(r_{\Gamma_0,Y}/4)}\right) + \frac{k-1}{4\pi} \sum_{e_j\in\mathcal{E}_{\Gamma_0}} (m_{e_j}-1),\end{aligned} \tag{36}$$

where

$$r_{\Gamma_0,Y} = \inf\left\{\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau) \,\middle|\, \tau\in\mathcal{F}_Y,\, \gamma\in\Gamma_0 \setminus \bigcup_{e_j\in\mathcal{E}_{\Gamma_0}} \Gamma_{0,e_j}\right\} > 0.$$

From this, we immediately conclude that

$$\sup_{\tau\in\mathcal{F}_Y} \|f(\tau)\|^2_{\mathrm{Pet}} \leq \sup_{\tau\in\mathcal{F}_Y} \|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}} = O_{\Gamma_0,Y}(k), \tag{37}$$

where the implied constant depends on Γ_0 and the choice of Y.

We are left to provide bounds for the pointwise Petersson norm of f, when τ ranges across the neighborhoods $\mathcal{F}_j^Y$ of the cusps. For this, we will have to distinguish between the two cases $Y > \frac{k}{2\pi}$ and $Y < \frac{k}{2\pi}$. Without loss of generality, we can assume that $j = 1$, when $p_1 \in \mathcal{P}_\Gamma$ is the cusp at infinity for Γ lying above the cusp p at infinity for Γ_0 with ramification index $[\Gamma_{0,p} : \Gamma_{p_1}]$ (note that $\Gamma_{0,p}$ denotes the stabilizer subgroup of p in Γ_0). With the above notations, we obtain the inclusion $\Gamma \setminus \Gamma_{p_1} \subseteq \Gamma_0 \setminus \Gamma_{0,p}$.

We first treat the case $Y > \frac{k}{2\pi}$, which implies that $\mathcal{F}_1^Y \subset \mathcal{F}_1^{k/(2\pi)}$. Arguing as in the proof of Theorem 6.1 in [FJK16], we deduce, recalling the inclusion $\Gamma \setminus \Gamma_{p_1} \subseteq \Gamma_0 \setminus \Gamma_{0,p}$, that

$$\sup_{\tau\in\mathcal{F}_1^Y} \|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}} \leq \sup_{\tau\in\mathcal{F}_1^{k/(2\pi)}} \|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}} \leq \sup_{\tau\in\partial\mathcal{F}_1^{k/(2\pi)}} \|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}}$$
$$\leq \frac{k-1}{4\pi}\left(\sup_{\tau\in\partial\mathcal{F}_1^{k/(2\pi)}} \sum_{\gamma\in\Gamma\setminus\Gamma_{p_1}} \frac{1}{\cosh^k(\mathrm{dist_{hyp}}(\tau,\gamma\tau)/2)} + \sup_{\tau\in\partial\mathcal{F}_1^{k/(2\pi)}} \sum_{\gamma\in\Gamma_{p_1}} \frac{1}{\cosh^k(\mathrm{dist_{hyp}}(\tau,\gamma\tau)/2)}\right)$$
$$\leq \frac{k-1}{4\pi}\left(\sup_{\tau\in\partial\mathcal{F}_1^{k/(2\pi)}} \sum_{\gamma\in\Gamma_0\setminus\Gamma_{0,p}} \frac{1}{\cosh^k(\mathrm{dist_{hyp}}(\tau,\gamma\tau)/2)} + \sup_{\tau\in\partial\mathcal{F}_1^{k/(2\pi)}} \sum_{\gamma\in\Gamma_{p_1}} \frac{1}{\cosh^k(\mathrm{dist_{hyp}}(\tau,\gamma\tau)/2)}\right). \tag{38}$$

Arguments similar to the ones used to derive the bound (36), lead for the first term of (38) to the bound

$$\frac{k-1}{4\pi} \sup_{\tau\in\partial\mathcal{F}_1^{k/(2\pi)}} \sum_{\gamma\in\Gamma_0\setminus\Gamma_{0,p}} \frac{1}{\cosh^k(\mathrm{dist_{hyp}}(\tau,\gamma\tau)/2)}$$
$$\leq \frac{k-1}{2\pi} + \frac{3(k-1)}{\pi\cosh^{k-4}(r_{\Gamma_0,k/(2\pi)}/4)}\left(1+\frac{1}{\sinh^2(r_{\Gamma_0,k/(2\pi)}/4)}\right) + \frac{k-1}{4\pi}\sum_{e_j\in\mathcal{E}_{\Gamma_0}}(m_{e_j}-1),$$

where

$$r_{\Gamma_0,k/(2\pi)} = \inf\left\{\mathrm{dist_{hyp}}(\tau,\gamma\tau)\,\middle|\,\tau\in\partial\mathcal{F}_1^{k/(2\pi)},\ \gamma\in\Gamma_0\setminus\left(\Gamma_{0,p}\cup\bigcup_{e_j\in\mathcal{E}_{\Gamma_0}}\Gamma_{0,e_j}\right)\right\} > 0.$$

Since it is easy to see that

$$\frac{1}{\sinh^2(r_{\Gamma_0,k/(2\pi)})} = O_{\Gamma_0}(1),$$

we arrive for the first term of (38) at the bound

$$\frac{k-1}{4\pi} \sup_{\tau\in\partial\mathcal{F}_1^{k/(2\pi)}} \sum_{\gamma\in\Gamma_0\setminus\Gamma_{0,p}} \frac{1}{\cosh^k(\mathrm{dist_{hyp}}(\tau,\gamma\tau)/2)} = O_{\Gamma_0}(k). \tag{39}$$

Using the same arguments as in the proof of Proposition 3.2, we derive for the second term of (38) the bound

$$\frac{k-1}{4\pi} \sup_{\tau\in\partial\mathcal{F}_1^{k/(2\pi)}} \sum_{\gamma\in\Gamma_{p_1}} \frac{1}{\cosh^k(\mathrm{dist_{hyp}}(\tau,\gamma\tau)/2)} = O\big(k^{\frac{3}{2}}\big). \tag{40}$$

By means of (38), we thus deduce from (39) and (40) in the case $Y > \frac{k}{2\pi}$ the bound

$$\sup_{\tau\in\mathcal{F}_1^Y} \|f(\tau)\|^2_{\mathrm{Pet}} \leq \sup_{\tau\in\mathcal{F}_1^Y} \|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}} = O_{\Gamma_0}\big(k^{\frac{3}{2}}\big). \tag{41}$$

We finally turn to the case $Y < \frac{k}{2\pi}$, which implies that $\mathcal{F}_1^{k/(2\pi)} \subset \mathcal{F}_1^Y$. Here we find, arguing as in the preceding case that

$$\begin{aligned}\sup_{\tau\in\mathcal{F}_1^Y} \|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}} &\leq \sup_{\tau\in\mathcal{F}_1^Y\setminus\mathcal{F}_1^{k/(2\pi)}} \|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}} \\ &\leq \frac{k-1}{4\pi} \sup_{\tau\in\mathcal{F}_1^Y\setminus\mathcal{F}_1^{k/(2\pi)}} \sum_{\gamma\in\Gamma_0\setminus\Gamma_{0,p}} \frac{1}{\cosh^k(\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau)/2)} \\ &\quad + \frac{k-1}{4\pi} \sup_{\tau\in\mathcal{F}_1^Y\setminus\mathcal{F}_1^{k/(2\pi)}} \sum_{\gamma\in\Gamma_{p_1}} \frac{1}{\cosh^k(\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau)/2)}. \end{aligned} \tag{42}$$

As before, we now obtain the bounds

$$\begin{aligned}&\frac{k-1}{4\pi} \sup_{\tau\in\mathcal{F}_1^Y\setminus\mathcal{F}_1^{k/(2\pi)}} \sum_{\gamma\in\Gamma_0\setminus\Gamma_{0,p}} \frac{1}{\cosh^k(\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau)/2)} \\ &\leq \frac{k-1}{2\pi} + \frac{3(k-1)}{\pi\cosh^{k-4}(r'_{\Gamma_0,Y}/4)}\left(1+\frac{1}{\sinh^2(r'_{\Gamma_0,Y}/4)}\right) + \frac{k-1}{4\pi}\sum_{e_j\in\mathcal{E}_{\Gamma_0}}(m_{e_j}-1) \\ &= O_{\Gamma_0,Y}(k), \end{aligned} \tag{43}$$

noting that

$$r'_{\Gamma_0,Y} = \inf\left\{\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau) \,\middle|\, \tau\in\mathcal{F}_1^Y,\, \gamma\in\Gamma_0\setminus\left(\Gamma_{0,p}\cup\bigcup_{e_j\in\mathcal{E}_{\Gamma_0}}\Gamma_{0,e_j}\right)\right\} > 0,$$

as well as the bound

$$\frac{k-1}{4\pi} \sup_{\tau\in\mathcal{F}_1^Y\setminus\mathcal{F}_1^{k/(2\pi)}} \sum_{\gamma\in\Gamma_{p_1}} \frac{1}{\cosh^k(\mathrm{dist}_{\mathrm{hyp}}(\tau,\gamma\tau)/2)} = O\big(k^{\frac{3}{2}}\big). \tag{44}$$

By means of (42), we thus deduce from (43) and (44) in the case $Y < \frac{k}{2\pi}$ the bound

$$\sup_{\tau\in\mathcal{F}_1^Y}\|f(\tau)\|^2_{\mathrm{Pet}} \leq \sup_{\tau\in\mathcal{F}_1^Y}\|B_{k,\chi}(\tau,\tau)\|_{\mathrm{Pet}} = O_{\Gamma_0,Y}\big(k^{\frac{3}{2}}\big). \tag{45}$$

Since Y has been fixed, the claim of the theorem follows from (37), (41), and (45). □

Remark 3.6. If $f \in S_{k,\chi}(\Gamma)$ is not a Hecke eigenform, then there is no evidence from the literature to suggest that the estimates (32) and (33) can be improved. Thus the estimates (32) and (33) are expected to be optimal.

4. Sup-norm bounds for Jacobi cusp forms

For $k \in \mathbb{Z}_{\geq 5}$ and $m \in \mathbb{Z}_{\geq 1}$, let $f \in J^{\mathrm{cusp}}_{k,m}(\Gamma_0)$ be a Jacobi cusp form of weight k and index m for the full modular group $\Gamma_0 = \mathrm{SL}_2(\mathbb{Z})$, which is normalized with respect to the Petersson inner product defined by (12). We now aim at bounding the quantity

$$\sup_{(\tau,z)\in\mathbb{H}\times\mathbb{C}} \|f(\tau,z)\|_{\mathrm{Pet}}.$$

For this we will employ the decomposition (13), namely

$$f(\tau,z) = \sum_{\mu=0}^{2m-1} \varphi_\mu(\tau)\vartheta_{\mu,m}(\tau,z),$$

where the functions φ_μ are cusp forms of weight $(k-\frac{1}{2})$ with respect to the finite index subgroup $\Gamma_1 = \Gamma_0(4m)$ of Γ_0 and the theta functions $\vartheta_{\mu,m}$ are defined in (14).

For a fixed $\tau = \xi + i\eta \in \mathbb{H}$, consider the elliptic curve $E_\tau := \mathbb{C}/\Lambda_\tau$ with $\Lambda_\tau := \mathbb{Z} \oplus \tau\mathbb{Z}$. Let O_τ denote the identity element of E_τ, when considered as an abelian group with $\oplus_\tau$ denoting the group operation, and let

$$[2]\colon E_\tau \longrightarrow E_\tau$$

be multiplication by 2, given by the assignment $z \mapsto 2z := z \oplus_\tau z$, which is an isogeny of degree 4. Let $\mathcal{M}_\tau$ be the line bundle associated to the divisor O_τ. Then, the theorem of the cube gives the isomorphism

$$[2]^*\mathcal{M}_\tau \cong \mathcal{M}_\tau^{\otimes 4}.$$

We then put $\mathcal{L}_\tau := \mathcal{M}_\tau^{\otimes 2}$ and find that the theta functions $\vartheta_{\mu,m}(\tau,\cdot)$ ($\mu = 0,\ldots,2m-1$) arise as global holomorphic sections of the line bundle $\mathcal{L}_\tau^{\otimes m}$.

The pointwise norm of $\vartheta_{\mu,m}(\tau,\cdot) \in H^0(E_\tau, \mathcal{L}_\tau^{\otimes m})$ at the point $z = x + iy \in E_\tau$ (identifying E_τ with its universal cover $\mathbb{C}$) is given by the following formula

$$\|\vartheta_{\mu,m}(\tau,z)\|^2_{\mathcal{L}_\tau^{\otimes m}} := |\vartheta_{\mu,m}(\tau,z)|^2 \sqrt{\eta}\, e^{-\frac{4\pi m y^2}{\eta}}. \tag{46}$$

Let μ_{eucl} denote the Euclidean metric on E_τ; at the point $z = x + iy \in E_\tau$, it is given by the formula

$$\mu_{\mathrm{eucl}}(z) = \frac{i}{2} \cdot \frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{\eta} = \frac{\mathrm{d}x \wedge \mathrm{d}y}{\eta}. \tag{47}$$

Furthermore, the L^2-inner product of $\vartheta_{\mu,m}(\tau,\cdot), \vartheta_{\mu',m}(\tau,\cdot) \in H^0(E_\tau, \mathcal{L}_\tau^{\otimes m})$ is given by the formula

$$\begin{aligned} \langle \vartheta_{\mu,m}(\tau,\cdot), \vartheta_{\mu',m}(\tau,\cdot)\rangle_{\mathrm{L}^2,\mathcal{L}_\tau^{\otimes m}} &:= \int_{E_\tau} \vartheta_{\mu,m}(\tau,z)\,\overline{\vartheta_{\mu',m}(\tau,z)}\,\sqrt{\eta}\, e^{-\frac{4\pi m y^2}{\eta}}\,\mu_{\mathrm{eucl}}(z) \\ &= \int_0^\eta \int_0^1 \vartheta_{\mu,m}(\tau,z)\,\overline{\vartheta_{\mu',m}(\tau,z)}\,\sqrt{\eta}\, e^{-\frac{4\pi m y^2}{\eta}}\,\frac{\mathrm{d}x \wedge \mathrm{d}y}{\eta}. \end{aligned} \tag{48}$$

The L^2-norm of $\vartheta_{\mu,m}(\tau,\cdot) \in H^0(E_\tau, \mathcal{L}_\tau^{\otimes m})$ induced by the L^2-inner product defined above is thus given by

$$\|\vartheta_{\mu,m}(\tau,\cdot)\|^2_{\mathrm{L}^2,\,\mathcal{L}_\tau^{\otimes m}} = \langle \vartheta_{\mu,m}(\tau,\cdot), \vartheta_{\mu,m}(\tau,\cdot)\rangle_{\mathrm{L}^2,\mathcal{L}_\tau^{\otimes m}}. \tag{49}$$

We next recall the well-known fact about the orthogonality of the Jacobi theta functions (which was already implicitly used in formula (15)).

Lemma 4.1. *The set of Jacobi theta functions $\{\vartheta_{\mu,m}(\tau,\cdot)\}_{\mu=0}^{2m-1}$ constitutes an orthogonal basis for the space $H^0(E_\tau, \mathcal{L}_\tau^{\otimes m})$ with respect to the L^2-inner product (48).*

Proof. As above, we write $\tau = \xi + i\eta \in \mathbb{H}$ and for $\mu, \mu' = 0,\ldots,2m-1$ and $n \in \mathbb{Z}$, we define

$$a(\mu,n) := n - \frac{\mu}{2m} \quad \text{and} \quad b(\mu,n) := 2mn - \mu. \tag{50}$$

Recalling the definition of the Jacobi theta function (14), we compute for the L^2-inner product (48)

$$\begin{aligned} &\langle \vartheta_{\mu,m}(\tau,\cdot), \vartheta_{\mu',m}(\tau,\cdot)\rangle_{\mathrm{L}^2,\,\mathcal{L}_\tau^{\otimes m}} \\ &= \int_0^\eta \sum_{n\in\mathbb{Z}} \sum_{n'\in\mathbb{Z}} e^{2\pi i m\tau a(\mu,n)^2}\,\overline{e^{2\pi i m\tau a(\mu',n')^2}} \left(\int_0^1 e^{2\pi i z b(\mu,n)}\,\overline{e^{2\pi i z b(\mu',n')}}\,\mathrm{d}x \right) e^{-\frac{4\pi m y^2}{\eta}}\,\frac{\mathrm{d}y}{\sqrt{\eta}}. \end{aligned} \tag{51}$$

Observing that the inequalities $-2m < \mu - \mu' < 2m$ imply that the equality

$$b(\mu,n) = b(\mu',n') \quad \Longleftrightarrow \quad 2m(n-n') = \mu - \mu'$$

can only hold if $n = n'$ and $\mu = \mu'$, we deduce

$$\int_0^1 e^{2\pi i z b(\mu,n)}\overline{e^{2\pi i z b(\mu',n')}}\,\mathrm{d}x = e^{-2\pi y(b(\mu,n)+b(\mu',n'))}\int_0^1 e^{2\pi i x(b(\mu,n)-b(\mu',n'))}\,\mathrm{d}x$$

$$= \begin{cases} e^{-4\pi y b(\mu,n)}, & \text{if } \mu=\mu' \text{ and } n=n', \\ 0, & \text{else.}\end{cases} \tag{52}$$

This proves the pairwise orthogonality of the $2m$ Jacobi theta functions $\{\vartheta_{\mu,m}(\tau,\cdot)\}_{\mu=0}^{2m-1}$ and thus their linear independence. Now, since we have $\deg(\mathcal{L}_\tau^{\otimes m}) = 2m$, the Riemann–Roch Theorem on the elliptic curve E_τ shows that

$$\dim H^0(E_\tau, \mathcal{L}_\tau^{\otimes m}) = 2m,$$

from which the claim follows. □

Proposition 4.2. *With notations as above, we have for $\mu = 0,\ldots,2m-1$, the bound*

$$\sup_{\tau\in\mathbb{H}} \|\vartheta_{\mu,m}(\tau,\cdot)\|^2_{\mathrm{L}^2,\mathcal{L}_\tau^{\otimes m}} = O\bigg(\frac{1}{\sqrt{m}}\bigg).$$

Proof. Let $\tau = \xi + i\eta \in \mathbb{H}$, $z = x+iy \in \mathbb{C}$, and $\mu = 0,\ldots,2m-1$. For $\tau\in\mathbb{H}$, we need to bound the L^2-norm (49) of $\vartheta_{\mu,m}(\tau,\cdot)\in H^0(E_\tau,\mathcal{L}_\tau^{\otimes m})$, namely the quantity

$$\|\vartheta_{\mu,m}(\tau,\cdot)\|^2_{\mathrm{L}^2,\mathcal{L}_\tau^{\otimes m}} = \langle \vartheta_{\mu,m}(\tau,\cdot),\vartheta_{\mu,m}(\tau,\cdot)\rangle_{\mathrm{L}^2,\mathcal{L}_\tau^{\otimes m}}.$$

Recalling the notation (50) and introducing $a(n) := a(\mu,n)$ as well as $b(n) := b(\mu,n)$ as a short-hand, we arrive as in (51) at the equality

$$\|\vartheta_{\mu,m}(\tau,\cdot)\|^2_{\mathrm{L}^2,\mathcal{L}_\tau^{\otimes m}} = \int_0^\eta \sum_{n\in\mathbb{Z}}\sum_{n'\in\mathbb{Z}} e^{2\pi i m\tau a(n)^2}\overline{e^{2\pi i m\tau a(n')^2}}\bigg(\int_0^1 e^{2\pi i z b(n)}\overline{e^{2\pi i z b(n')}}\,\mathrm{d}x\bigg)\, e^{-\frac{4\pi m y^2}{\eta}}\,\frac{\mathrm{d}y}{\sqrt{\eta}}. \tag{53}$$

Since (by (52))

$$\int_0^1 e^{2\pi i z b(n)}\overline{e^{2\pi i z b(n')}}\,\mathrm{d}x = \begin{cases} e^{-4\pi y b(n)}, & \text{if } n=n', \\ 0, & \text{else,}\end{cases}$$

we arrive at

$$\sum_{n\in\mathbb{Z}}\sum_{n'\in\mathbb{Z}} e^{2\pi i m\tau a(n)^2}\overline{e^{2\pi i m\tau a(n')^2}}\int_0^1 e^{2\pi i z b(n)}\overline{e^{2\pi i z b(n')}}\,\mathrm{d}x = \sum_{n\in\mathbb{Z}} e^{-4\pi m\eta\left(n-\frac{\mu}{2m}\right)^2 - 4\pi y(2mn-\mu)}. \tag{54}$$

Substituting (54) into (53) and using an integral test, we find the bound

$$\|\vartheta_{\mu,m}(\tau,\cdot)\|^2_{\mathrm{L}^2,\mathcal{L}_\tau^{\otimes m}} = \sum_{n\in\mathbb{Z}} \int_0^\eta e^{-4\pi m\eta\left(n-\frac{\mu}{2m}\right)^2-4\pi y(2mn-\mu)}\, e^{-\frac{4\pi m y^2}{\eta}}\,\frac{\mathrm{d}y}{\sqrt{\eta}}$$

$$\leq \int_0^\eta \int_{-\infty}^\infty e^{-4\pi m\eta\left(\nu-\frac{\mu}{2m}\right)^2-4\pi y(2m\nu-\mu)-\frac{4\pi m y^2}{\eta}}\,\mathrm{d}\nu\,\frac{\mathrm{d}y}{\sqrt{\eta}} \tag{55}$$

$$+ \int_0^\eta e^{-\frac{\pi\eta\mu^2}{m}+4\pi y\mu-\frac{4\pi m y^2}{\eta}}\,\frac{\mathrm{d}y}{\sqrt{\eta}}. \tag{56}$$

Now, we rewrite the exponent of the integrand in (55) in the form

$$4\pi m\eta\left(\nu-\frac{\mu}{2m}\right)^2 + 4\pi y(2m\nu-\mu) + \frac{4\pi m y^2}{\eta}$$
$$= \left(\sqrt{\frac{\pi\eta}{m}}(2m\nu-\mu)\right)^2 + 4\pi y(2m\nu-\mu) + \left(2\sqrt{\frac{\pi m}{\eta}}y\right)^2$$
$$= \left(\sqrt{\frac{\pi\eta}{m}}(2m\nu-\mu) + 2\sqrt{\frac{\pi m}{\eta}}y\right)^2.$$

Substituting

$$\rho := \sqrt{\frac{\pi\eta}{m}}(2m\nu-\mu) + 2\sqrt{\frac{\pi m}{\eta}}y$$

into (55), we obtain for the inner integral

$$\int_{-\infty}^\infty e^{-4\pi m\eta\left(\nu-\frac{\mu}{2m}\right)^2-4\pi y(2m\nu-\mu)-\frac{4\pi m y^2}{\eta}}\,\mathrm{d}\nu = \frac{1}{2\sqrt{\pi m\eta}}\int_{-\infty}^\infty e^{-\rho^2}\,\mathrm{d}\rho = \frac{1}{2\sqrt{m}}\frac{1}{\sqrt{\eta}}.$$

From this, we compute the double integral (55) as

$$\int_0^\eta \int_{-\infty}^\infty e^{-4\pi m\eta\left(\nu-\frac{\mu}{2m}\right)^2-4\pi y(2m\nu-\mu)-\frac{4\pi m y^2}{\eta}}\,\mathrm{d}\nu\,\frac{\mathrm{d}y}{\sqrt{\eta}} = \frac{1}{2\sqrt{m}}\frac{1}{\sqrt{\eta}}\int_0^\eta \frac{\mathrm{d}y}{\sqrt{\eta}} = \frac{1}{2\sqrt{m}}. \tag{57}$$

For the integral (56), we find in a similar way

$$\int_0^\eta e^{-\frac{\pi\eta\mu^2}{m}+4\pi y\mu-\frac{4\pi m y^2}{\eta}}\,\frac{\mathrm{d}y}{\sqrt{\eta}} \leq \int_{-\infty}^\infty e^{-\left(-\sqrt{\frac{\pi\eta}{m}}\mu+2\sqrt{\frac{\pi m}{\eta}}y\right)^2}\,\frac{\mathrm{d}y}{\sqrt{\eta}} = \frac{1}{2\sqrt{m}}. \tag{58}$$

Adding up the bounds (57) and (58), yields

$$\|\vartheta_{\mu,m}(\tau,\cdot)\|^2_{\mathrm{L}^2,\mathcal{L}_\tau^{\otimes m}} \leq \frac{1}{\sqrt{m}},$$

which proves the claim. □

Proposition 4.3. *With notations as above, we have the bound*

$$\sup_{(\tau,z)\in\mathbb{H}\times\mathbb{C}} \sum_{\mu=0}^{2m-1} \|\vartheta_{\mu,m}(\tau,z)\|^2_{\mathcal{L}_\tau^{\otimes m}} = O(\sqrt{m}). \tag{59}$$

Proof. We start by considering the Bergman kernel $B_{\mathcal{L}_\tau^{\otimes m}}(z)$ associated to the line bundle $\mathcal{L}_\tau^{\otimes m}$ on the elliptic curve E_τ equipped with the Kähler form $\omega = \mu_{\mathrm{eucl}}$ (see formula (47)). From the bound (17), we recall

$$B_{\mathcal{L}_\tau^{\otimes m}}(z) = \det\nolimits_\omega\big(\mathrm{c}_1(\mathcal{L}_\tau, \|\cdot\|_{\mathcal{L}_\tau})(z)\big)\, m + O(1),$$

where $\mathrm{c}_1(\mathcal{L}_\tau, \|\cdot\|_{\mathcal{L}_\tau})(z)$ is the curvature form of $\mathcal{L}_\tau$ at the point $z \in E_\tau$. Using equations (16) and (46), the curvature form $\mathrm{c}_1(\mathcal{L}_\tau, \|\cdot\|_{\mathcal{L}_\tau})(z)$ is given by the formula (identifying again E_τ with its universal cover $\mathbb{C}$ and writing $z = x + iy$)

$$\begin{aligned}\mathrm{c}_1(\mathcal{L}_\tau, \|\cdot\|_{\mathcal{L}_\tau})(z) &= -\frac{i}{2\pi}\partial_z\partial_{\overline{z}} \log \|\vartheta_{\mu,1}(\tau, z)\|^2_{\mathcal{L}_\tau} \\ &= -\frac{i}{2\pi}\partial_z\partial_{\overline{z}}\bigg(-\frac{4\pi y^2}{\eta}\bigg) = 2\mu_{\mathrm{eucl}}(z) = 2\omega(z).\end{aligned}$$

This yields $\det_\omega\big(\mathrm{c}_1(\mathcal{L}_\tau, \|\cdot\|_{\mathcal{L}_\tau})(z)\big) = 2$, and therefore we get

$$B_{\mathcal{L}_\tau^{\otimes m}}(z) = 2m + O(1). \tag{60}$$

Setting

$$\Theta_{\mu,m}(\tau, z) := \frac{\vartheta_{\mu,m}(\tau, z)}{\|\vartheta_{\mu,m}(\tau, \cdot)\|_{\mathrm{L}^2,\mathcal{L}^{\otimes m}}} \qquad (\mu = 0, \ldots, 2m-1)$$

and using Lemma 4.1, we deduce that the set $\{\Theta_{\mu,m}(\tau,\cdot)\}_{\mu=0}^{2m-1}$ constitutes an orthonormal basis for the space $H^0(E_\tau, \mathcal{L}_\tau^{\otimes m})$ with respect to the L^2-inner product (48). By the very definition of the Bergman kernel we thus have

$$B_{\mathcal{L}_\tau^{\otimes m}}(z) = \sum_{\mu=0}^{2m-1} \|\Theta_{\mu,m}(\tau, z)\|^2_{\mathcal{L}_\tau^{\otimes m}}$$

for any $z \in E_\tau$. Finally, using Proposition 4.2 and observing that

$$\sum_{\mu=0}^{2m-1} \|\vartheta_{\mu,m}(\tau, z)\|^2_{\mathcal{L}_\tau^{\otimes m}} \leq \frac{1}{\sqrt{m}} \sum_{\mu=0}^{2m-1} \|\Theta_{\mu,m}(\tau, z)\|^2_{\mathcal{L}_\tau^{\otimes m}} = \frac{1}{\sqrt{m}} B_{\mathcal{L}_\tau^{\otimes m}}(z),$$

the claim follows by means of (60). □

Theorem 4.4. *For $k \in \mathbb{Z}_{\geq 5}$ and $m \in \mathbb{Z}_{\geq 1}$, let $f \in J^{\mathrm{cusp}}_{k,m}(\Gamma_0)$ be a Jacobi cusp form of weight k and index m for the full modular group $\Gamma_0 = \mathrm{SL}_2(\mathbb{Z})$, which is normalized with respect to the Petersson inner product. Then, we have the bound*

$$\sup_{(\tau,z)\in\mathbb{H}\times\mathbb{C}} \|f(\tau, z)\|^2_{\mathrm{Pet}} = O_\epsilon\big(k^{\frac{3}{2}}\, m^{2+\epsilon}\big), \tag{61}$$

where the implied constant depends only on the choice of $\epsilon > 0$.

Proof. Substituting the decomposition (13) of the Jacobi form $f(\tau, z)$ into its pointwise Petersson norm and applying the Cauchy–Schwartz inequality, we compute

$$\begin{aligned}\|f(\tau,z)\|^2_{\mathrm{Pet}} &= \Bigg|\sum_{\mu=0}^{2m-1}\varphi_\mu(\tau)\vartheta_{\mu,m}(\tau,z)\Bigg|^2 \eta^k\, e^{-\frac{4\pi m y^2}{\eta}} \\ &\leq \Bigg(\sum_{\mu=0}^{2m-1}|\varphi_\mu(\tau)|^2\,\eta^{k-1/2}\Bigg)\Bigg(\sum_{\mu=0}^{2m-1}|\vartheta_{\mu,m}(\tau,z)|\,\eta^{1/2}\,e^{-\frac{4\pi m y^2}{\eta}}\Bigg) \\ &= \Bigg(\sum_{\mu=0}^{2m-1}\|\varphi_\mu(\tau)\|^2_{\mathrm{Pet}}\Bigg)\Bigg(\sum_{\mu=0}^{2m-1}\|\vartheta_{\mu,m}(\tau,z)\|^2_{\mathcal{L}_\tau^{\otimes m}}\Bigg).\end{aligned}$$

Thus

$$\sup_{(\tau,z)\in\mathbb{H}\times\mathbb{C}}\|f(\tau,z)\|^2_{\mathrm{Pet}} \leq \sup_{\tau\in\mathbb{H}}\sum_{\mu=0}^{2m-1}\|\varphi_\mu(\tau)\|^2_{\mathrm{Pet}}\cdot \sup_{(\tau,z)\in\mathbb{H}\times\mathbb{C}}\sum_{\mu=0}^{2m-1}\|\vartheta_{\mu,m}(\tau,z)\|^2_{\mathcal{L}_\tau^{\otimes m}}. \tag{62}$$

Since f is normalized with respect to the Petersson inner product, we have from (15) that

$$\langle f, f\rangle_{\mathrm{Pet}} = \frac{1}{\sqrt{4m}\,[\Gamma_0:\Gamma_1]}\sum_{\mu=0}^{2m-1}\langle\varphi_\mu,\varphi_\mu\rangle_{\mathrm{Pet}} = 1. \tag{63}$$

Combining (63) with the bound (35) given in Theorem 3.5, we find the bound

$$\sum_{\mu=0}^{2m-1}\|\varphi_\mu(\tau)\|^2_{\mathrm{Pet}} = \sum_{\mu=0}^{2m-1}\langle\varphi_\mu,\varphi_\mu\rangle_{\mathrm{Pet}}\,\frac{\|\varphi_\mu(\tau)\|^2_{\mathrm{Pet}}}{\langle\varphi_\mu,\varphi_\mu\rangle_{\mathrm{Pet}}} \leq C\,\sqrt{4m}\,[\Gamma_0:\Gamma_1]\,k^{\frac{3}{2}}, \tag{64}$$

where C is a positive constant depending on Γ_0. Recalling now that

$$[\Gamma_0:\Gamma_1] = [\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(4m)] = O_\epsilon\big(m^{1+\epsilon}\big), \tag{65}$$

where the implied constant depends only on the choice of $\epsilon > 0$, we derive from the bounds (64) and (65), the bound

$$\sup_{\tau\in\mathbb{H}}\sum_{\mu=0}^{2m-1}\|\varphi_\mu(\tau)\|^2_{\mathrm{Pet}} = O_\epsilon\big(k^{\frac{3}{2}}\,m^{\frac{3}{2}+\epsilon}\big), \tag{66}$$

where the implied constant depends only on the choice of $\epsilon > 0$. The claimed bound (61) finally follows from (62) by combining the bound (66) with the bound (59) established in Proposition 4.3. □

Remark 4.5. The bound (61) is polynomial in k and m, and thus improves W. Kohnen's bound (1), which is exponential in k. Comparing our bound with the one obtained by P. Anamby and S. Das in [AD23], there is an improvement with regard to the polynomial growth in k, while the polynomial growth in m is slightly worse.

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